Branching Rules of Classical Lie Groups in Two Ways

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Contents

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Notations

$\mathrm{Mat}_n(\mathbb{C})$	associative algebraic of $n \times n$ complex matrices
$\mathrm{GL}(n,\mathbb{C})$	group of invertible $\mathbb{C}$-linear transformations on $\mathbb{C}^n$
$\mathrm{GL}(n,\mathbb{R})$	group of invertible $\mathbb{R}$-linear transformations on $\mathbb{R}^n$
$\mathrm{O}(n,\mathbb{C})$	complex orthogonal group consisting of $\{g \in \mathrm{GL}(n,\mathbb{C}) : g^t g = I\}$, where I is the $n \times n$ identity matrix and g^t is the transpose of g
$\mathrm{U}(n)$	unitary group consisting of $\{g \in \mathrm{GL}(n,\mathbb{C}) : g^* g = I\}$, where g^* is the conjugate transpose of g
$\mathrm{SO}(n)$	real special orthogonal group $\{g \in \mathrm{GL}(n,\mathbb{R}) : g^t g = 1, \det(g) = 1\}$
S^1	group of unit circle $\{z \in \mathbb{C}^\times : \|z\| = 1\}$
S_n	nth symmetric group
$G \downarrow H$	branching rule from G to its subgroup H, where the inclusion $H \hookrightarrow G$ is understood
T	maximal torus of a compact connected Lie group
W	Weyl group corresponding to a maximal torus
$\mathrm{X}(T)$	$\mathrm{Hom}_{\mathrm{Lie}}(T, S^1)$, the character lattice of T
Λ	integral lattice, or kernel of the exponential map of a torus
Λ^*	lattice of integral froms
Φ	roots of a compact connected Lie group with a choice of maximal torus
K	Weyl chambers corresponding to Φ
Φ_+	subset of Φ consisting of positive roots for a choice of K
ρ	half sum of the roots
χ_π	character of a representation π
π_λ^G	continuous irreducible representation of G with highest weight λ
$[\pi_\lambda^G : \pi_\mu^H]$	multiplicity of π_μ^H in π_λ^G restricting to H
$\ell(\lambda)$	length of a partition λ
$\|\lambda\|$	sum of $\lambda_1 + \lambda_2 + \cdots$ for a partition λ
$s_\lambda^{(n)}$	Schur polynomial of n variables with signature λ
$h_k^{(n)}$	complete symmetric polynomial in n variables
$\Sigma(n)$	Weyl group of $\mathrm{SO}(2n+1)$
$\Sigma_0(n)$	Weyl group of $\mathrm{SO}(2n)$
$\mathbb{D}(V)$	Weyl algebra associated to V
$\mathcal{P}(V)$	space of polynomials on V
$S(V)$	symmteric algebra of V

1 Introduction

Suppose we have a group G and a subgroup $H \subset G$. Given an irreducible representation (π, V) of G, it is not necessarily true that the restricted representation $(\pi|_H, V)$ of H is irreducible. In the case where G is a compact Lie group and H a closed subgroup, the restricted representation $\pi|_H$ will decompose into a sum of irreducible representations of the subgroup H. The rules for such decomposition are called branching rules. Branching rules have important applications in physics, for example, in the case of explicit symmetric breaking. In representation theory, we often need to know the decomposition of the tensor product of irreducible representations into a sum of irreducible representations; such tensor product rule is the same as the branching rule for G as the diagonally embedded subgroup into $G \times G$. We will see examples of this for $G = \mathrm{GL}(n, \mathbb{C})$ and $\mathrm{O}(n, \mathbb{C})$.

In the first half of this paper, we present an original combinatorial proof of the branching rules for the successive unitary groups and for the successive special orthogonal groups. The proof of $\mathrm{U}(n) \downarrow \mathrm{U}(n-1)$ branching relies on the Weyl character formula and the ingredients from symmetric function theory that involves proper cancellation of Schur polynomials of horizontal strips. The branching rule for orthogonal group, on the other hand, involves two different cases, $\mathrm{SO}(2n+1) \downarrow \mathrm{SO}(2n)$ and $\mathrm{SO}(2n) \downarrow \mathrm{SO}(2n-1)$. The branching rule for $\mathrm{SO}(2n+1) \downarrow \mathrm{SO}(2n)$ is easier to obtain since both groups share the same maximal torus. To get the branching rule for $\mathrm{SO}(2n) \downarrow \mathrm{SO}(2n-1)$, we make use of the branching rule for unitary group along with careful combinatorial manipulation. Two remarkable things come out of these branching rules. First, the branching rules share an interleaving pattern. Secondly, all these branching rules have multiplicity one. If we keep branching down an irreducible representation of $\mathrm{U}(n)$ or $\mathrm{SO}(n)$ all the way to $\mathrm{U}(1)$ or $\mathrm{SO}(2)$, the irreducible summands will terminante in one-dimensional representations. In this way, we can get a basis labelled by a chain of interleaved signatures, called *Gelfand-Tsetlin basis*, of any irreducible representations of $\mathrm{U}(n)$ or $\mathrm{SO}(n)$.

The second half of the paper is an exposition on the existing literature of obtaining branching rules via duality. Our exposition is based on Chapter 4 and 5 of [GW09] and [HTW05]. First we setup the language of linear algebraic groups over $\mathbb{C}$ and introduce the more general context of representation theory of algebras. Then we prove a general theorem on the duality between the irreducible regular representations of a linear algebraic group G and the irreducible representations of the commuting algebra of G. As an application, we next prove a duality theorem when the commuting algebra comes from the Weyl algebra which acts as polynomial differential operators on polynomials on a complex vector space. This will give us $\mathrm{GL}(n,\mathbb{C})$-$\mathrm{GL}(k,\mathbb{C})$ duality and $\mathrm{O}(n,\mathbb{C})$-$\mathfrak{sp}(k,\mathbb{C})$ Howe duality. By combining Schur-Weyl duality and the $\mathrm{GL}(n,\mathbb{C})$-$\mathrm{GL}(k,\mathbb{C})$ duality, we give three interpretations of the Littlewood-Richardson coefficients as branching rules of three different group/subgroup pairs. Finally, we prove the branching rule for $\mathrm{O}(n,\mathbb{C}) \times \mathrm{O}(n,\mathbb{C}) \downarrow \mathrm{O}(n,\mathbb{C})$ that is only valid for irreducible $\mathrm{O}(n,\mathbb{C})$-representations in the *stable range*. This proof will make use of most of the dualities we have developed, following [HTW05].

2 Branching Rule of Successive Unitary Groups

2.1 Weyl Character Formula

We briefly introduce the notations and conventions we use to describe the root system of a compact connected Lie group. Our conventions follow Chapter V of [BtD85].

Let G be a compact connected Lie group, and fix T to be a maximal torus of G with $T \cong (S^1)^n$. We define the *Weyl group* of G (with fixed choice of T) to be $W = N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. Then W acts on T naturally by conjugation. We denote the *character lattice* of G to be $\mathrm{X}(T) = \mathrm{Hom}_{\mathrm{Lie}}(T, S^1)$, and we let W act on $\mathrm{X}(T)$ via $(g.\lambda)(t) = \lambda(g^{-1}t)$. We can identify $\mathrm{X}(T) \cong \mathbb{Z}^n$ as follows. If $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$, then the corresonding element in $\mathrm{X}(T)$ is the continuous map sending $(t_1, \dots, t_n) \in T$ to $\prod_{i=1}^n t_i^{\mu_i}$. We write $t^\mu = \prod_{i=1}^n t_i^{\mu_i}$. This way, the multiplication on $\mathrm{X}(T)$ becomes addition on $\mathbb{Z}^n$.

Let (π, V) be a finite-dimensional continuous representation of G over $\mathbb{C}$. If we restrict the representation to T, then $(\pi|_T, V)$ will decompose as a sum of one-dimensional irreducible representations of T. In terms of characters, we have $\chi_\pi(t) = \sum_{\mu \in \mathrm{X}(T)} m_\pi(\mu) t^\mu$, where $m_\pi(\mu)$ is the dimension of the *weight space* $V_\mu = \{v \in V : \pi(t)v = t^\mu v, \forall t \in T\}$. We say $\lambda \in \mathrm{X}(T)$ is a *weight* of V if $m_\pi(\lambda) > 0$.

Let $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation of G, and let $\mathrm{Ad}_{\mathbb{C}}$ denote the complexified representation. We use Φ to denote the set of nonzero weights of $\mathrm{Ad}_{\mathbb{C}}$, and we call them *roots* of G. We choose an Ad-invariant inner product $\langle -, - \rangle$ on $\mathfrak{g}$, which is possible since G is compact. The induced action of W on $\mathfrak{t}$ agrees with $\mathrm{Ad}|_{N_G(T)}$, so the inner product is also W-invariant. We identify $\mathfrak{t}$ with $\mathfrak{t}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{t}, \mathbb{R})$ via this inner product. The kernel Λ of the exponential map $\exp : \mathfrak{t} \to T$ is called the *integral lattice*, and the group $\Lambda^* = \{\alpha \in \mathfrak{t}^* : \alpha\Lambda \subset \mathbb{Z}\}$ is called the *lattice of integral forms*. We will always choose the coordinates of T so that Λ is simply $\mathbb{Z}^n$ and $\langle -, - \rangle$ is the standard Euclidean metric, so that $\Lambda^* = \mathbb{Z}^n$ as well. Thus given $a \in \Phi$, we can view it as an element of Λ^* via $\mathrm{X}(T) \cong \mathbb{Z}^n$.

Given $a \in \Phi$, we can associate a hyperplane $\mathscr{H}_a \subset \mathfrak{t}$ with $\mathscr{H}_a = \mathrm{Lie}(\ker a)$. These hyperplanes will divide $\mathfrak{t}$ into finitely many convex regions, called *Weyl chambers*. We will fix a choice of K for each G, and we call it the *fundamental Weyl chamber*. For such K, we can assign to it a set of *positive roots* $\Phi_+ = \{a \in \Phi : \langle a, t \rangle > 0 \text{ for all } t \in K\}$. Then we can define a partial order on $\mathfrak{t}^*$. We write $\gamma \le \lambda$ for $\gamma, \lambda \in \mathfrak{t}^*$ if $\langle \gamma, \tau \rangle \le \langle \lambda, \tau \rangle$ for every $\tau \in K$. In this case we say λ is *higher* than γ. We call elements in $\overline{K} \cap \Lambda^*$ *dominant weights*.

The gem of the theory of compact Lie group is the following celebrated character formula by Weyl that describes the character of all irreducible representation of a compact connected

Lie group. A proof of it can be found in Section VI.1, VI.2 of [BtD85], or Chapter 22 of [Bum04].

Theorem 2.1.1 (Weyl character formula)**.** *Suppose* (π, V) *is an irreducible continuous representation of a compact connected Lie group* G*. Then*

(i) π *has a unique highest weight* λ *with* $m_\pi(\lambda) = 1$*.*

(ii) π *is the unique irreducible continuous representation of* G *with highest weight* λ*. In addition, every dominant weight is the highest weight of some irreducible continuous representation.*

(iii) For $t \in T$*,*

$$\chi_\pi(t) = \frac{\sum_{w\in W} \det(w) t^{w(\lambda+\rho)}}{\prod_{a\in\Phi_+}(t^{a/2} - t^{-a/2})},$$

where $\det(w)$ *is the determinant of the action of* w *on* $\mathfrak{t}$*, and* $\rho = \frac{1}{2}\sum_{a\in\Phi_+} a$*, the half sum if positive roots. We call the denominator the* Weyl denominator with respect to G*. Moreover,*

$$\prod_{a\in\Phi_+}(t^{a/2} - t^{-a/2}) = \sum_{w\in W} \det(w) t^{w(\rho)}.$$

Note that since W preserves the inner product $\langle -, - \rangle$ on $\mathfrak{t}$, we have $\det(w) = \pm 1$. It is a fact that elements of W are generated by hyperplane reflections on $\mathfrak{t}$, so we also write $\det(w) = (-1)^{\ell(w)}$, where $\ell(w)$ denote the smallest number of elementary reflections used to generate w (see Chapter 20 of [Bum04]). The two equivalent descriptions of the denominator in the Weyl character formula will turn out to be useful in different scenarios.

2.2 Maximal Torus and Root System of U(n)

We use the following description of the root system of U(n), following the convention in Section V.6 of [BtD85]. We assume $n \geq 2$. We choose T to be the maximal torus of U(n) consisting of diagonal matrices. That is, $T = \operatorname{diag}(t_1, \ldots, t_n)$ for $t_i \in S^1$. The Weyl group of U(n) turns out to be the nth symmetric group $W \cong S_n$, and W acts on T by permuting t_i's. The induced W-action on $\mathfrak{t} \cong \mathbb{R}^n$ is, for $\sigma \in W$ and $\mu \in \mathfrak{t}$,

$$\sigma(\mu_1, \mu_2, \ldots, \mu_n) = (\mu_{\sigma(1)}, \mu_{\sigma(2)}, \ldots, \mu_{\sigma(n)}).$$

The determinant $\det(w) = (-1)^{\ell(w)}$ in the character formula is the same as the sign of the permuation w. The roots of U(n), identified with elements in $\Lambda^* \cong \mathbb{Z}^n$, are $e_i - e_j$, $1 \leq i, j \leq n$. Here $e_i \in \mathbb{Z}^n$ is the basis element with 1 on its ith entry and zeros elsewhere, and it correpsonds to the weight $T \to S^1$ that projects $t \mapsto t_i$. The fundamental weyl chamber we choose is $\overline{K} = \{(\mu_1, \ldots, \mu_n) \in \mathfrak{t} : \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n\}$, so the positive roots are $e_i - e_j$ for $i < j$. The dominant weights are therefore $\mu \in \mathbb{Z}^n$ with $\mu_1 \geq \cdots \geq \mu_n$.

Hence by Theorem 2.1.1, given a dominant weight λ, the unique irreducible representation of $\mathrm{U}(n)$ with highest weight λ, denoted as $\pi_\lambda^{\mathrm{U}(n)}$, has character

$$\chi_\pi(t) = \frac{\sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} t^{\sigma(\lambda+\rho)}}{\sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} t^{\sigma\rho}}, \tag{2.1}$$

where $\rho = \frac{1}{2} \sum_{a \in \Phi_+} a = (\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{-(n-1)}{2})$.

2.3 Statement of $\mathrm{U}(n) \downarrow \mathrm{U}(n-1)$ Branching Rule

Suppose we are given an irreducible representation $\pi_\lambda^{\mathrm{U}(n)}$ where λ is a dominant weight. We realize $\mathrm{U}(n-1)$ as a subgroup of $\mathrm{U}(n)$ by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

which induces an inclusion of the maximal tori $T' \hookrightarrow T$. This gives a surjection $\mathrm{X}(T) \cong \mathbb{Z}^n \to \mathrm{X}(T') \cong \mathbb{Z}^{n-1}$ by projecting the first $n-1$ entries. With such embedding, we can restrict $\pi_\lambda^{\mathrm{U}(n)}$ to $\pi_\lambda^{\mathrm{U}(n)}|_{\mathrm{U}(n-1)}$. Then as a representation of $\mathrm{U}(n-1)$, it decomposes into the direct sum of irreducible representations of the subgroup $\mathrm{U}(n-1)$, each indexed by its highest weight:

$$\pi_\lambda^{\mathrm{U}(n)}\Big|_{\mathrm{U}(n-1)} = \bigoplus_\mu [\pi_\lambda^{\mathrm{U}(n)} : \pi_\mu^{\mathrm{U}(n-1)}] \pi_\mu^{\mathrm{U}(n-1)}.$$

The sum is over dominant weights of $\mathrm{U}(n-1)$, and we use $[\pi_\lambda^{\mathrm{U}(n)} : \pi_\mu^{\mathrm{U}(n-1)}]$ to denote the multiplicity of $\pi_\mu^{\mathrm{U}(n-1)}$ in the decomposition. Our goal is to determine $[\pi_\lambda^{\mathrm{U}(n)} : \pi_\mu^{\mathrm{U}(n-1)}]$ for all pairs of $\lambda \in \mathbb{Z}^n$ and $\mu \in \mathbb{Z}^{n-1}$. It turns out that this multiplicity has a very clean description.

Theorem 2.3.1 ($\mathrm{U}(n) \downarrow \mathrm{U}(n-1)$ branching rule)**.** *Suppose λ is a dominant weight of $\mathrm{U}(n)$, and μ is a dominant weight of $\mathrm{U}(n-1)$. Then $[\pi_\lambda^{\mathrm{U}(n)} : \pi_\mu^{\mathrm{U}(n-1)}] = 1$ if λ and μ interleave. That is,*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n,$$

and 0 otherwise.

2.4 Schur Polynomials and Pieri's Formula

Before proceeding further, we first modify the character formula (2.1) for $\pi_\lambda^{\mathrm{U}(n)}$ so that both the numerator and the denominator can be expressed as elements in the ring $\mathbb{Z}[t_1, \ldots, t_n]$. To do this, we simply multiply both the numerator and the denominator by

$$t^{(\frac{n-1}{2})^n} = t_1^{\frac{n-1}{2}} t_2^{\frac{n-1}{2}} \cdots t_n^{\frac{n-1}{2}}.$$

Since the action of $\sigma \in S_n$ does nothing to such half-weight, in the formula (2.1) we may replace ρ with $\rho' = (n-1, n-2, \ldots, 0)$. Then we can rewrite the character formula using determinants:

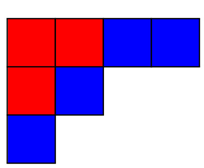
$$\chi_\pi(t) = \frac{\det(t_i^{\lambda_j+n-j})}{\det(t_i^{n-j})}.$$

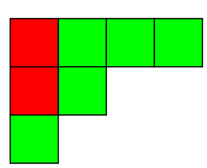
We recognize the denominator to be the determinant of a Vandermonde matrix, and we denote it as $\Delta_n(t) = \det(t_i^{n-j}) = \prod_{1\le i<j\le n}(t_i - t_j)$. Note that the numerator $\det(t_i^{\lambda_j+n-j})$ divides each of $t_i - t_j$ in $\mathbb{Z}[t_1, \ldots, t_n]$, so it divides $\Delta_n(t)$. Hence $\chi_\pi(t)$ is a symmetric homogeneous polynomial in $\mathbb{Z}[t_1, \ldots, t_n]$. It is called the *Schur polynomial* corresponding to λ, and we write it alternatively as $s_\lambda = s_\lambda(t_1, \ldots, t_n)$. When we want to make the number of indeterminants explicit, we write $s_\lambda^{(n)}$ instead of s_λ.

We first restrict the attention to the case when λ is a partition; i.e., it does not contain negative entries. Recall that a *partition* λ is a sequence $(\lambda_1, \ldots, \lambda_m)$ such that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m \ge 0$. We shall identify two partitions to be the same if they only differ by trailing zeros. We define the *length* of a partition λ, denoted $\ell(\lambda)$, to be $\ell(\lambda) = \sup\{n \in \mathbb{N} : \lambda_n > 0\}$, and $\ell(\lambda) = 0$ if $\lambda = 0$. We also denote $|\lambda| = \lambda_1 + \lambda_2 + \cdots$. Then a partition λ with $\ell(\lambda) \le n$ is a dominant weight for $\mathrm{U}(n)$. However, not all dominant weights are partitions since they may contain negative entries. We say λ is a partition of n, denoted $\lambda \vdash n$, if $|\lambda| = n$.

For partitions λ, μ, we write inclusion $\lambda \subset \mu$ if $\lambda_i \le \mu_i$ for all $i \in \mathbb{N}$. If $\lambda \subset \mu$, then we define μ/λ to be the nonnegative integer sequence $(\mu_1 - \lambda_1, \mu_2 - \lambda_2, \ldots)$. Naturally we shall define $|\mu/\lambda| = |\mu| - |\lambda|$. We may associate each partition λ with a Young diagram $\mathrm{YD}(\lambda)$. Then $\mathrm{YD}(\mu/\lambda)$ is defined to be the set theoretical difference $\mathrm{YD}(\mu) \setminus \mathrm{YD}(\lambda)$. We say μ/λ is a *horizontal strip* if $\mathrm{YD}(\mu/\lambda)$ has no two boxes in the same column.

For example, let $\mu = (4, 2, 1)$ and $\lambda = (2, 1)$. We lay $\mathrm{YD}(\lambda)$ on top of $\mathrm{YD}(\mu)$ so there top left corners coincide. We mark $\mathrm{YD}(\mu)$ in red. Then μ/λ is a horizontal strip, marked in blue. However, if $\lambda' = (1, 1)$, then μ/λ' is not a horitonal strip (marked in green):





Lemma 2.4.1. *If λ, μ are partitions of length $\leq n$, then $\lambda \subset \mu$ and μ/λ is a horizontal strip if and only if*

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \mu_n \geq \lambda_n.$$

Proof. Clearly $\mu_1 \geq \lambda_1 \geq \cdots$ is sufficient for μ/λ to be a horizontal strip. On the other hand, if μ/λ is a horizontal strip, then we need to require $\mu_i \geq \lambda_i$ for $\lambda \subset \mu$. If $\mu_i > \lambda_{i-1}$, then the ith and $(i-1)$th rows of $\mathrm{YD}(\mu/\lambda)$ will contain a box in the same column. So the condition is sufficient as well. □

Our main tool to prove the branching rule of $\mathrm{U}(n) \downarrow \mathrm{U}(n-1)$ is Pieri's formula. Let $h_k^{(n)}$ be the kth *complete symmetric polynomial* in n variables. That is,

$$h_k^{(n)}(t_1, \ldots, t_n) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} t_{i_1} \cdots t_{i_k}.$$

Observe that $h_k^{(n)}$ is precisely the character of $S^k\mathbb{C}^n$, the symmetric k-tensor of the standard representation of $\mathrm{U}(n)$. Pieri's formula gives the decomposition of the product of a Schur polynomial with a complete symmetric polynomial as a sum of Schur polynomials. Equivalently, in the language of representations, it gives the multiplicity of $\pi_\mu^{\mathrm{U}(n)}$ in $\pi_\lambda^{\mathrm{U}(n)} \otimes S^k\mathbb{C}^n$.

Theorem 2.4.2 (Pieri's formula)**.**

$$s_\lambda^{(n)} h_k^{(n)} = \sum_\mu s_\mu^{(n)},$$

where h_k is the kth complete symmetric polynomial in n variables, and the summation is over all partitions μ with $\ell(\mu) \leq n$ such that $\lambda \subset \mu$ and μ/λ is a horizontal strip with $|\mu/\lambda| = k$.

Proof. The proof presented here is based on the proof of Theorem B.7 of [BS17]. Define $M = \{\nu \in \mathbb{Z}^n : \sum_{i=1}^n \nu_i = k, \nu_i \geq 0, \forall i\}$. Then $h_k^{(n)} = \sum_{\nu \in M} t^\nu$. Let $M_1 = \{\nu \in M : \lambda + \nu \text{ is a partition and } (\lambda+\nu)/\lambda \text{ is a horizontal strip}\}$, and denote $M_2 = M \setminus M_1$. Write $s_\lambda^{(n)} h_k^{(n)}$ as

$$s_\lambda^{(n)} h_k^{(n)} = \Delta_n^{-1} \sum_{\nu \in M} t^\nu \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} t^{\sigma(\lambda+\rho)},$$

Since for any $\sigma \in S_n$, we have $\sum_{\nu \in M} t^\nu = \sum_{\nu \in M} t^{\sigma\nu}$, we have

$$s_\lambda^{(n)} h_k^{(n)} = \Delta_n^{-1} \sum_{\nu \in M} \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} t^{\sigma(\lambda+\nu+\rho)}.$$

Define $X(\nu) = \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} t^{\sigma(\lambda+\nu+\rho)}$, so that $s_\lambda^{(n)} h_k^{(n)} = \Delta_n^{-1} \sum_{\nu \in M} X(\nu)$. Our goal is to show $\sum_{\nu \in M} X(\nu) = \sum_{\nu \in M_1} X(\nu)$. To this end, we seek a involution $\nu \mapsto \nu'$ such that $X(\nu) =$

$-X(\nu')$ for all $\nu \in M_2$, so that $\sum_{\nu \in M_2} X(\nu) = 0$, since if $\nu \neq \nu'$, they would cancel each other; otherwise $X(\nu) = 0$.

Observe that $\nu \in M_2$ if and only if $(\lambda + \nu)/\lambda$ not a horizontal strip (including the cases where $\lambda + \nu$ is not a partition), and if and only if there exists some i such that $\lambda_{i+1} + \nu_{i+1} > \lambda_i$. For a given ν, we let i be the largest of such integer. Then we define ν' by letting $\nu'_j = \nu_j$ for $j \neq i$ and $j \neq i+1$, and

$$\nu'_i = \lambda_{i+1} + \nu_{i+1} - \lambda_i - 1$$
$$\nu'_{i+1} = \lambda_i + \nu_i - \lambda_{i+1} + 1.$$

Since $\lambda_{i+1} + \nu_{i+1} > \lambda_i$ and $\lambda_i \geq \lambda_{i+1}$, we have $\nu'_i \geq 0$ and $\nu'_{i+1} \geq 0$. Adding two equations we see that $\nu'_i + \nu'_{i+1} = \nu_i + \nu_{i+1}$, so $\sum_i \nu'_i = k$. Hence $\nu' \in M$. Note the second equation implies $\lambda_{i+1} + \nu'_{i+1} > \lambda_i$, so $\nu' \in M_2$. Finally for $j > i$, we have $\lambda_{j+1} + \nu'_{j+1} = \lambda_{j+1} + \nu_{j+1} \leq \lambda_j$, so i is also the largest integer for which $\lambda_{i+1} + \nu'_{i+1} > \lambda_i$. It follows that $\nu \mapsto \nu'$ is an involution.

Finally we notice that $s_i(\lambda + \nu + \rho) = \lambda + \nu' + \rho$, where $s_i \in S_n$ swaps ith and $(i+1)$th entries. By making a change of variable $\sigma \mapsto \sigma s_i$, we conclude that $X(\nu) = -X(\nu')$ for all $\nu \in M_2$. □

2.5 Proof of U(n) ↓ U($n-1$) Branching Rule

We first prove a special case of Theorem 2.3.1 when the highest weights of the irreducible representations we are considering are partitions.

Proposition 2.5.1. *Suppose λ, μ are partitions with $\ell(\lambda) \leq n$ and $\ell(\mu) \leq n-1$. Then $[\pi_\lambda^{\mathrm{U}(n)} : \pi_\mu^{\mathrm{U}(n-1)}] = 1$ if λ and μ interleave. That is,*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n,$$

and 0 otherwise.

Proof. The character of $\pi_\lambda^{\mathrm{U}(n)}|_{\mathrm{U}(n-1)}$ is just

$$\chi_{\pi_\lambda^{\mathrm{U}(n)}}(t_1, \ldots, t_{n-1}, 1) = s_\lambda(t_1, \ldots, t_{n-1}, 1)$$

because of the way we embed $\mathrm{U}(n-1) \hookrightarrow \mathrm{U}(n)$. The strategy is then to decompose $s_\lambda(t_1, \ldots, t_{n-1}, 1)$ into sum of the form $\sum s_\mu(t_1, \ldots, t_{n-1})$, which will then tell us the multiplicities of each s_μ with $\ell(\mu) \leq n-1$.

Observe that

$$s_\lambda(t_1, \ldots, t_{n-1}, 1) = \frac{\det \begin{bmatrix} t_1^{\lambda_1+n-1} & t_2^{\lambda_1+n-1} & \cdots & 1 \\ t_1^{\lambda_2+n-2} & t_2^{\lambda_2+n-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{\lambda_n} & t_2^{\lambda_n} & \cdots & 1 \end{bmatrix}}{\Delta_n(t_1, \ldots, t_{n-1}, 1)}. \tag{2.2}$$

We write the denominator as

$$\begin{aligned}\Delta_n(t_1,\dots,t_{n-1},1) &= \prod_{1\le j<k\le n-1}(t_j-t_k)\prod_{1\le j\le n-1}(t_j-1)\\ &= \Delta_{n-1}(t_1,\dots,t_{n-1})\cdot(-1)^{n-1}\prod_{1\le j\le n-1}(1-t_j).\end{aligned}$$

Expanding the determinant in (2.2) along the rightmost column, then dividing out Δ_{n-1}, we have

$$\begin{aligned}&s_\lambda(t_1,\dots,t_{n-1},1)\frac{\Delta_n(t_1,\dots,t_{n-1},1)}{\Delta_{n-1}}\\ &= (-1)^{n-1}\left(\prod_{1\le j\le n-1}(1-t_j)\right)s_\lambda(t_1,\dots,t_{n-1},1)\\ &= s^{(n-1)}_{(\lambda_1+1,\dots,\lambda_{n-1}+1)} - s^{(n-1)}_{(\lambda_1+1,\dots,\lambda_{n-2}+1,\lambda_n)} + s^{(n-1)}_{(\lambda_1+1,\dots,\lambda_{n-3}+1,\lambda_{n-1},\lambda_n)} - \cdots\\ &= \sum_{i=1}^{n}(-1)^{n-i}s^{(n-1)}_{\lambda^{i*}},\end{aligned}$$

where we denote

$$\lambda^{i*} = (\lambda_1+1,\dots,\lambda_{i-1}+1,\widehat{\lambda_i},\lambda_{i+1},\dots,\lambda_n),$$

and we use hat to mean omission of the term. On the other hand, as formal series, we have

$$\prod_{1\le j\le n-1}(1-t_j)^{-1} = \prod_{1\le j\le n-1}\sum_{i=0}^{\infty}t_j^i = \sum_{k=0}^{\infty}h_k(t_1,\dots,t_{n-1}).$$

Hence

$$\begin{aligned}s_\lambda(t_1,\dots,t_{n-1},1) &= (-1)^{n-1}\sum_{k=0}^{\infty}h_k^{(n-1)}\cdot\sum_{i=1}^{n}(-1)^{n-i}s^{(n-1)}_{\lambda^{i*}}\\ &= \sum_{i=1}^{n}(-1)^{i+1}\sum_{k=0}^{\infty}h_k^{(n-1)}s^{(n-1)}_{\lambda^{i*}}.\end{aligned}$$

To simplify notation, we use $\Xi(\lambda)$ to denote the set of partitions μ with $\ell(\mu)\le n-1$, $\mu\supset\lambda$ and that μ/λ is a horizontal strip. By Pieri's formula,

$$\sum_{k=0}^{\infty}h_k^{(n-1)}s^{(n-1)}_{\lambda^{i*}} = \sum_{k=0}^{\infty}\sum_{\substack{\mu\in\Xi(\lambda^{i*})\\ |\mu/\lambda^{i*}|=k}}s^{(n-1)}_\mu = \sum_{\mu\in\Xi(\lambda^{i*})}s^{(n-1)}_\mu.$$

Note that we get rid of the infinite sum over k by simply removing the constraint on $|\mu/\lambda^{i*}|$. Hence we can write $s_\lambda(t_1,\dots,t_{n-1},1)$ as an alternating sum:

$$s_\lambda(t_1,\dots,t_{n-1},1) = \sum_{i=1}^{n}(-1)^{i+1}\sum_{\mu\in\Xi(\lambda^{i*})}s^{(n-1)}_\mu.$$

We will show that with proper cancellation, the summation can be made to have the desired interleaving pattern. Let

$$q_k = \sum_{i=k}^{n} (-1)^{i+1} \sum_{\mu \in \Xi(\lambda^{i*})} s_\mu^{(n-1)} = q_{k+1} + (-1)^{k+1} \sum_{\mu \in \Xi(\lambda^{k*})} s_\mu^{(n-1)}$$

so that $q_1 = s_\lambda(t_1, \ldots, t_{n-1}, 1)$.

Lemma 2.5.2.

$$q_k = (-1)^{k+1} \sum_{\mu} s_\mu^{(n-1)},$$

where we sum over all μ such that $\mu_1 \geq \lambda_1 + 1 \geq \mu_2 \geq \lambda_2 + 1 \geq \cdots \geq \mu_{k-1} \geq \lambda_{k-1} + 1 \geq \lambda_k \geq \mu_k \geq \lambda_{k+1} \geq \cdots \geq \mu_{n-1} \geq \lambda_n$. If an index goes out of range, we simply omit it in the chain of inequality.

Proof. We shall prove this lemma by induction. For $k = n$ case, we have

$$q_n = \sum_{\Xi(\lambda^{n*})} s_\mu^{(n-1)},$$

where $\lambda^{n*} = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_{n-1} + 1)$, so this case follows directly from Lemma 2.4.1.

We proceed the inductive case backward. Suppose

$$q_{k+1} = (-1)^{k+2} \sum_{\mu'} s_{\mu'}^{(n-1)}$$

for μ' such that

$$\mu'_1 \geq \lambda_1 + 1 \geq \cdots \geq \mu'_{k-1} \geq \lambda_{k-1} + 1 \geq \mu'_k \geq \lambda_k + 1 \geq \lambda_{k+1} \geq \mu'_{k+1} \geq \cdots \geq \lambda_n. \qquad (2.3)$$

Then

$$q_k = (-1)^{k+1} \left(\sum_{\Xi(\lambda^{k*})} s_\mu^{(n-1)} - \sum_{\mu'} s_{\mu'}^{(n-1)} \right),$$

where the range of μ' is (2.3). Since $\lambda^{k*} = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_{k-1} + 1, \lambda_{k+1}, \ldots, \lambda_n)$, applying Lemma 2.4.1, we see the range of summation for μ is

$$\mu_1 \geq \lambda_1 + 1 \geq \cdots \geq \mu_{k-1} \geq \lambda_{k-1} + 1 \geq \mu_k \geq \lambda_{k+1} \geq \mu_{k+1} \geq \cdots \geq \mu_{n-1} \geq \lambda_n. \qquad (2.4)$$

Comparing the range (2.4) with (2.3), we see that they only differ on the range of μ'_k and μ_k, where

$$\lambda_{k-1} + 1 \geq \mu'_k \geq \lambda_k + 1$$
$$\lambda_{k-1} + 1 \geq \mu_k \geq \lambda_{k+1}.$$

Since $\lambda_k + 1 > \lambda_{k+1}$, after cancellation, we get $\lambda_k \geq \mu_k \geq \lambda_{k+1}$. This agrees with the range given in the lemma. □

Since $q_1 = s_\lambda(t_1, \ldots, t_{n-1}, 1)$, by the lemma, we have

$$s_\lambda(t_1, \ldots, t_{n-1}, 1) = \sum_\mu s_\mu^{(n-1)},$$

where $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$. This precisely means that λ and μ interleave, finishing the proof of Proposition 2.5.1. $\square$

It is now easy to obtain the branching rule of $\mathrm{U}(n) \downarrow \mathrm{U}(n-1)$ for an irreducible representation whose highest weight can be any dominant weight, not just a partition.

Proof of Theorem 2.3.1. Let $\det : \mathrm{U}(n) \to S^1$ be the determinant function. Then it is the character of an one-dimensional representation of $\mathrm{U}(n)$, denoted as $\pi_{\det}^{\mathrm{U}(n)}$. Note that $\chi_{\pi_{\det}^{\mathrm{U}(n)}}(t) = t^{1^n} = t_1 \cdots t_n$. We can reduce the problem to the case when λ is a partition, i.e., $\lambda_n \geq 0$, by looking at $\pi_\lambda^{\mathrm{U}(n)} \otimes (\pi_{\det}^{\mathrm{U}(n)})^{\otimes r}$, for some $r \in \mathbb{N}$. Let $\gamma = (r^n) = (r, r, \ldots, r)$, so the character of $(\pi_{\det}^{\mathrm{U}(n)})^{\otimes r}$ is $t \mapsto t^\gamma$. Then the character the tensor product $\pi_\lambda^{\mathrm{U}(n)} \otimes (\pi_{\det}^{\mathrm{U}(n)})^{\otimes r}$ is

$$\frac{\sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} t^{\sigma(\lambda+\gamma+\rho)}}{\Delta_n(t)}.$$

We choose r big enough so that $\lambda + \gamma$ is a partition. Then for $t = (t_1, \ldots, t_{n-1}, 1)$, we can apply Proposition 2.5.1 to get

$$\frac{\sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} t^{\sigma(\lambda+\gamma+\rho)}}{\Delta_n(t)} = \sum_\mu \frac{\sum_{\sigma \in S_{n-1}} (-1)^{\ell(w)} t^{\sigma(\mu+\rho')}}{\Delta_{n-1}(t)},$$

where μ interleaves with $\lambda + \gamma$. Since $\mathbb{Z}[t_1, t_2, \ldots, t_{n-1}]$ is a domain, we can divide both sides by t^γ. Let $\mu' = \mu - \gamma$. It follows that $\pi_{\mu'}^{\mathrm{U}(n-1)}$ appears in $\pi_\lambda^{\mathrm{U}(n)}$ with multiplicity 1 if and only if λ and μ' interleaves. $\square$

3 Branching Rule of Successive Orthogonal Groups

3.1 Maximal Tori and Root Systems of SO$(2n+1)$ and SO$(2n)$

Like with the U(n) case, we first give a description of the maximal torus and the root system of SO$(2n+1)$ and SO$(2n)$, following Section V.6 of [BtD85]. We will always assume $n \geq 2$.

For SO(2), we can identify SO$(2) \cong S^1$ by

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}.$$

Let $T = T(n) = (\mathrm{SO}(2))^n \subset \mathrm{SO}(2n) \subset \mathrm{SO}(2n+1)$, where we include $T(n)$ as n matrices in SO(2) along the diagonal of SO$(2n)$, and the inclusion SO$(2n) \hookrightarrow$ SO$(2n+1)$ is via

$$A \mapsto \begin{pmatrix} A & \\ & 1 \end{pmatrix}.$$

Then $T(n)$ is a maximal torus in both SO$(2n)$ and SO$(2n+1)$.

The Weyl group of SO$(2n+1)$ is the semidirect product $\Sigma(n) = \mathbb{Z}_2^n \rtimes S_n$ where we use $\mathbb{Z}_2$ to denote the cyclic group of order 2, and S_n acts on $\mathbb{Z}_2^n$ by permuting the entries. The action of $(\tau, \sigma) \in \Sigma(n)$ with $\tau \in \mathbb{Z}_2^n$ and $\sigma \in S_n$ on $\mu \in \mathfrak{t} \cong \mathbb{R}^n$ is

$$(\tau, \sigma).(\mu_1, \dots, \mu_n) = ((-1)^{\tau_1}\mu_{\sigma(1)}, \dots, (-1)^{\tau_n}\mu_{\sigma(n)}).$$

Thinking of elements in $\Sigma(n)$ as product of elementary reflections, it is evident that $\det(\tau, \sigma) = (-1)^{\ell(\tau,\sigma)} = (-1)^{\sum \tau_i}(-1)^{\ell(\sigma)}$. Define a homomorphism $\delta : \Sigma(n) \to \mathbb{Z}_2$ by $\delta(\tau, \sigma) = \sum_{i=1}^n \tau_i \bmod 2$. Then it turns out the Weyl group of SO$(2n)$ is $\Sigma_0(n) = \ker \delta$. The action of $\Sigma_0(n)$ on $\mathfrak{t}$ inherits from that of $\Sigma(n)$.

The root systems for SO$(2n+1)$, with a fixed choice of the fundamental Weyl chamber, is:

- Fundamental Weyl chamber: $\overline{K} = \{(\mu_1, \dots, \mu_n) \in \mathfrak{t} : \mu_1 \geq \cdots \geq \mu_n \geq 0\}$.
- Positive roots: $e_i \pm e_j$, $i < j$, and e_k, $1 \leq i, j, k \leq n$.
- Dominant weights: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ with $\lambda \in \mathbb{Z}^n$.
- Half sum of positive roots: $(n - \frac{1}{2}, n - \frac{3}{2}, \dots, \frac{1}{2})$.

The root system for SO$(2n)$ is given by:

- Fundamental Weyl chamber: $\overline{K} = \{(\mu_1, \ldots, \mu_n) \in \mathfrak{t} : \mu_1 \geq \cdots \geq |\mu_n|\}$.
- Positive roots: $e_i \pm e_j$, $1 \leq i < j \leq n$.
- Dominant weights: $\lambda_1 \geq \lambda_2 \geq \cdots \geq |\lambda_n|$ with $\lambda \in \mathbb{Z}^n$.
- Half sum of positive roots: $(n-1, n-2, \ldots, 0)$.

By Theorem 2.1.1, knowing the information about root systems allows us to compute the character of any irreducible representation indexed by its highest weight.

3.2 Branching Rule for SO$(2n+1) \downarrow$ SO$(2n)$

We begin with deriving the branching rule for SO$(2n+1) \downarrow$ SO$(2n)$, which is relatively more straightforward, as the maximal tori of both SO$(2n)$ and SO$(2n+1)$ have the same rank n.

Theorem 3.2.1 (SO$(2n+1) \downarrow$ SO$(2n)$ branching rule)**.** *Let λ be a dominant weight of* SO$(2n+1)$ *and μ a dominant weight of* SO$(2n)$*. Then* $[\pi_\lambda^{\mathrm{SO}(2n+1)} : \pi_\mu^{\mathrm{SO}(2n)}] = 1$ *if*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_n \geq |\mu_n|,$$

and 0 *otherwise.*

Proof. By the Weyl character formula (Theorem 2.1.1),

$$\chi_{\pi_\lambda^{\mathrm{SO}(2n+1)}} = \frac{\sum_{w \in \Sigma(n)} (-1)^{\ell(w)} t^{w(\lambda+\rho)}}{\Delta_{\mathrm{SO}(2n+1)}},$$

where $\Delta_{\mathrm{SO}(2n+1)}$ is the Weyl denominator. Notice the positive roots of SO$(2n)$ is a subset of that of SO$(2n+1)$. Using the product formula for the Weyl denominator, we find

$$\Delta_{\mathrm{SO}(2n+1)} = \Delta_{\mathrm{SO}(2n)} \prod_{1 \leq i \leq n} (t_i^{1/2} - t_i^{-1/2}).$$

Hence

$$
\begin{aligned}
&\Delta_{\mathrm{SO}(2n)}\chi_{\pi_\lambda^{\mathrm{SO}(2n+1)}} \\
&= \sum_{w\in\Sigma(n)} (-1)^{\ell(w)} t^{w(\lambda+\rho)} \prod_{i=1}^{n} \frac{t_i^{1/2}}{t_i - 1} \\
&= \sum_{\sigma\in S_n} (-1)^{\ell(\sigma)} \sum_{\tau\in\mathbb{Z}_2^n} (-1)^{\ell(\tau)} t^{\tau\sigma(\lambda+\rho)} \prod_{i=1}^{n} \frac{t_i^{1/2}}{t_i - 1} \\
&= \sum_{\sigma\in S_n} (-1)^{\ell(\sigma)} \sum_{\tau\in\mathbb{Z}_2^{n-1}} (-1)^{\ell(\tau)} \prod_{j=1}^{n-1} t_j^{[\tau\sigma(\lambda+\rho)]_j} \prod_{i=1}^{n-1} \frac{t_i^{1/2}}{t_i - 1} \left\{ (t_n^{[\sigma(\lambda+\rho)]_n} - t_n^{-[\sigma(\lambda+\rho)]_n}) \frac{t_n^{1/2}}{t_n - 1} \right\}.
\end{aligned}
$$

Observe that the bracket is the geometric series

$$
\sum_{|k|\le[\sigma(\lambda+\rho')]_n} t_n^k,
$$

where $k \in \mathbb{Z}$ and the new ρ' is the half-sum of SO$(2n)$; i.e., $\rho' = \rho - (\frac{1}{2})^n \in \mathbb{Z}^n$. So by induction we get

$$
\begin{aligned}
\Delta_{\mathrm{SO}(2n)}\chi_{\pi_\lambda^{\mathrm{SO}(2n+1)}} &= \sum_{\sigma\in S_n} (-1)^{\ell(\sigma)} \prod_{i=1}^{n} \left(\sum_{|k|\le[\sigma(\lambda+\rho')]_i} t_i^k \right) \\
&= \sum_{\sigma\in S_n} (-1)^{\ell(\sigma)} \prod_{i=1}^{n} \left(\sum_{|k|\le\lambda_i+n-i} t_{\sigma(i)}^k \right),
\end{aligned}
$$

where we used $[\sigma(\lambda+\rho')]_i = (\lambda+\rho')_{\sigma^{-1}(i)}$ and a change of variable $i \mapsto \sigma(i)$. This can be rewritten as the determinant of the matrix whose ith row and jth column is $\sum_{|k|\le\lambda_i+n-i} t_j^k$. Since the determinant remains the same when subtracting one row from another row, by subtracting the $(i+1)$th row from the ith row of this matrix, we see that

$$
\begin{aligned}
\Delta_{\mathrm{SO}(2n)}\chi_{\pi_\lambda^{\mathrm{SO}(2n+1)}} &= \det \left(\sum_{\lambda_{i+1}+n-i\le|k|\le\lambda_i+n-i} t_j^k \right) \\
&= \sum_{\sigma\in S_n} (-1)^{\ell(\sigma)} \prod_{i=1}^{n} \left(\sum_{\lambda_{i+1}+n-i\le|k|\le\lambda_i+n-i} t_{\sigma(i)}^k \right),
\end{aligned}
$$

Note that we may assume $\lambda_{n+1} = 0$.

Now we fix $\sigma \in S_n$. Interchanging product and summation, we can combine the enumerated integers k's into a tuple $p \in \mathbb{Z}^n$:

$$\prod_{i=1}^{n} \left(\sum_{\lambda_{i+1}+n-i \le |k| \le \lambda_i+n-i} t_{\sigma(i)}^{k} \right) = \sum_{\lambda_{i+1}+n-i \le |p_i| \le \lambda_i+n-i} \left(\prod_{j=1}^{n} t_{\sigma(j)}^{p_j} \right)$$
$$= \sum_{\lambda_{i+1}+n-i \le |p_i| \le \lambda_i+n-i} t^{\sigma(p)}.$$

For a given p, define $\tau \in \mathbb{Z}_2^n \cap \Sigma_0(n)$ by $\tau_i = 0$ if $p_i \ge 0$ and $\tau_i = 1$ if $p_i < 0$ for $i \le n-1$. That is, $(-1)^{\tau_i}$ reflects the sign of p_i for $i \le n-1$. This uniquely determines τ_n for it to be in $\Sigma_0(n)$ by requiring $\sum \tau_i \bmod 2 = 0$. Let $s \in \mathbb{Z}^n$ be $s = \tau p$, where the action of τ sends $p_i \mapsto (-1)^{\tau_i} p_i$. Then $s_i \ge 0$ for $i \le n-1$. Since $\tau^2 = 1$, we have $p = \tau s$. Summing over τ and s instead of p, we obtain

$$\sum_{\lambda_{i+1}+n-i \le |p_i| \le \lambda_i+n-i} t^{\sigma(p)} = \sum_{\substack{\tau \in \mathbb{Z}_2^n \cap \Sigma_0(n) \\ \lambda_{i+1}+n-i \le s_i \le \lambda_i+n-i \\ |s_n| \le \lambda_n}} t^{\sigma\tau s}.$$

Let $\mu = s - \rho'$. Then

$$\sum_{\substack{\tau \in \mathbb{Z}_2^n \cap \Sigma_0(n) \\ \lambda_{i+1}+n-i \le s_i \le \lambda_i+n-i \\ |s_n| \le \lambda_n}} t^{\sigma\tau s} = \sum_{\tau \in \mathbb{Z}_2^n \cap \Sigma_0(n)} \sum_{\substack{\lambda_{i+1} \le \mu_i \le \lambda_i \\ |\mu_n| \le \lambda_n}} t^{\sigma\tau(\mu+\rho')}.$$

Putting everything together,

$$\Delta_{\mathrm{SO}(2n)} \chi_{\pi_\lambda^{\mathrm{SO}(2n+1)}} = \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \sum_{\tau \in \mathbb{Z}_2^n \cap \Sigma_0(n)} \sum_{\substack{\lambda_{i+1} \le \mu_i \le \lambda_i \\ |\mu_n| \le \lambda_n}} t^{\sigma\tau(\mu+\rho')}$$
$$= \sum_{w \in \Sigma_0(n)} (-1)^{\ell(w)} \sum_{\substack{\lambda_{i+1} \le \mu_i \le \lambda_i \\ |\mu_n| \le \lambda_n}} t^{w(\mu+\rho')}$$
$$= \sum_{\substack{\lambda_{i+1} \le \mu_i \le \lambda_i \\ |\mu_n| \le \lambda_n}} \Delta_{\mathrm{SO}(2n)} \chi_{\pi_\mu^{\mathrm{SO}(2n)}}.$$

Since $\mathbb{Z}[t_1, \ldots, t_n]$ is a domain, diving out $\Delta_{\mathrm{SO}(2n)}$, we have

$$\chi_{\pi_\lambda^{\mathrm{SO}(2n+1)}} = \sum_{\substack{\lambda_{i+1} \le \mu_i \le \lambda_i \\ |\mu_n| \le \lambda_n}} \chi_{\pi_\mu^{\mathrm{SO}(2n)}}.$$

This gives the branching rule $\mathrm{SO}(2n+1) \downarrow \mathrm{SO}(2n)$. $\square$

3.3 Branching Rule for SO$(2n) \downarrow$ SO$(2n-1)$

Like the ones we have seen so far, the branching rule for $\mathrm{SO}(2n) \downarrow \mathrm{SO}(2n-1)$ also has an interleaving pattern.

Theorem 3.3.1 (Branching rule for $\mathrm{SO}(2n) \downarrow \mathrm{SO}(2n-1)$)**.** *Let λ be a dominant weight of* $\mathrm{SO}(2n)$ *and μ a dominant weight of* $\mathrm{SO}(2n-1)$*. Then* $[\pi_\lambda^{\mathrm{SO}(2n)} : \pi_\mu^{\mathrm{SO}(2n-1)}] = 1$ *if*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq |\lambda_n|,$$

and 0 *otherwise.*

To obtain this branching rule, we will make use of Theorem 2.3.1, the unitary group branching rule. This will guide us when restricting the maximal torus to one with less rank, which is the main obstacle in this case. We apply the branching rule of $\mathrm{U}(n) \downarrow \mathrm{U}(n-1)$ to get the following lemma concerning determinants.

Lemma 3.3.2. *Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_n$. Then for indeterminants $t_1, \dots, t_{n-1}$ and $t_n = 1$, we have*

$$\frac{\det(t_j^{\alpha_i})}{\prod_{1 \leq i \leq n-1}(t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}})} = \sum_\beta \det(t_j^{\beta_i}),$$

where the summation is over all $\beta = (\beta_1, \dots, \beta_{n-1})$ such that $\beta_i \in \mathbb{Z} + \frac{1}{2}$ and $\alpha_{i+1} < \beta_i < \alpha_i$ for all i. The determinant on the left is of an $n \times n$ matrix, while the one on the right is of an $(n-1) \times (n-1)$ matrix.

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_i = \alpha_i - (n-i)$. Then $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, so λ is a dominant weight of $\mathrm{U}(n)$. Writing the branching rule $\mathrm{U}(n) \downarrow \mathrm{U}(n-1)$ in terms of the Weyl character formula, we have

$$\frac{\sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} t^{\sigma(\lambda+\rho)}}{\Delta_{\mathrm{U}(n)}(t_1, \dots, t_{n-1}, 1)} = \sum_\mu \frac{\sum_{\sigma \in S_{n-1}} (-1)^{\ell(\sigma)} t^{\sigma(\mu+\rho')}}{\Delta_{\mathrm{U}(n-1)}(t_1, \dots, t_{n-1})}, \tag{3.1}$$

where $\rho = (\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{-(n-1)}{2})$ and $\rho' = (\frac{n-2}{2}, \frac{n-4}{2}, \dots, \frac{-(n-2)}{2})$, and the summation is over all μ such that $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$. By multiplying both sides by $t^{\frac{n-1}{2}} = \prod_{1 \leq i \leq n-1} t_i^{\frac{n-1}{2}}$, we may assume $\rho = (n-1, n-2, \dots, 0)$ and $\rho' = (n - \frac{3}{2}, n - \frac{5}{2}, \dots, \frac{1}{2})$. Then $\alpha = \lambda + \rho$. Obeserve that the set of positive roots of $\mathrm{U}(n-1)$ is contained in that of $\mathrm{U}(n)$, so we have $\Delta_{\mathrm{U}(n)}(t_1, \dots, t_{n-1}, 1) = \Delta_{\mathrm{U}(n-1)}(t_1, \dots, t_{n-1}) \prod_{1 \leq i \leq n-1}(t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}})$. Multiplying both sides of (3.1) by $\Delta_{\mathrm{U}(n-1)}$ and writing in terms of determinants, we get

$$\frac{\det(t_j^{[\lambda+\rho]_i})}{\prod_{1 \leq i \leq n-1}(t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}})} = \sum_\mu \det(t_j^{[\mu+\rho']_i}).$$

Since $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ is equivalent to $\lambda_i + n - i > \mu_i + n - i - \frac{1}{2} > \lambda_{i+1} + n - i - 1$, which is the same as $[\lambda+\rho]_i > [\mu+\rho']_i > [\lambda+\rho]_{i+1}$, with a change of variable $\beta = \mu + \rho'$, we have

$$\frac{\det(t_j^{\alpha_i})}{\prod_{1\leq i\leq n-1}(t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}})} = \frac{\det(t_j^{[\lambda+\rho]_i})}{\prod_{1\leq i\leq n-1}(t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}})} = \sum_\beta \det(t_j^{\beta_i}),$$

where $\alpha_{i+1} < \beta_i < \alpha_i$ and $\beta_i \in \mathbb{Z} + \frac{1}{2}$ for all i. □

We are now ready to prove the branching rule of $\mathrm{SO}(2n) \downarrow \mathrm{SO}(2n-1)$.

Proof of Theorem 3.3.1. We embed $\mathrm{SO}(2n-1) \hookrightarrow \mathrm{SO}(2n)$ in the natural way such that the embedding of the corresponding maximal torus is $(t_1, t_2, \ldots, t_{n-1}) \mapsto (t_1, t_2, \ldots, t_{n-1}, 1)$. Comparing the sets of positive roots of $\mathrm{SO}(2n)$ and of $\mathrm{SO}(2n-1)$, by the Weyl denominator formula, we have

$$\Delta_{\mathrm{SO}(2n)}(t_1, \ldots, t_{n-1}, 1) = \prod_{i<j\leq n-1} (t_i^{\frac{1}{2}} t_j^{-\frac{1}{2}} - t_i^{-\frac{1}{2}} t_j^{\frac{1}{2}})(t_i^{\frac{1}{2}} t_j^{\frac{1}{2}} - t_i^{-\frac{1}{2}} t_j^{-\frac{1}{2}}) \prod_{1\leq i\leq n-1} (t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}})^2$$
$$\Delta_{\mathrm{SO}(2n-1)}(t_1, \ldots, t_{n-1}) = \prod_{i<j\leq n-1} (t_i^{\frac{1}{2}} t_j^{-\frac{1}{2}} - t_i^{-\frac{1}{2}} t_j^{\frac{1}{2}})(t_i^{\frac{1}{2}} t_j^{\frac{1}{2}} - t_i^{-\frac{1}{2}} t_j^{-\frac{1}{2}}) \prod_{1\leq i\leq n-1} (t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}}).$$

Hence

$$\Delta_{\mathrm{SO}(2n)}(t_1, \ldots, t_{n-1}, 1) = \prod_{1\leq i\leq n-1} (t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}}) \cdot \Delta_{\mathrm{SO}(2n-1)}(t_1, \ldots, t_{n-1}).$$

Let $A_\lambda^{\mathrm{SO}(2n)}$ be the numerator in $\chi_{\pi_\lambda^{\mathrm{SO}(2n)}}$ according to the Weyl character formula, and similarly let $A_\mu^{\mathrm{SO}(2n-1)}$ be the enumerator of $\chi_{\pi_\mu^{\mathrm{SO}(2n-1)}}$. Let $m(\mu) = [\pi_\lambda^{\mathrm{SO}(2n)} : \pi_\mu^{\mathrm{SO}(2n-1)}]$. Then $\pi_\lambda^{\mathrm{SO}(2n)}|_{\mathrm{SO}(2n-1)} = \bigoplus_\mu m(\mu)\pi_\mu^{\mathrm{SO}(2n-1)}$ translates to

$$\frac{A_\lambda^{\mathrm{SO}(2n)}(t_1, \ldots, t_{n-1}, 1)}{\Delta_{\mathrm{SO}(2n)}(t_1, \ldots, t_{n-1}, 1)} = \sum_\mu m(\mu) \frac{A_\mu^{\mathrm{SO}(2n-1)}(t_1, \ldots, t_{n-1})}{\Delta_{\mathrm{SO}(2n-1)}(t_1, \ldots, t_{n-1})}.$$

Multiply both sides by $\Delta_{\mathrm{SO}(2n-1)}$, we get

$$\frac{A_\lambda^{\mathrm{SO}(2n)}(t_1, \ldots, t_{n-1}, 1)}{\prod_{1\leq i\leq n-1}(t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}})} = \sum_\mu m(\mu) A_\mu^{\mathrm{SO}(2n-1)}(t_1, \ldots, t_{n-1}). \tag{3.2}$$

Denote $\rho = (n-1, n-2, \ldots, 0)$ and $\rho' = (n - \frac{3}{2}, n - \frac{5}{2}, \ldots, \frac{1}{2})$; i.e., half-sums of positive roots of $\mathrm{SO}(2n)$ and $\mathrm{SO}(2n-1)$. We continue to denote the Weyl group of $\mathrm{SO}(2n)$ as $\Sigma_0(n)$. Observe that

$$A_\lambda^{\mathrm{SO}(2n)}(t_1, \ldots, t_{n-1}, 1) = \sum_{w\in\Sigma_0(n)} (-1)^{\ell(w)} t^{w(\lambda+\rho)} = \sum_{\substack{\tau\in\mathbb{Z}_2^n \\ \ell(\tau)=0}} \sum_{\sigma\in S_n} (-1)^{\ell(\sigma)} t^{\sigma\tau(\lambda+\rho)}.$$

Then the left hand side of equation (3.2) becomes

$$\frac{A_\lambda^{\mathrm{SO}(2n)}(t_1,\dots,t_{n-1},1)}{\prod_{1\le i\le n-1}(t_i^{\frac{1}{2}}-t_i^{-\frac{1}{2}})} = \sum_{\substack{\tau\in\mathbb{Z}_2^n\\ \ell(\tau)=0}} \frac{\sum_{\tau\in S_n}(-1)^{\ell(\sigma)}t^{\sigma\tau(\lambda+\rho)}}{\prod_{1\le i\le n-1}(t_i^{\frac{1}{2}}-t_i^{-\frac{1}{2}})} = \sum_{\substack{\tau\in\mathbb{Z}_2^n\\ \ell(\tau)=0}} \frac{\det(t_j^{[\tau(\lambda+\rho)]_i})}{\prod_{1\le i\le n-1}(t_i^{\frac{1}{2}}-t_i^{-\frac{1}{2}})}.$$

Once fixed τ, we can compute $\frac{\det(t_j^{[\tau(\lambda+\rho)]_i})}{\prod_{1\le i\le n-1}(t_i^{\frac{1}{2}}-t_i^{-\frac{1}{2}})}$ via Lemma 3.3.2 by first sorting $\tau(\lambda+\rho)$ in decreasing order and then apply the lemma. Sorting can be achieved by properly swapping rows of the matrix $(t_j^{[\tau(\lambda+\rho)]_i})$, which may introduce a sign change. Notice that $\tau(\lambda+\rho)$ contains distinct entries, and $\lambda+\rho$ is decreasing. Let $\kappa(\tau)$ denote the number of ordered pairs in $\tau(\lambda+\rho)$, that is, the number of (i,j) pairs, $i<j$, such that $[\tau(\lambda+\rho)]_i < [\tau(\lambda+\rho)]_j$. Clearly κ depends only on τ, as $\lambda+\rho$ is decreasing and $[\lambda+\rho]_n$ has the least absolute value. Let $\alpha = \mathrm{sort}(\tau(\lambda+\rho))$ so that $\alpha_1 > \alpha_2 > \cdots > \alpha_n$. Then $\kappa(\tau)$ will have the same parity as the number of swaps needed to sort $\tau(\lambda+\rho)$ into α. By Lemma 3.3.2,

$$\sum_{\substack{\tau\in\mathbb{Z}_2^n\\ \ell(\tau)=0}} \frac{\det(t_j^{[\tau(\lambda+\rho)]_i})}{\prod_{1\le i\le n-1}(t_i^{\frac{1}{2}}-t_i^{-\frac{1}{2}})} = \sum_{\substack{\tau\in\mathbb{Z}_2^n\\ \ell(\tau)=0}} (-1)^{\kappa(\tau)} \sum_{\substack{\alpha=\mathrm{sort}(\tau(\lambda+\rho))\\ \alpha_{i+1}<\beta_i<\alpha_i\\ \beta_i\in\mathbb{Z}+\frac{1}{2}}} \det(t_j^{\beta_i}). \tag{3.3}$$

Lemma 3.3.3. *In the right side of* (3.3)*, it is enough to sum over β such that for every $1\le i\le n-1$, there exists some j such that $|\lambda+\rho|_{i+1} < |\beta_j| < |\lambda+\rho|_i$.*

Proof. Denote $\theta = |\lambda+\rho|$ (note that the only term that could be negative is $(\lambda+\rho)_n = \lambda_n$). First fix some τ. Then fix some β in the summation that does not satisfy the condition in the lemma. That is, there exists a smallest integer k such that the interval (θ_{k+1},θ_k) does not contain any $|\beta_j|$. Then exact one of $[\tau(\lambda+\rho)]_k$, $[\tau(\lambda+\rho)]_{k+1}$ is negative, for otherwise there would be more than one β_j lying inside either (θ_{k+1},θ_k) or $(-\theta_k,-\theta_{k+1})$. Let $\tau'=\tau$ except that $\tau'_k = \tau_{k+1}$ and $\tau'_{k+1}=\tau_k$ (i.e. negating τ_k, τ_{k+1}). Then $\ell(\tau')=\ell(\tau)=0$, and the same β will appear in the summation of τ'. Moreover, $(-1)^{\kappa(\tau')} \ne (-1)^{\kappa(\tau)}$ since number of ordered pairs in $\tau'\theta$ will be offset by one compared to that in $\tau\theta$. Also it is clear that the map $\tau\mapsto\tau'$ is an involution (i.e. $\tau''=\tau$) since we are looking at the smallest k. Hence the term $\det(t_j^{\beta_i})$ that appears in the summation of τ will be cancelled by the same term appearing in the summation of τ'. □

Lemma 3.3.4. *For any β that satisfies the condition of Lemma 3.3.3, it appears in* (3.3) *with multiplicity* 1.

Proof. We will show that any such β appears in the summation for a unique $\tau\in\mathbb{Z}_2^n$ with $\ell(\tau)=0$. If $\beta_k>0$ for all k, then clearly $(-1)^{\tau_i}=1$ for $1\le i\le n-1$, and τ_n can be either

1 or -1. Let k be the smallest integer such that $\beta_k < 0$. Then for all $i < k$, we have $\beta_i > 0$. This implies that $(-1)^{\tau_i} = 1$ and $\tau_i = 0$, for otherwise there will be more than one β in some (θ_{i+1}, θ_i). Since $\beta_k < 0$, we must set $\tau_k = 1$ so that $(-1)^{\tau_k} = -1$. Next we find the smallest $k' > k$ such that $\beta_{k'} > 0$, and by the same reasoning we require $(-1)^{\tau_{k'}} = 1$ and $(-1)^{\tau_i} = -1$ for $k < i < k'$. If we keep doing this, we will see that for $i < n$, $(-1)^{\tau_i} = 1$ if $\beta_i > 0$ and $(-1)^{\tau_i} = -1$ if $\beta_i < 0$. The value of τ_n is then uniquely determined to make $\ell(\tau) = 0$. □

By the two lemmas, we can write the inner summation on right side of (3.3) by first enumerating the signs of β. Let $\overline{\tau}$ denote the first $n-1$ components of τ, which acts on β by negating corresponding components. Hence

$$\sum_{\substack{\tau \in \mathbb{Z}_2^n \\ \ell(\tau)=0}} \frac{\det(t_j^{[\tau(\lambda+\rho)_i]})}{\prod_{1\le i\le n-1}(t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}})} = \sum_{\substack{\tau \in \mathbb{Z}_2^n \\ \ell(\tau)=0}} (-1)^{\kappa(\tau)} \sum_{\substack{\beta_i \in \mathbb{Z}+\frac{1}{2} \\ |\lambda+\rho|_{i+1} < \beta_i < |\lambda+\rho|_i}} \det(t_j^{\mathrm{sort}(\overline{\tau}\beta)}).$$

Lemma 3.3.5. $\det(t_j^{[\mathrm{sort}(\overline{\tau}\beta)]_i}) = (-1)^{\ell(\overline{\tau})+\kappa(\tau)} \det(t_j^{[\overline{\tau}\beta]_i})$.

Proof. Since $\det(t_j^{[\mathrm{sort}(\overline{\tau}\beta)]_i}) = (-1)^{\kappa(\overline{\tau})} \det(t_j^{[\overline{\tau}\beta]_i})$, it is enough to show that

$$(-1)^{\kappa(\overline{\tau})} = (-1)^{\ell(\overline{\tau})+\kappa(\tau)}.$$

By definition, $\kappa(\tau)$ is the number of ordered pairs in $\tau\theta$, and $\kappa(\overline{\tau})$ is the number of ordered pairs in $\overline{\tau\theta}$, consisting of the first $n-1$ components of $\tau\theta$, for some θ sorted decreasingly as $\theta_1 > \theta_2 > \cdots > \theta_n$. Hence the difference $\kappa(\tau) - \kappa(\overline{\tau})$ equals the number of ordered pairs formed by $[\tau\theta]_i$ and $[\tau\theta]_n$ for all $i < n$, which is equal to the number of negations among $\tau_1, \ldots, \tau_{n-1}$, i.e., $\ell(\overline{\tau})$. Hence the equality follows. □

Therefore,

$$\sum_{\substack{\tau \in \mathbb{Z}_2^n \\ \ell(\tau)=0}} \frac{\det(t_j^{[\tau(\lambda+\rho)_i]})}{\prod_{1\le i\le n-1}(t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}})} = \sum_{\overline{\tau} \in \mathbb{Z}_2^{n-1}} (-1)^{\ell(\overline{\tau})} \sum_{\substack{\beta_i \in \mathbb{Z}+\frac{1}{2} \\ |\lambda+\rho|_{i+1} < \beta_i < |\lambda+\rho|_i}} \det(t_j^{[\overline{\tau}\beta]_i}).$$

Let $\mu = \beta - \rho'$, so $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_{n-1} \ge 0$. We get

$$\begin{aligned}
\sum_{\substack{\tau \in \mathbb{Z}_2^n \\ \ell(\tau)=0}} \frac{\det(t_j^{[\tau(\lambda+\rho)_i]})}{\prod_{1\le i\le n-1}(t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}})} &= \sum_{\overline{\tau} \in \mathbb{Z}_2^{n-1}} (-1)^{\ell(\overline{\tau})} \sum_{\substack{\mu_i \in \mathbb{Z} \\ |\lambda+\rho|_{i+1} < [\mu+\rho']_i < |\lambda+\rho|_i}} \det(t_j^{[\overline{\tau}(\mu+\rho')]_i}) \\
&= \sum_{\overline{\tau} \in \mathbb{Z}_2^{n-1}} (-1)^{\ell(\overline{\tau})} \sum_{\substack{\mu_i \in \mathbb{Z} \\ |\lambda|_{i+1} \le \mu_i \le |\lambda|_i}} \det(t_j^{[\overline{\tau}(\mu+\rho')]_i}) \\
&= \sum_{\substack{\mu_i \in \mathbb{Z} \\ |\lambda|_{i+1} \le \mu_i \le |\lambda|_i}} A_\mu^{\mathrm{SO}(2n-1)}(t_1, \ldots, t_{n-1}).
\end{aligned}$$

Comparing this with (3.2) yields the branching rule of $\mathrm{SO}(2n) \downarrow \mathrm{SO}(2n-1)$. □

4 Branching Rules via Duality

In this chapter, we will show an algebraic way of obtaining new branching rules from the old ones via duality. Unlike our previous proofs that involve careful combinatorial manipulations of the character formula, the new approach we will see is more systematic, in the sense that once the general theory is developed, many branching rules will follow.

We will shift our focus from compact Lie groups to linear algebraic groups that are *reductive*, the analogue of complete reducibility of compact Lie groups. The algebraic setup will be more suitable to present the duality theorems. A brief discussion of the link between the complete reducibility of certain linear algebraic groups and that of compact Lie groups is discussed at Section 4.3.

This exposition is based on Chapter 4 and 5 of [GW09], the appendix of [BS17], and the paper by [HTW05].

4.1 Algebraic Setup

We consider the following algebraic setup.

Definition 4.1.1. A subgroup G of $\mathrm{GL}(n,\mathbb{C})$ is a *linear algebraic group* if $G = \bigcap_{f\in A} f^{-1}(0)$ where A is a set of polynomial functions on $\mathrm{Mat}_n(\mathbb{C})$.

Definition 4.1.2. A $\mathbb{C}$-valued function g on $\mathrm{GL}(n,\mathbb{C})$ is *regular* if

$$g \in \mathbb{C}[x_{11}, \dots, x_{nn}, \det(x)^{-1}].$$

A $\mathbb{C}$-valued function on a linear algebraic group $G \subset \mathrm{GL}(n,\mathbb{C})$ is *regular* if it is the restriction of a regular function on $\mathrm{GL}(n,\mathbb{C})$. Let $\mathcal{O}[G]$ denote the set of regular functions on G, which forms a commutative algebra.

Definition 4.1.3. Let G, H be linear algebraic gorups. We say a map $\varphi : G \to H$ is a *regular map* if $\varphi^*(\mathcal{O}[H]) \subset \mathcal{O}[G]$, where $\varphi^*(f)(g) = f(\varphi(g))$ for all $f \in \mathcal{O}[H]$, $g \in G$.

Definition 4.1.4. Let G be a linear algebraic group. A *representation* of G is a pair (ρ, V) where V is a complex vector space and $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism.

Definition 4.1.5. We say (ρ, V) is a *regular representation* if $\dim V < \infty$ and after fixing a basis for V, each coordinate-wise ρ_{ij} is a regular function for all i, j.

We also need to look at infinite-dimensional representations, that are locally regular:

Definition 4.1.6. A *locally regular* representation of G is a representation (ρ, V) such that every finite-dimensional subspace of V is contained in a finite-dimensional G-invariant subspace where the restriction of ρ is regular.

4.2 Representations of Algebras

For our purpose, we look at a more general notion of representation at the level of associative algebras. We will only encounter algebras with identity, so we always assume this.

Definition 4.2.1. Let $\mathcal{A}$ be an associative algebra. A *representation*, or *module*, of $\mathcal{A}$ is a pair (ρ, V) where $\rho : \mathcal{A} \to \mathrm{End}(V)$ is an algebra homomorphism.

Most of the time we will simply write V instead of (ρ, V) as a representation of $\mathcal{A}$, where the action ρ will be understood.

Note there is a bijection between the representations (not just the regular ones) of any group G and the representations of the group algebra $\mathbb{C}[G]$. We will use representations of groups and representations of the corresponding group algebras interchangably.

Definition 4.2.2. An $\mathcal{A}$-module (possibly infinite-dimensional) V is *irreducible* if the only $\mathcal{A}$-invariant subspaces are $\{0\}$ and V.

Definition 4.2.3. Let (ρ, V) and (τ, W) be representations of an associative algebra $\mathcal{A}$. We use $\mathrm{Hom}_{\mathcal{A}}(V, W)$ denote the linear subspace of the $\mathrm{Hom}(V, W)$ that commute with the action of $\mathcal{A}$. Such map is called an $\mathcal{A}$-*module homomorphism*.

We say a vector space V has *countable dimension* if the cardinality of every linearly independent set of V is countable. Schur's lemma asserts the only $\mathcal{A}$-module homomorphisms between two irreducible (possibly infinite-dimensional but with countable dimension) representations are multiplication by scalars. The proof can be found at Lemma 4.1.4 of [GW09].

Lemma 4.2.4 (Schur's Lemma)**.** *Let (ρ, V) and (τ, W) be irreducible representations of an associative algebra $\mathcal{A}$. Suppose V and W have countable dimension over $\mathbb{C}$. Then*

$$\dim \mathrm{Hom}_{\mathcal{A}}(V, W) = \begin{cases} 1 & \text{if } (\rho, V) \cong (\tau, W) \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.2.5. A finite-dimensional $\mathcal{A}$-module V is *completely reducible* if for every $\mathcal{A}$-invariant subspace $W \subset V$, there exists a complementary $\mathcal{A}$-invariant subspace $U \subset V$ such that $V = W \oplus U$.

An equivalent definition for complete reducibility of a finite-dimensional $\mathcal{A}$-module V is if it has a decomposition $V = W_1 \oplus \cdots \oplus W_n$ with each W_i a finite-dimensional irreducible $\mathcal{A}$-module.

We also want to extend the notion of complete reducibility to infintie-dimensional representations.

Definition 4.2.6. An $\mathcal{A}$-module V is *locally completely reducible* if for every $v \in V$, the cyclic submodule $\mathcal{A}v$ is finite-dimensional and completely reducible.

We want to characterize the decomposition into irreducibles for a locally completely reducible representation. Let $\mathcal{A}$ be an associative algebra. Let $\widehat{\mathcal{A}}$ be the set of all equivalence classes of finite-dimensional irreducible $\mathcal{A}$-modules. Suppose V is an $\mathcal{A}$-module. For each $\lambda \in \widehat{\mathcal{A}}$, define the λ-*isotypic subspace* to be $V_{(\lambda)} = \sum_{U \subset V, [U]=\lambda} U$. Fix a module F^λ to be a representative for each $\lambda \in \widehat{\mathcal{A}}$. Then there are tautological maps

$$S_\lambda : \mathrm{Hom}_{\mathcal{A}}(F^\lambda, V) \otimes F^\lambda \to V, \quad S_\lambda(u \otimes w) = u(w).$$

We make $\mathrm{Hom}_{\mathcal{A}}(F^\lambda, V) \otimes F^\lambda$ into an $\mathcal{A}$-module by $x.(u \otimes w) = u \otimes (x.w)$.

Proposition 4.2.7 (Proposition 4.1.15 of [GW09])**.** *Let V be a locally completely reducible $\mathcal{A}$-module. Then S_λ is an $\mathcal{A}$-module isomorphism onto $V_{(\lambda)}$. Furthermore,*

$$V \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}} V_{(\lambda)} \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}} \mathrm{Hom}_{\mathcal{A}}(F^\lambda, V) \otimes F^\lambda.$$

4.3 Reductivity of Classical Groups

Now back to the algebraic group setting, we record without proof several fundational results regarding the representation theory of linear algebrac group.

Definition 4.3.1. A linear algebraic group G is *reductive* if every regular representation of G is completely reducible.

Let (σ, V) and (τ, W) be irreducible regular representations of a linear algebraic reductive groups H, K, respectively. Let $G = H \times K$. Then it turns out G is reductive as well, and $(\sigma \otimes \tau, V \otimes W)$ is an irreducible regular representation of G, where $\sigma \otimes \tau$ denote the representation where $(h, k) \in G$ acts by $\sigma(h) \otimes \tau(k)$. Moreover, all irreducible regular representations of G arise this way.

In this paper, the only linear algebraic groups we are concerned about are $\mathrm{GL}(n, \mathbb{C})$, the general linear group over $\mathbb{C}$, and the orthogonal group $\mathrm{O}(n, \mathbb{C})$, where $\mathrm{O}(n, \mathbb{C}) = \{g \in \mathrm{GL}(n, \mathbb{C}) : g^t g = I_n\}$. Here is the reason why both groups are reductive.

Let G be a linear algebraic group. It can then be viewed as a real Lie group. Let τ be a Lie group automorphism of G with $\tau^2 = \mathrm{id}$. If for any $f \in \mathcal{O}[G]$ we have $[g \mapsto \overline{f(\tau(g))}] \in \mathcal{O}[G]$, then we call τ a *complex conjugation* on G. Let $K = \{g \in G : \tau(g) = g\}$. Then K is a closed Lie group, and we call K a *real form* of G. Then it is a general fact that (Proposition 1.7.7 of [GW09])

Proposition 4.3.2. *Let K be a real form of a linear algebraic group G, and let $\mathfrak{k}$ be the Lie algebra of K inside $\mathfrak{g}$, the Lie algebra of G. Let (ρ, V) be a regular representation of G. Then a linear subspace $W \subset V$ is invariant under $\mathrm{d}\rho(\mathfrak{k})$ if and only if W is invariant under G^0, the connected component of the identity of G (as a real Lie group).*

For $G = \mathrm{GL}(n, \mathbb{C})$, let τ be the involution on G sending $g \mapsto (g^*)^{-1}$, where g^* denotes the conjugate transpose of g. Then τ fixes precisely $K = \mathrm{U}(n) \subset \mathrm{GL}(n, \mathbb{C})$. But we know any finite-dimensional continuous representation of $\mathrm{U}(n)$ is completely reducible, by constructing a $\mathrm{U}(n)$-invariant Hermitian form using integration over an invariant Haar measure on $\mathrm{U}(n)$. Since $\mathrm{U}(n)$ is connected, the induced representations on $\mathfrak{k}$ are also completely reducible. Since regular representations are continuous and $\mathrm{GL}(n, \mathbb{C})$ is connected, by Proposition 4.3.2, it follows that $\mathrm{GL}(n, \mathbb{C})$ is reductive. The proposition also implies that a regular representation of $\mathrm{GL}(n, \mathbb{C})$ is irreducible if and only if the restriction to $\mathrm{U}(n)$ as a continuous representation is irreducible. Through a process called *complexification*, any continous representation of $\mathrm{U}(n)$ can be extended to a complex analytic representation of $\mathrm{GL}(n, \mathbb{C})$ with its complex manifold structure. Moreover, the complex analytic representation that an irreducible continuous representation of $\mathrm{U}(n)$ extends to turns out to be automatically algebraic (i.e. a regular representation of $\mathrm{GL}(n, \mathbb{C})$), so it is irreducible. Complexification of a compact Lie group is discussed in Chapter 24 of [Bum04] and III.8 of [BtD85]. Hence the Weyl character formula (Theorem 2.1.1) gives a characterization of all irreducible regular representations of $\mathrm{GL}(n, \mathbb{C})$ by their highest weights (as continuous representations restricted to $\mathrm{U}(n)$).

To see that $\mathrm{O}(n, \mathbb{C})$ is reductive, our strategy is to show the index-2 subgroup $\mathrm{SO}(n, \mathbb{C}) = \{g \in \mathrm{O}(n, \mathbb{C}) : \det(g) = 1\}$ is reductive. Let τ be the involution on $\mathrm{SO}(n, \mathbb{C})$ sending $g \mapsto \overline{g}$, so τ is a complex conjugation. Then the fixed subgroup of τ is $\mathrm{SO}(n)$. In the same argument as in previous paragraph, by Proposition 4.3.2, $\mathrm{SO}(n, \mathbb{C})$ is reductive. Using an averaging argument as in proving the complete reducibility over a finite group, it can be shown that if a finite-index linear algebraic subgroup is reductive, then the original group is reductive (see Proposition 3.3.5 of [GW09]). Thus $\mathrm{O}(n, \mathbb{C})$ is reductive.

Example 4.3.3. To appreciate the reductivity of classical groups $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{O}(n, \mathbb{C})$, let's consider a group that is not reductive. Let $G \subset \mathrm{GL}_2(\mathbb{C})$ be the "$ax + b$" group; i.e.,

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times, b \in \mathbb{C} \right\}.$$

This is clearly a linear algebraic group. We let G act on $\mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ in the standard way by multiplication. Then $\mathbb{C}e_1$ is a G-invariant subspace. However, $\mathbb{C}e_1$ does not have a G-invariant complement, since

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} e_2 = e_1 + e_2 \notin \mathbb{C}e_2.$$

4.4 General Duality Theorem

Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a reductive linear algebraic group. Let $\widehat{G}$ denote the set of equivalence classes of the irreducible regular representations of G, and we fix (π^λ, F^λ) to be a representative for each $\lambda \in \widehat{G}$. Let (ρ, L) be a locally regular representation of G, and we assume that L has countable dimension. We hope to apply Proposition 4.2.7, and to do so we need to check two things, which we put as the next two propositions.

Proposition 4.4.1. *L is a locally completely reducible $\mathbb{C}[G]$-module.*

Proof. For any $v \in L$, since L is locally regular, we have $\mathbb{C}v \subset W$ for some regular G-invariant subspace $W \subset L$. Then $\mathbb{C}[G].v \subset \mathbb{C}[G].W \subset W$, so the cyclic module $\mathbb{C}[G].v$ is regular, and in particular finite-dimensional. Since G is reductive, it follows that $\mathbb{C}[G].v$ is completely reducible. □

Proposition 4.4.2. *Let $V \subset L$ be a finite-dimensional G-invariant subspace that is irreducible. Then $(\rho|_V, V)$ is a regular representation of G.*

Proof. Since L is locally regular, there exists $W \subset L$ with $V \subset W$ where $\rho|_W$ is regular. Since V is G-invariant, after a change of basis, we can view $\rho|_V$ as a submatrix of $\rho|_W$. Hence by our definition, $\rho|_V$ is a regular representation. □

Thus by Proposition 4.2.7, as G-modules we have

$$L \cong \bigoplus_{\lambda \in \mathrm{Spec}(\rho)} \mathrm{Hom}_G(F^\lambda, L) \otimes F^\lambda,$$

where $\mathrm{Spec}(\rho) \subset \widehat{G}$ denote the set of irreducible representation classes that appear in L, and $g \in G$ acts by $I \otimes \pi^\lambda(g)$.

Assume $\mathcal{R} \subset \mathrm{End}(L)$ is a subalgebra such that

(i) $\mathcal{R}$ acts irreducibly on L.

(ii) If $g \in G$ and $T \in \mathcal{R}$, then $\rho(g) T \rho(g)^{-1} \in \mathcal{R}$, so G acts on $\mathcal{R}$.

(iii) The action in (ii) is locally regular.

Let $\mathcal{R}^G = \{T \in \mathcal{R} : \rho(g)T = T\rho(g) \text{ for all } g \in G\}$. Since the action of $\mathcal{R}^G$ commutes with that of G, this makes L into an $(\mathcal{R}^G \otimes \mathbb{C}[G])$-module Let $E^\lambda = \mathrm{Hom}_G(F^\lambda, L)$ for $\lambda \in \mathrm{Spec}(\rho)$. We can define an action of $\mathcal{R}^G$ on the component E^λ by $T \cdot u = T \circ u$ for $T \in \mathcal{R}^G$, $u \in \mathrm{Hom}_G(F^\lambda, L)$. This makes sense because for $u \in E^\lambda$, $x \in L$, $g \in G$,

$$(Tu)(\pi^\lambda(g)x) = T(u(\pi^\lambda(g)x)) = T(\rho(g)u(x)) = \rho(g)(Tu(x)),$$

so that $Tu \in E^\lambda$. Note that this action is compatible with the action of $\mathcal{R}^G \subset \mathcal{R}$ on L, since for any $T \in \mathcal{R}^G$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_G(F^\lambda, L) \otimes F^\lambda & \xrightarrow{S_\lambda} & L \\ \downarrow{\scriptstyle T\otimes I} & & \downarrow{\scriptstyle T} \\ \mathrm{Hom}_G(F^\lambda, L) \otimes F^\lambda & \xrightarrow{S_\lambda} & L \end{array}$$

Hence as a module for $\mathcal{R}^G \otimes \mathbb{C}[G]$, we have the decomposition

$$L \cong \bigoplus_{\lambda \in \mathrm{Spec}(\rho)} E^\lambda \otimes F^\lambda$$

where $T \in \mathcal{R}^G$ acts by $T \otimes I$ on each summand.

Theorem 4.4.3 (Duality)**.** *With the above setup, as an $\mathcal{R}^G$ module, each E^λ above is irreducible for $\lambda \in \mathrm{Spec}(\rho)$. If $E^\lambda \cong E^\mu$ as $\mathcal{R}^G$ modules for $\lambda, \mu \in \mathrm{Spec}(\rho)$, then $\lambda = \mu$.*

To prove the duality theorem, we will assume the following version of Jacobson Density Theorem (Theorem 4.1.5 of [GW09]).

Theorem 4.4.4 (Jacobson Density Theorem)**.** *Let V be a countable-dimensional vector space of $\mathbb{C}$. Let $\mathcal{R}$ be a subalgebra of $\mathrm{End}(V)$ with identity that acts irreducibly on V. If $v_1, \ldots, v_n$ are linearly independent in V, then for any $w_1, \ldots, w_n \in V$, there exists $T \in \mathcal{R}$ with $Tv_i = w_i$ for $1 \le i \le n$.*

As a first step to prove the duality theorem, we prove the following lemma.

Lemma 4.4.5. *Let X be a finite-dimensional G-invariant subspace of L. Then $\mathrm{Hom}_G(X, L) = \mathcal{R}^G|_X$.*

Proof. Clearly $\mathcal{R}^G|_X \subset \mathrm{Hom}_G(X, L)$, so we only need to show the other direction. Let $T \in \mathrm{Hom}_G(X, L)$. Let $\{v_1, \ldots, v_n\}$ be a basis of X. Since $\mathcal{R}$ acts irreducibly on L (this is assumption (i) of $\mathcal{R}$), by Theorem 4.4.4, there exists $r \in \mathcal{R}$ such that $Tv_i = rv_i$ for all

$1 \le i \le n$. Hence $T = r|_X$. By hypothesis, the action of G on $\mathcal{R}$ by conjugation is locally regular (this is assumption (ii)(iii) of $\mathcal{R}$). By Proposition 4.4.1 with $L = \mathcal{R}$, and the by Proposition 4.2.7, there exists a G-intertwining projection $p : \mathcal{R} \to \mathcal{R}^G$. Since the restriction $\mathcal{R} \to \mathrm{Hom}(X, L)$ also commutes with G actions, as X is G-invariant, we have $T = p(r)|_X$. □

Proof of Theorem 4.4.3. First we show that $E^\lambda = \mathrm{Hom}_G(F^\lambda, L)$ is irreducible under the action of $\mathcal{R}^G$. Let $T, S \in \mathrm{Hom}_G(F^\lambda, L)$ be nonzero elements, and we look for $r \in \mathcal{R}^G$ such that $r \circ T = S$. Note that TF^λ and SF^λ are isomorphic irreducible representations of G with class λ. By Lemma 4.4.5, there exists $r \in \mathcal{R}^G$ such that $r|_{TF^\lambda}$ implements such isomorphism. By Schur's lemma (Lemma 4.2.4), there exists $c \in \mathbb{C}^\times$ such that $r|_{TF^\lambda} \circ T = cS$. Hence the action of $c^{-1}r$ sends T to S.

Now suppose $\lambda \neq \mu$ for $\lambda, \mu \in \mathrm{Spec}(\rho)$. Suppose $\varphi : \mathrm{Hom}_G(F^\lambda, L) \to \mathrm{Hom}_G(F^\mu, L)$ is an $\mathcal{R}^G$-module map, and we will show $\varphi = 0$. Let $T \in \mathrm{Hom}_G(F^\lambda, L)$, and set $S = \varphi(T)$. Let $U = TF^\lambda + SF^\mu$. Since $\lambda \neq \mu$, the sum is direct. Let $p : U \to SF^\mu$ be the projection relative to such direct sum decomposition. By Lemma 4.4.5, there exists $r \in \mathcal{R}^G$ such that $r|_U = p$. As a $F^\lambda \to SF^\mu$ map, $pT = 0$, so that $rT = 0$. Then $rS = r\varphi(T) = \varphi(rT) = 0$. But $rS = pS = S$, so $S = 0$.

□

4.5 Schur-Weyl Duality

Let's now look at an application of the duality theorem in a case where $\dim L < \infty$. Consider $G = \mathrm{GL}(n, \mathbb{C})$, and let $\mathbb{C}^n$ be the standard representation of G. Let S_k denote the kth symmetric group. We can extend the actions of G and S_k to $(\mathbb{C}^n)^{\otimes k}$ by letting $g \in G$ act diagonally:

$$g.(v_1 \otimes \cdots \otimes v_k) = gv_1 \otimes \cdots \otimes gv_k,$$

and by letting $\sigma \in S_k$ act by permuting the entries:

$$\sigma.(v_1 \otimes \cdots \otimes v_k) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

It is clear that the action of G commutes with that of S_k. It is not so obvious that the algebra generated by the action of S_k is the commuting ring of the action of G in $\mathrm{End}((\mathbb{C}^n)^{\otimes k})$.

Proposition 4.5.1. *Let $\mathcal{A} \subset \mathrm{End}((\mathbb{C}^n)^{\otimes k})$ denote the algebra generated by the action of S_k of permuting the entries. Then $\mathcal{A} = \mathrm{End}((\mathbb{C}^n)^{\otimes k})^G$.*

For a proof of the proposition, see Proposition A.8 of [BS17].

Applying Theorem 4.4.3 to the case when $\mathcal{R} = \mathrm{End}(L)$, we have a decomposition

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda \in \mathcal{S} \subset \widehat{G}} G_k^\lambda \otimes F_n^\lambda, \tag{4.1}$$

where F_n^λ denote an irreducible regular representation of $G = \mathrm{GL}(n, \mathbb{C})$ with highest weight λ (see Section 4.3 about using using highest weights to index irreducible representations of G), and G_k^λ denote a irreducible representation of S_k uniquely determined by λ. The set $\mathcal{S}$ of highest weights λ that appear in the above decomposition can be further determined:

Theorem 4.5.2 (Schur-Weyl duality)**.** *In (4.1), $\mathcal{S} = \mathrm{Par}(n, k)$, where $\mathrm{Par}(n, k)$ is the set of all partitions of k of length $\leq n$.*

For the proof of the theorem, see Theorem A.10 of [BS17].

4.6 Weyl Algebra Duality

Next we will apply the general duality theorem (Theorem 4.4.3) to the Weyl algebra acting on the space of polynomials, which we now define.

Definition 4.6.1. Let V be an n-dimensional vector space over $\mathbb{C}$. Fix a set of basis $\{e_1, \ldots, e_n\}$ for V, and let $\{x_1, \ldots, x_n\}$ be the corersponding dual basis for V^*. Let $\mathcal{P}(V) \cong S(V^*) \cong \mathbb{C}[x_1, \ldots, x_n]$ be the space of polynomials on V. We define the *Weyl algebra* of V to be the subspace $\mathbb{D}(V) \subset \mathrm{End}(\mathcal{P}(V))$ generated by the operators $D_i = \frac{\partial}{\partial x_i}$ and $M_i =$ multiplication by x_i, for $i = 1, \ldots, n$.

Proposition 4.6.2. *A $\mathbb{C}$-basis for $\mathbb{D}(V)$ is $\{M^\alpha D^\beta : \alpha, \beta \in \mathbb{N}^n\}$, where we use the multi-index notation $M^\alpha = M_1^{\alpha_1} \cdots M_n^{\alpha_n}$ and $D^\beta = D_1^{\beta_1} \cdots D_n^{\beta_n}$.*

Proof. Note that $[D_i, M_j] = \delta_{ij} I$ for all $i, j = 1, \ldots, n$. Hence $\mathbb{D}(V) = \mathrm{Span}_{\mathbb{C}}\{M^\alpha D^\beta : \alpha, \beta \in \mathbb{N}^n\}$. It remains to show linear independence. Suppose for $c_i \in \mathbb{C}^\times$, $1 \leq i \leq n$, we have $\sum_{i=1}^n c_i M^{\alpha^i} D^{\beta^i} = 0$ for $\alpha^i, \beta^i \in \mathbb{N}^n$. Suppose for $i \neq j$, either $\alpha^i \neq \alpha^j$ or $\beta^i \neq \beta^j$. We use $|\beta^i|$ to denote $\sum_{j=1}^n \beta_j^i$. Pick i such that $|\beta^i|$ is the smallest (if there are multiple ones, just pick any). Then $M^{\alpha^i} D^{\beta^i} x^{\beta^i} = k x^{\alpha^i}$ for some positive integer k. Now we look at the other contributions to the coefficient of x^{α^i}, which must sum up to 0. For all j, since $|\beta^j| \geq |\beta^i|$, if $\beta^j \neq \beta^i$, then $D^{\beta^j} x^{\beta^i} = 0$, so there is no contribution from the action of $c_j M^{\alpha^j} D^{\beta^j}$. If $\beta^j = \beta^i$, then we need $\alpha^j = \alpha^i$ to have $M^{\alpha^j} D^{\beta^j} x^{\beta^i} = k' x^{\alpha'}$ for some k'. Hence $i = j$, so we need $c_i = 0$, which is a contradiction.

□

Let $G \subset \mathrm{GL}(V)$ be an reductive linear algebraic group. Let G act on $\mathcal{P}(V)$ by a representation ρ where $(\rho(g)f)(x) = f(g^{-1}x)$. We want to let G act on $\mathbb{D}(V)$ by $g.T = \rho(g) T \rho(g)^{-1}$ for $T \in \mathbb{D}(V)$ and $g \in G$, so we need to check $g.T \in \mathbb{D}(V)$.

Lemma 4.6.3. *For $g \in G \subset \mathrm{GL}(V)$ with matrix (g_{ij}) relative to $\{e_1, \ldots, e_n\}$, we have*

$$\rho(g)D_j\rho(g^{-1}) = \sum_{i=1}^{n} g_{ij}D_i$$
$$\rho(g)M_i\rho(g^{-1}) = \sum_{j=1}^{n} (g^{-1})_{ij}M_j.$$

Proof. Let $f(x) \in \mathcal{P}(V)$. Then by using chain rule,

$$\left(\rho(g)\frac{\partial}{\partial x_j}\rho(g^{-1})f\right)(x) = \rho(g)\left(\frac{\partial}{\partial x_j}f(gx)\right) = \rho(g)\left(\sum_i \frac{\partial f}{\partial x_i}(gx)\frac{\partial (gx)_i}{\partial x_j}\right)$$
$$= \rho(g)\left(\sum_i \frac{\partial f}{\partial x_i}(gx)g_{ij}\right) = \left(\sum_i g_{ij}\frac{\partial}{\partial x_i}\right)f,$$

which proves the first identity. Similarly,

$$\left(\rho(g)x_i\rho(g^{-1})f\right)(x) = \rho(g)\left(x_if(gx)\right) = (g^{-1}x)_if(x) = \sum_j (g^{-1})_{ij}x_jf(x).$$

□

Combining with the characterization of the basis for $\mathbb{D}(V)$, we see that the action of G on $\mathbb{D}(V)$ is well-defined. Let $\mathbb{D}_k(V) = \mathrm{Span}_{\mathbb{C}}\{M^\alpha D^\beta : |\alpha| + |\beta| \le k\}$, which is finite-dimensional. Then the previous lemma shows that $\mathbb{D}_k(V)$ is stable under the action of G. Since $g^{-1} = \det(g)^{-1}\mathrm{adj}(g)$, where $\mathrm{adj}(g)$ is the adjugate matrix whose entries are polynomials in matrix entires x_{ij}, it follows that the action of G on each $\mathbb{D}_k(V)$ is regular. Hence G acts locally regular on $\mathbb{D}(V)$.

Theorem 4.6.4 (Weyl algebra duality)**.** *Let $G \subset \mathrm{GL}(V)$ be a reductive linear algebraic group. Let G acts on $\mathcal{P}(V)$ by $\rho(g)f(x) = f(g^{-1}x)$ for $f \in \mathcal{P}(V)$ and $g \in G$. Then*

$$\mathcal{P}(V) \cong \bigoplus_{\lambda \in \mathcal{S}} E^\lambda \otimes F^\lambda$$

as a $\mathbb{D}(V)^G \otimes \mathbb{C}[G]$ module, where $\mathcal{S} \subset \widehat{G}$, F^λ is an irreducible regular representation of G with class λ, and E^λ is an irreducible representation of $\mathbb{D}(V)^G$ determined uniquely by λ.

Proof. We will apply Theorem 4.4.3 with $\mathcal{R} = \mathbb{D}(V)$ and $L = \mathcal{P}(V)$. Then L has a countable basis B of products of coordinates. If we have any uncountable linearly independent set A of L, for each $b \in B$ it can be written as a linear combination of finitely many elements in A, so by keeping those elements in A that appear in this way we can reduce A to a countable set of linearly independent vectors, which is a contradiction. Hence L has countable dimension. We also note ρ is locally regular since $\mathcal{P}^k(V)$, the degree k homogeneous polynomials, is a regular G-module.

We have already shown that conditions (ii) and (iii) of the hypothesis of duality theorem have been met. It remains to prove (i), the irreducibility of $L = \mathcal{P}(V)$ as $\mathbb{D}(V)$ module. Let $f \in \mathcal{P}(V)$ that is not zero. Then there is some $\alpha \in \mathbb{N}^n$ such that $D^\alpha f \in \mathbb{C}^\times$. Hence for any $g \in \mathcal{P}(V)$, if we let M_g be the multiplication-by-g operator in $\mathbb{D}(V)$, then $g \in \mathbb{C}M_g D^\alpha f$. Hence $\mathbb{D}(V).f = \mathcal{P}(V)$, so $\mathbb{D}(V)$ acts irreducibly on $\mathcal{P}(V)$. □

The goal of the remaining section is to realize $\mathbb{D}(V)^G$ as a generating Lie subalgebra of it. We define an action of G on $\mathcal{P}(V \oplus V^*)$ in the natural way by

$$g.f(v, w^*) = f(g^{-1}v, w^* \circ g),$$

for $g \in G$, $v \in V$, $w^* \in V^*$. Liet $x_1, \dots, x_n$ be a basis of V, and let $\xi_1, \dots, \xi_n$ be the corresponding dual basis. Note that $g.x_i = \sum_{j=1}^n (g^{-1})_{ij} x_j$ and $g.\xi_j = \sum_{i=1}^n g_{ij}\xi_i$.

The *first fundamental theorems* (FFT) of invariant theory give a finite generating set (as algebras) for $\mathcal{P}(V \oplus V^*)^G$ when G is a classical group. We record the results for $G = \mathrm{GL}(n, \mathbb{C})$ and $G = \mathrm{O}(n, \mathbb{C})$, which will be sufficient for our need.

Theorem 4.6.5 (Polynomial FFT for $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{O}(n, \mathbb{C})$)**.** *Let $V = \mathbb{C}^n$. Then as associative algebras,*

- *$\mathcal{P}(V^k \oplus (V^*)^k)^{\mathrm{GL}(n,\mathbb{C})}$ is generated by $\sum_{a=1}^n x_{ai}\xi_{aj}$, for all $1 \le i, j \le k$, where ξ_{aj}'s are the coordinates on $(V^*)^k$ dual to the coordinates x_{ai}'s on V^k.*

- *$\mathcal{P}(V^k \oplus (V^*)^k)^{\mathrm{O}(n,\mathbb{C})}$ is generated by $\sum_{a=1}^n x_{ai}x_{aj}, \sum_{a=1}^n \xi_{a_i}\xi_{a_j}, \sum_{a=1}^n \xi_{a_i}x_{a_j}$ for all $1 \le i, j \le k$.*

Proof. See Theorem 5.2.1 and 5.2.2 of [GW09]. □

We wish to relate $\mathbb{D}(V)^G$ to $\mathcal{P}(V \oplus V^*)^G$, which we know the generating set of for $G = \mathrm{GL}(n, \mathbb{C})$ or $\mathrm{O}(n, \mathbb{C})$.

Define a map $\sigma : \mathbb{D}(V) \to \mathcal{P}(V \oplus V^*)$ as follows. Let $T = \sum_{|\alpha|+|\beta| \le k} c_{\alpha\beta} M^\alpha D^\beta \in \mathbb{D}_k(V)$ with $c_{\alpha\beta} \neq 0$ for some α, β such that $|\alpha| + |\beta| = k$. Then

$$\sigma(T) = \sum_{|\alpha|+|\beta|=k} c_{\alpha\beta} x^\alpha \xi^\beta \in \mathcal{P}^k(V \oplus V^*).$$

We call $\sigma(T)$ the *Weyl symbol* of T. Note that σ is not linear, but it is multiplicative, in the sense that $\sigma(ST) = \sigma(S)\sigma(T)$ for $S, T \in \mathbb{D}(V)$. Let $\sigma_k : \mathbb{D}_k(V) \to \mathcal{P}^k(V \oplus V^*)$ be the restriction $\sigma_k = \sigma|_{\mathbb{D}_k(V)}$. Then σ_k descends to $\overline{\sigma_k} : \mathbb{D}_k(V)/\mathbb{D}_{k-1}(V) \to \mathcal{P}^k(V \oplus V^*)$, which is then linear. Since $\mathbb{D}_k(V)$ is stable under conjugation by G, we can view $\mathbb{D}_k(V)/\mathbb{D}_{k-1}(V)$ as a G-module with the induced action. By comparing basis, we see such G-action agrees with the G-conjugation on $\mathbb{D}(V)$. Hence $\overline{\sigma_k}$ is an isomorphism of G-modules for each k.

The next theorem gives a sufficient condition for computing a generating set of the algebra $\mathbb{D}(V)^G$.

Theorem 4.6.6. *Let $\{\psi_1, \ldots, \psi_r\}$ be a set of polynomials that generates the algebra $\mathcal{P}(V \oplus V^*)^G$, and suppose for $T_1, \ldots, T_r \in \mathbb{D}(V)^G$ with $\sigma(T_j) = \psi_j$ for all j. Then $\{T_1, \ldots, T_r\}$ generates the algebra $\mathbb{D}(V)^G$.*

Proof. Let $\mathcal{J}$ be the subalgebra generated by $\{T_1, \ldots, T_r\}$. Since $\mathbb{C} \subset \mathcal{P}(V \oplus V^*)^G$, one of ψ_j must be an element of $\mathbb{C}^\times$, and so is T_j, so $\mathbb{C} = \mathbb{D}_0(V)^G \subset \mathcal{J}$. Next for $k \geq 1$, by induction suppose $\mathbb{D}_{k-1}(V)^G \subset \mathcal{J}$. Let $S \in \mathbb{D}_k(V)^G \setminus \mathbb{D}_{k-1}(V)$. Then $S + \mathbb{D}_{k-1}(V) \neq 0 \in \mathbb{D}_k(V)/\mathbb{D}_{k-1}(V)$. So $\overline{\sigma_k}(S + \mathbb{D}_{k-1}(V)) = \sum c_{j_1 \cdots j_k} \psi_1^{j_1} \cdots \psi_r^{j_r} \neq 0 \in \mathcal{P}^k(V \oplus V^*)^G$. Let $R = \sum c_{j_1 \cdots j_k} T_1^{j_1} \cdots T_r^{j_r} \in \mathbb{D}(V)$. Since $\sigma(R) = \overline{\sigma_k}(S + \mathbb{D}_{k-1}(V)) \in \mathcal{P}^k(V \oplus V^*)^G$, it follows that $R \in \mathbb{D}_k(V)$. Hence $\sigma(R) = \overline{\sigma_k}(R + \mathbb{D}_{k-1}(V)) = \overline{\sigma_k}(S + \mathbb{D}_{k-1}(V))$. Since $\overline{\sigma_k}$ is an isomorphism, we have $R - S \in \mathbb{D}_{k-1}(V)^G \subset \mathcal{J}$. Thus $S \in \mathcal{J}$, completing the induction. ☐

Corollary 4.6.7. *Under the hypothesis of the previous theorem, suppose additionally $\mathfrak{g}' = \mathrm{Span}_{\mathbb{C}}\{T_1, \ldots, T_r\}$ is a Lie subalgebra of $\mathbb{D}(V)^G$. Then in the decomposition of Theorem 4.6.4, E^λ is an irreducible $\mathfrak{g}'$-module where the action of $\mathfrak{g}'$ is just the restriction of that of $\mathbb{D}(V)^G$.*

Proof. Since $\mathfrak{g}'$ generates $\mathbb{D}(V)^G$ as algebra, we have $\mathbb{D}(V)^G \cong U(\mathfrak{g}')$, where $U(\mathfrak{g}')$ denote the universal enveloping algebra of $\mathfrak{g}'$. Then the claim follows since there is a correspondence between the representations of $\mathbb{D}(V)^G$ and representations of $\mathfrak{g}'$ by the universal property of $U(\mathfrak{g}')$. ☐

In the following sections, we consider applying Theorem 4.6.4 and Corollary 4.6.7 to particular choices of V and $G \subset \mathrm{GL}(V)$ to get concrete forms of decomposition of $\mathcal{P}(V) \cong \bigoplus E^\lambda \otimes F^\lambda$ where F^λ is an irreducible finite-dimensional G-module and E^λ is an irreducible (usually infinite-dimensional) $\mathfrak{g}'$-module.

4.7 GL($n,\mathbb{C}$)-GL($k,\mathbb{C}$) Duality

Let $G = \mathrm{GL}(n, \mathbb{C})$ and $V = \mathrm{Mat}_{n,k}(\mathbb{C})$, the space of $n \times k$ matrices over $\mathbb{C}$. We let G act on V by left-multiplication, which induces an action ρ on $\mathcal{P}(V)$ by $\rho(g)f(v) = f(g^{-1}v)$.

For $1 \leq i, j \leq k$, define $E_{ij} \in \mathbb{D}(V)$ by $E_{ij} = \sum_{1 \leq a \leq n} x_{ai} \frac{\partial}{\partial x_{aj}}$.

Proposition 4.7.1. *Let $\mathfrak{g}' = \mathrm{Span}_{\mathbb{C}}\{E_{ij} : 1 \leq i, j \leq k\}$. Then $\mathfrak{g}'$ is a Lie subalgebra of $\mathbb{D}(V)^G$ that is isomorphic to $\mathfrak{gl}(k, \mathbb{C})$. In addition it generates the algebra $\mathbb{D}(V)^G$.*

Proof. Using the Weyl sympbol map $\sigma : \mathbb{D}(V) \to \mathcal{P}(V \oplus V^*)$ defined in the previous section, we see that $\sigma(E_{ij}) = \sum_{a=1}^n x_{ai}\xi_{a_j}$. By the FFT for $G = \mathrm{GL}(n,\mathbb{C})$ (Theorem 4.6.5), $\mathcal{P}(V \oplus V^*)^G$ is generated by $z_{ij} = \sum_{a=1}^n x_{ai}\xi_{aj}$ for all $1 \leq i,j \leq k$. Hence by Theorem 4.6.6, $\mathrm{Span}_{\mathbb{C}}\{E_{ij}\} \subset \mathbb{D}(V)^G$ generates $\mathbb{D}(V)^G$ as algebra.

To show $\mathfrak{g}' \cong \mathfrak{gl}(k,\mathbb{C})$, consider another $G' = \mathrm{GL}(k,\mathbb{C})$, and let G' acts on V by $g'.v = vg'^{-1}$. This induces an action ρ' on $\mathcal{P}(V)$ by $\rho'(g')f(v) = f(vg')$. The differential of ρ' (as a smooth representation) at the identity e gives an representation $\mathrm{L}\rho' : \mathfrak{gl}(k,\mathbb{C}) \to \mathrm{End}(\mathcal{P}(V))$ of the Lie algebra $\mathfrak{gl}(k,\mathbb{C})$. We will compute the effect of $\mathrm{L}\rho'$ on the standard basis $\{e_{ij}\}_{1\leq i,j\leq k}$ of $\mathfrak{gl}(k,\mathbb{C})$. For $X \in \mathfrak{gl}(n,\mathbb{C})$, its action on $f \in \mathcal{P}(V)$ is given by

$$\begin{aligned}(X.f)(v) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(f(ve^{tX})) = \sum_{\substack{1\leq a\leq n\\1\leq b\leq k}} \frac{\partial f}{\partial x_{ab}}(v)\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(ve^{tX})_{ab}\\ &= \sum_{\substack{1\leq a\leq n\\1\leq b\leq k}} (vX)_{ab}\frac{\partial f}{\partial x_{ab}}(v)\\ &= \sum_{\substack{1\leq a\leq n\\1\leq b,c\leq k}} v_{ac}X_{cb}\frac{\partial f}{\partial x_{ab}}(v).\end{aligned}$$

Taking $X = e_{ij}$, we find $(e_{ij}.f)(v) = \sum_{1\leq a\leq n} v_{ai}\frac{\partial f}{\partial x_{aj}}(v)$. Hence $\mathrm{L}\rho'(e_{ij}) = \sum_{1\leq a\leq n} x_{ai}\frac{\partial}{\partial x_{aj}} = E_{ij} \in \mathbb{D}(V)$. Note that $\mathrm{L}\rho'$ may be regarded as a map of complex Lie algebra. Since $\{x_{ai}\frac{\partial}{\partial x_{aj}}\}_{1\leq a\leq n, 1\leq i,j\leq k}$ is linearly independent in $\mathbb{D}(V)$, $\mathrm{L}\rho'$ is injective, so we have $\mathfrak{g}' \cong \mathfrak{gl}(k,\mathbb{C})$. □

Theorem 4.7.2 (GL($n,\mathbb{C}$)-GL($k,\mathbb{C}$) duality)**.** *Let* $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$ *act on* $\mathcal{P}(\mathrm{Mat}_{n,k}(\mathbb{C}))$ *by* $\rho(g',g)f(v) = f(g^{-1}vg')$ *for* $g' \in \mathrm{GL}(k,\mathbb{C})$, $g \in \mathrm{GL}(n,\mathbb{C})$. *Then as a* $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$ *modules, we have the decomposition*

$$\mathcal{P}(\mathrm{Mat}_{n,k}(\mathbb{C})) \cong \bigoplus_{\lambda\in\mathcal{S}\subset\widehat{G}} E^\lambda \otimes F^\lambda,$$

where each $E^\lambda \otimes F^\lambda$ *is irreducible under the action of* $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$.

Proof. By Corollary 4.6.7, we have the decomposition

$$\mathcal{P}(\mathrm{Mat}_{n,k}(\mathbb{C})) \cong \bigoplus E^\lambda \otimes F^\lambda,$$

where E^λ is an irreducible representation of $\mathfrak{g}'$ and F^λ is an irreducible representation of $\mathrm{GL}(n,\mathbb{C})$. The irreducibility of E^λ as a $\mathfrak{g}'$ representation implies it is irreducible as a $G' = \mathrm{GL}(k,\mathbb{C})$ representation, since G' is connected. As in the proof of Proposition 4.7.1, the action of $g \in \mathrm{GL}(k,\mathbb{C})$ on $\mathcal{P}(\mathrm{Mat}_{n,k}(\mathbb{C}))$ is induced by right multiplying g^{-1}. Hence the above decomposition can also be regarded as an isomorphism of $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$ representations. □

The next theorem gives a description for the representations that occur in the decomposition of Theorem 4.7.2. We use F_n^λ to denote the irreducible representation of $\mathrm{GL}(n,\mathbb{C})$ with highest weight λ. For nonnegative weight λ, as in the previous chapters, we continue to use $\ell(\lambda)$ to denote the length of λ, i.e., the largest index j such that $\lambda_j > 0$. We can improve the duality in Theorem 4.7.2 by describing the exact irreducible representations appearing in the decomposition.

Theorem 4.7.3. *With notations as in Theorem 4.7.2,*

$$\mathcal{P}(\mathrm{Mat}_{n,k}(\mathbb{C})) \cong \bigoplus_\mu (F_k^\mu)^* \otimes F_n^\mu,$$

where the sum is over all nonnegative dominant weights μ with $\ell(\mu) \le \min\{k,n\}$.

Proof. See the proof of Theorem 5.6.7 of [GW09]. Roughly speaking, we seek polynomials in $\mathcal{P}(\mathrm{Mat}_{n,k}(\mathbb{C}))$ with weights $(-\mu,\mu)$ that are invariant under the action of a Borel subgroup $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$. Then these weights are precisely the highest weights. □

Corollary 4.7.4. *As a* $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$ *module,*

$$S(\mathbb{C}^k \otimes \mathbb{C}^n) \cong \bigoplus_\mu F_k^\mu \otimes F_n^\mu,$$

where the sum is over all nonnegative dominant weights μ with $\ell(\mu) \le \min\{k,n\}$. Here $S(\mathbb{C}^k \otimes \mathbb{C}^n)$ denote the symmetric algebra of $\mathbb{C}^k \otimes \mathbb{C}^n$ with action induced by the diagonal action of $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$ *on the tensor product $\mathbb{C}^k \otimes \mathbb{C}^n$.*

Proof. The symmetric algebra $S(\mathbb{C}^k \times \mathbb{C}^n)$ is related to $\mathcal{P}(\mathrm{Mat}_{n,k}(\mathbb{C}))$ in Theorem 4.7.2 as follows. Consider another representation σ of $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$ on $\mathcal{P}(\mathrm{Mat}_{k,n}(\mathbb{C}))$ by $\sigma(g',g)f(v) = f(g^t v g')$. This is the induced action of $v \mapsto (g^t)^{-1} v g'^{-1}$, under which $\mathrm{Mat}_{k,n}(\mathbb{C}) \cong (\mathbb{C}^k)^* \otimes (\mathbb{C}^n)^*$. Hence $\mathcal{P}(\mathrm{Mat}_{k,n}(\mathbb{C})) \cong S(\mathbb{C}^k \otimes \mathbb{C}^n)$. But σ differs from ρ only by precomposing $g \mapsto (g^t)^{-1}$, which amounts to passing to the dual representation from F_k^μ to $(F_k^\mu)^*$. □

4.8 O$(n,\mathbb{C})$-$\mathfrak{sp}(k,\mathbb{C})$ Duality

Let $G = \mathrm{O}(n,\mathbb{C}) = \{A \in \mathrm{GL}(n,\mathbb{C}) : A^t A = I\}$. That is, G is the group preserveing the symmetric bilinear form $(x,y) = \sum_{i=1}^n x_i y_i$. Let $V = \mathrm{Mat}_{n,k}(\mathbb{C})$ with G acting by left multiplication. For $1 \le i,j \le k$, define $\Delta_{ij}, M_{ij} \in \mathbb{D}(V)$ by

$$\Delta_{ij} = \sum_{p=1}^n \frac{\partial^2}{\partial x_{pi} \partial x_{pj}}, \quad M_{ij} = \text{multiplication by} \sum_{p=1}^n x_{pi} x_{pj}.$$

We continue to denote $E_{ij} = \sum_{1\le p\le n} x_{pi}\frac{\partial}{\partial x_{pj}} \in \mathbb{D}(V)$ for $1 \le i,j \le k$. We have seen that $E_{ij} \in \mathbb{D}(V)^{\mathrm{GL}(n,\mathbb{C})}$, so $E_{ij} \in \mathbb{D}(V)^G$.

Lemma 4.8.1. $\Delta_{ij}, M_{ij} \in \mathbb{D}(V)^G$.

Proof. First we compute the effect of $\rho(g)\frac{\partial}{\partial x_{pi}}\rho(g^{-1})$ on $\mathcal{P}(V)$ for $1 \le p \le n$, $1 \le i \le k$. Applying Lemma 4.6.3, we have

$$\rho(g)\frac{\partial}{\partial x_{pi}}\rho(g)^{-1} = \sum_{ab}(ge_{pi})_{ab}\frac{\partial}{\partial x_{ab}} = \sum_{a=1}^{n} g_{ap}\frac{\partial}{\partial x_{ai}}.$$

Hence

$$\begin{aligned}\rho(g)\Delta_{ij}\rho(g)^{-1} &= \sum_{p=1}^{n}\left(\rho(g)\frac{\partial}{\partial x_{pi}}\rho(g^{-1})\right)\left(\rho(g)\frac{\partial}{\partial x_{pj}}\rho(g)^{-1}\right)\\ &= \sum_{p=1}^{n}\left(\sum_a g_{ap}\frac{\partial}{\partial x_{ai}}\right)\left(\sum_b g_{bp}\frac{\partial}{\partial x_{bj}}\right)\\ &= \sum_{1\le a,b\le n}\left(\sum_{p=1}^{n} g_{ap}g_{bp}\right)\frac{\partial^2}{\partial x_{ai}x_{bj}}\\ &= \Delta_{ij}.\end{aligned}$$

Since M_{ij} is multiplication by (x_i, x_j) where we view x_i, x_j as column vectors with x_{ip}, x_{jp} on the ith component, and since the bilinear form is invariant under G, it follows that M_{ij} is invariant conjugation by $\rho(g)$. □

Proposition 4.8.2. *Let* $\mathfrak{g}' = \mathrm{Span}_{\mathbb{C}}\{E_{ij} + (n/2)\delta_{ij}, M_{ij}, \Delta_{ij} : 1 \le i,j \le k\}$. *Then* $\mathfrak{g}'$ *is a Lie subalgebra of* $\mathbb{D}(V)^G$ *that is isomorphic to* $\mathfrak{sp}(k,\mathbb{C})$. *In addition* $\mathfrak{g}'$ *generates the algebra* $\mathbb{D}(V)^G$.

Proof. We first need to check the given spanning set of $\mathfrak{g}'$ is closed under the Lie bracket. Clearly $[M_{ij}, M_{rs}] = 0$ and $[\Delta_{ij}, \Delta_{rs}] = 0$. It is then purely technical to verify the remaining pairs. We show for instance the computation of $[\Delta_{ij}, M_{rs}]$. For a second order differential operator $\frac{\partial^2}{\partial x\partial y}$, for polynomial φ and f, we have

$$\frac{\partial^2}{\partial x\partial y}(\varphi f) = \left(\frac{\partial^2}{\partial x\partial y}\varphi\right)f + \frac{\partial}{\partial x}\varphi\frac{\partial}{\partial y}f + \frac{\partial}{\partial y}\varphi\frac{\partial}{\partial x}f + \varphi\frac{\partial^2}{\partial x\partial y}f.$$

Hence

$$\left[\frac{\partial^2}{\partial x\partial y}, \varphi\right] = \frac{\partial^2}{\partial x\partial y}\varphi + \frac{\partial}{\partial x}\varphi\frac{\partial}{\partial y} + \frac{\partial}{\partial y}\varphi\frac{\partial}{\partial x}.$$

For $\varphi = M_{rs} = (x_r, x_s)$, we find

$$[\Delta_{ij}, M_{rs}] = \sum_{p=1}^{n} \left\{ \frac{\partial^2}{\partial x_{pi}\partial x_{pj}}(x_r, x_s) + \left(\frac{\partial}{\partial x_{pi}}(x_r, x_s)\right)\frac{\partial}{\partial x_{pj}} + \left(\frac{\partial}{\partial x_{pj}}(x_r, x_s)\right)\frac{\partial}{\partial x_{pi}} \right\}.$$

Then

$$\begin{aligned}
\frac{\partial^2}{\partial x_{pi}\partial x_{pj}}(x_r, x_s) &= \frac{\partial^2}{\partial x_{pi}\partial x_{pj}}(x_{pr}x_{ps}) = \delta_{is}\delta_{jr} + \delta_{ir}\delta_{js} \\
\frac{\partial}{\partial x_{pi}}(x_r, x_s)\frac{\partial}{\partial x_{pj}} &= \frac{\partial}{\partial x_{pi}}(x_{pr}x_{ps})\frac{\partial}{\partial x_{pj}} = \delta_{ir}x_{ps}\frac{\partial}{\partial x_{pj}} + \delta_{is}x_{pr}\frac{\partial}{\partial x_{pj}} \\
\frac{\partial}{\partial x_{pj}}(x_r, x_s)\frac{\partial}{\partial x_{pi}} &= \frac{\partial}{\partial x_{pj}}(x_{pr}x_{ps})\frac{\partial}{\partial x_{pi}} = \delta_{jr}x_{ps}\frac{\partial}{\partial x_{pi}} + \delta_{js}x_{pr}\frac{\partial}{\partial x_{pi}}.
\end{aligned}$$

It follows that

$$[\Delta_{ij}, M_{rs}] = \delta_{ir}\left(E_{sj} + \frac{n}{2}\delta_{js}\right) + \delta_{is}\left(E_{rj} + \frac{n}{2}\delta_{jr}\right) + \delta_{jr}\left(E_{si} + \frac{n}{2}\delta_{is}\right) + \delta_{js}\left(E_{ri} + \frac{n}{2}\delta_{ir}\right).$$

Now assume we have checked that $\mathfrak{g}'$ defined as above is a Lie subalgebra. Note that since $E_{ij} + (n/2)\delta_{ij}, M_{ij}, \Delta_{ij}$ have different degrees as differential operators, they are necessarily linearly independent. Recall that $\mathfrak{sp}(k,\mathbb{C}) \subset \mathfrak{gl}(2k,\mathbb{C})$ is the space of matrices X such that $X^tJ + JX = 0$, where $J = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix}$. Equivalently, $X \in \mathfrak{sp}(k,\mathbb{C})$ if and only if X takes the form

$$X = \begin{bmatrix} A & B \\ C & -A^t \end{bmatrix} \text{ with } A, B, C \in \mathrm{Mat}_k(\mathbb{C}), B = B^t, C = C^t.$$

Consider the linear map $\varphi : \mathfrak{g}' \to \mathfrak{sp}(k,\mathbb{C})$ defined by sending, for $1 \le i, j \le k$,

$$E_{ij} + (\delta/2)\delta_{ij} \mapsto e_{ij} - e_{k+j,k+i}, \quad M_{ij} \mapsto e_{i,j+k} + e_{j,i+k}, \quad \Delta_{ij} \mapsto -e_{i+k,j} - e_{j+k,i}.$$

Then φ is a linear isomorphism. Once we have computed all the commutation relations among the chosen basis of $\mathfrak{g}'$, we can verify that φ preserves the Lie bracket, so it is an isomorphism of Lie algebra.

To show that $\mathfrak{g}'$ generates the algebra $\mathbb{D}(V)^G$, we again resort to Theorem 4.6.6 and compute the Weyl symbols as

$$\sigma(E_{ij} + \frac{n}{2}\delta_{ij}) = \sum_{p=1}^{n} x_{pi}\xi_{pj}, \quad \sigma(M_{ij}) = \sum_{p=1}^{n} x_{pi}x_{pj}, \quad \sigma(\Delta_{ij}) = \sum_{p=1}^{n} \xi_{pi}\xi_{pj}.$$

By the FFT for $G = \mathrm{O}(n,\mathbb{C})$ (Theorem 4.6.5), $\mathcal{P}(V \oplus V^*)^G$ is generated by these polynomials, so $\mathfrak{g}'$ indeed generates $\mathbb{D}(V)^G$. □

Applying Corollary 4.6.7, we get

Theorem 4.8.3 ($\mathrm{O}(n,\mathbb{C})$-$\mathfrak{sp}(k,\mathbb{C})$ duality). *Let $V = \mathrm{Mat}_{n,k}(\mathbb{C})$. Let $\mathrm{O}(n,\mathbb{C})$ act on V by left multiplication, and let $\mathfrak{sp}(k,\mathbb{C}) \subset \mathbb{D}(V)^G$ act on $\mathcal{P}(V)$ by the action of the Weyl algebra. Then the action of $\mathrm{O}(n,\mathbb{C})$ commutes with that of $\mathfrak{sp}(k,\mathbb{C})$, and we have the decomposition*

$$\mathcal{P}(\mathrm{Mat}_{n,k}(\mathbb{C})) \cong \bigoplus_{\lambda \in \mathcal{S} \subset \widehat{G}} E^\lambda \otimes F^\lambda,$$

where each E^λ is an irreducible representation of $\mathfrak{sp}(k,\mathbb{C})$ and F^λ is an irreducible regular representation of $\mathrm{O}(n,\mathbb{C})$.

4.9 Seesaw Reciprocity

We consider a setup similar to the one in Section 4.4. Let G be a reductive group and let (ρ, L) be a locally regular representation of G. Let $\mathcal{R} \subset \mathrm{End}(L)$ be a subalgebra such that $\mathcal{R}$ acts irreducibly on L, and that G acts locally regularly on $\mathcal{R}$ by $g.T = \rho(g)T\rho(g)^{-1}$. Let $\mathcal{A} = \mathcal{R}^G$. Then by the general duality theorem (Theorem 4.4.3), we have

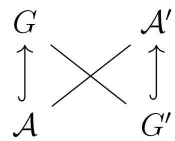
$$L \cong \bigoplus_{\lambda \in \mathrm{Spec}(\rho)} E^{\lambda^*} \otimes F^\lambda, \tag{4.2}$$

where E^{λ^*} is an irreducible representation of $\mathcal{A}$ uniquely determined by λ, and F^λ is an irreducible regular representation of G labeled by λ. Then $\lambda \Leftrightarrow \lambda^*$ gives a bijection between irreducible regular representations of G and irreducible representations of $\mathcal{A}$ appearing in the above decomposition. In this case, we say $(G, \mathcal{A})$ is a *dual reductive pair relative to* L. When $\mathcal{A}$ can be generated by the regular action of a linear algebraic group G' (resp. Lie algebra $\mathfrak{g}'$), we also call (G, G') (resp. $(G, \mathfrak{g}')$) a dual reductive pair.

Suppose $(G', \mathcal{A}')$ is another dual reductive pair acting on the same vector space L. Let (ρ', L) be the representation of G' on L. We assume the we have inclusions $G' \subset G$ and $\mathcal{A} \subset \mathcal{A}'$. Then the duality theorem gives

$$L \cong \bigoplus_{\mu \in \mathrm{Spec}(\rho')} E'^{\mu^*} \otimes F'^{\mu}, \tag{4.3}$$

where E'^{μ^*} is an irreducible representation of $\mathcal{A}'$ and F'^{μ} is an irreducible regular representation of G', and $\mu \Leftrightarrow \mu^*$ is a bijection between irreducibles. Assume further that the action of G restriction to G' on L is the same as the one from G', and also the action of $\mathcal{A}'$ is compatible with that of $\mathcal{A}$ for restriction. In this case, we call $(G, \mathcal{A})$ and $(G', \mathcal{A}')$ a *seesaw pair relative* *to* L, and we use denote the pair using the diagram



Theorem 4.9.1 (Seesaw Reciprocity)**.** *In the same notation, we have as vector spaces*

$$\mathrm{Hom}(E^{\lambda^*}, E'^{\mu^*}) \cong \mathrm{Hom}(F'^{\mu}, F^{\lambda}),$$

for $\lambda \in \mathrm{Spec}(\rho)$ *and* $\mu \in \mathrm{Spec}(\rho')$*. In terms of multiplicity formula, we can express this as*

$$[F^{\lambda} : F'^{\mu}] = [E'^{\mu^*} : E^{\lambda^*}].$$

Proof. By Proposition 4.2.7, as an $\mathcal{A}$-module, E'^{μ^*} in (4.3) decomposes as

$$E'^{\mu^*} \cong \bigoplus_{\sigma^* \in \widehat{\mathcal{A}}} \mathrm{Hom}_{\mathcal{A}}(E^{\sigma^*}, E'^{\mu^*}) \otimes E^{\sigma^*}.$$

Hence as $(\mathcal{A} \otimes \mathbb{C}[G'])$-modules,

$$L \cong \bigoplus_{\mu \in \mathrm{Spec}(\rho')} \bigoplus_{\sigma^* \in \widehat{\mathcal{A}}} \mathrm{Hom}_{\mathcal{A}}(E^{\sigma^*}, E'^{\mu^*}) \otimes E^{\sigma^*} \otimes F'^{\mu}. \tag{4.4}$$

Similarly, by Proposition 4.2.7, as a G'-module,

$$F^{\lambda} \cong \bigoplus_{\nu \in \widehat{G'}} \mathrm{Hom}_{G'}(F'^{\nu}, F^{\lambda}) \otimes F'^{\nu}.$$

Hence (4.2) becomes, as $(\mathcal{A} \otimes \mathbb{C}[G'])$-modules,

$$L \cong \bigoplus_{\lambda \in \mathrm{Spec}(\rho)} \bigoplus_{\nu \in \widehat{G'}} \mathrm{Hom}_{G'}(F'^{\nu}, F^{\lambda}) \otimes E^{\lambda^*} \otimes F'^{\nu}. \tag{4.5}$$

Comparing (4.4) and (4.5), by equating $\sigma^* = \lambda^*$ and $\nu = \mu$, we have our desired isomorphism. □

4.10 Littlewood-Richardson Coefficients

Now we apply seesaw reciprocity to the $\mathrm{GL}(n, \mathbb{C})$-$\mathrm{GL}(k, \mathbb{C})$ duality. Let $V = \mathrm{Mat}_{n,k+m}(\mathbb{C})$ and $Y = S(V)$. Let $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(k+m, \mathbb{C})$ act on Y by $(g, g').f(v) = f(g^t v g')$. By Corollary 4.7.4, we have

$$S(V) \cong \bigoplus_{\lambda} F_n^{\lambda} \otimes F_{k+m}^{\lambda},$$

for all dominant weight λ with $\ell(\lambda) \leq \min\{n, k+m\}$. Let $(\mathrm{GL}(n,\mathbb{C})\times\mathrm{GL}(n,\mathbb{C}))\times(\mathrm{GL}(k,\mathbb{C})\times \mathrm{GL}(m,\mathbb{C}))$ act on $S(V)$ by

$$((g_1, g_2), (h_1, h_2)).f(v_1, v_2) = f(g_1^t v_1 h_1, g_2^t v_2 h_2),$$

where we identify $\mathrm{Mat}_{n,k+m}(\mathbb{C}) \cong \mathrm{Mat}_{n,k}(\mathbb{C}) \oplus \mathrm{Mat}_{n,m}(\mathbb{C})$ for $v_1 \in \mathrm{Mat}_{n,k}(\mathbb{C})$, $v_2 \in \mathrm{Mat}_{n,m}(\mathbb{C})$. Applying Corollary 4.7.4 to $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(k,\mathbb{C})$ acting on the summand $\mathrm{Mat}_{n,k}(\mathbb{C})$ and to $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C})$ acting on the summand $\mathrm{Mat}_{n,m}(\mathbb{C})$, we have

$$\begin{aligned} S(\mathrm{Mat}_{n,k}(\mathbb{C}) \oplus \mathrm{Mat}_{n,m}(\mathbb{C})) &\cong S(\mathrm{Mat}_{n,k}(\mathbb{C})) \otimes S(\mathrm{Mat}_{n,m}(\mathbb{C})) \\ &\cong \bigoplus_{\mu,\nu} (F_n^\mu \otimes F_k^\mu) \otimes (F_n^\nu \otimes F_m^\nu) \\ &\cong \bigoplus_{\mu,\nu} (F_n^\mu \otimes F_n^\nu) \otimes (F_k^\mu \otimes F_m^\nu), \end{aligned}$$

for dominant weights μ, ν with $\ell(\mu) \leq \min\{n, k\}$ and $\ell(\nu) \leq \min\{n, m\}$.

This shows $(\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C}), \mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C}))$ is a dual reductive pair with respect to $S(V)$, so we can form the seesaw

$$\begin{array}{ccc} \mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C}) & & \mathrm{GL}(k+m,\mathbb{C}) \\ \uparrow & \times & \uparrow \\ \mathrm{GL}(n,\mathbb{C}) & & \mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C}) \end{array}$$

The inclusion $\mathrm{GL}(n,\mathbb{C}) \hookrightarrow \mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$ is $g \mapsto (g, g)$, and $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C}) \hookrightarrow \mathrm{GL}(k,m)(\mathbb{C})$ is given by $(g, h) \mapsto \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}$. It can be readily checked that both the representations from both dual reductive pairs agree when restricted to $\mathrm{GL}(n,\mathbb{C}) \times (\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C}))$. Hence the hypothesis of Theorem 4.9.1 is satisfied. By applying the theorem, we get

$$[F_n^\mu \otimes F_n^\nu : F_n^\lambda] = [F_{k+m}^\lambda : F_k^\mu \otimes F_m^\nu],$$

whenever $\ell(\lambda) \leq \min\{n, k+m\}$, $\ell(\mu) \leq \min\{n, k\}$ and $\ell(\nu) \leq \min\{n, m\}$. Notice that when n sufficiently large, the right hand side of the equality does not depend on n.

Definition 4.10.1. Define $c_{\mu\nu}^\lambda = [F_n^\mu \otimes F_n^\nu : F_n^\lambda]$ for $\lambda, \mu, \nu \in \mathbb{N}$ with n sufficiently large such that $\max\{\ell(\lambda), \ell(\mu), \ell(\nu)\} \leq n$. We call $c_{\mu\nu}^\lambda$ *Littlewood-Richardson* coefficients.

Hence the Littlewood-Richardson coefficients record simultaneously the branching rules of $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C}) \downarrow \mathrm{GL}(n,\mathbb{C})$ and $\mathrm{GL}(k+m,\mathbb{C}) \downarrow \mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C})$.

We can give a third interpretation of Littlewood-Richardson coefficients by resorting to the Schur-Weyl duality (Theorem 4.5.2).

Let $Y = (\mathbb{C}^n)^{\otimes(k+m)}$. Let $\mathrm{GL}(n,\mathbb{C})$ and S_{k+m} act on Y as in the Schur-Weyl duality. That is, $\mathrm{GL}(n,\mathbb{C})$ acts diagonally while S_{k+m} permutes the $k+m$ entries. This gives the decomposition

$$Y \cong \bigoplus_{\lambda \in \mathrm{Par}(n,k+m)} G_{k+m}^{\lambda} \otimes F_n^{\lambda}.$$

Write Y as $(\mathbb{C}^n)^{\otimes(k+m)} \cong (\mathbb{C}^n)^{\otimes k} \otimes (\mathbb{C}^n)^m$, we can let $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$ act on Y via

$$(g_1, g_2).(v_1 \otimes \cdots \otimes v_{k+m}) = g_1 v_1 \otimes \cdots \otimes g_1 v_k \otimes g_2 v_{k+1} \otimes \cdots \otimes g_2 v_{k+m},$$

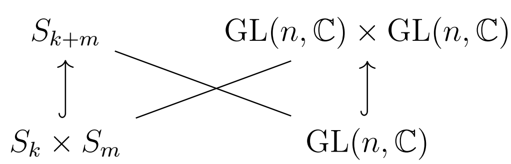
and we let $S_k \times S_m$ act by

$$(\sigma_1, \sigma_2).(v_1 \otimes \cdots \otimes v_{k+m}) = v_{\sigma_1(1)} \otimes \cdots \otimes v_{\sigma_1(k)} \otimes v_{k+\sigma_2(1)} \otimes \cdots \otimes v_{k+\sigma_2(m)}.$$

By applying Schur-Weyl duality to $(\mathbb{C}^n)^{\otimes k}$ and $(\mathbb{C}^n)^m$ separately, we get

$$\begin{aligned} Y &\cong \left(\bigoplus_{\mu \in \mathrm{Par}(n,k)} G_k^{\mu} \otimes F_n^{\mu} \right) \otimes \left(\bigoplus_{\nu \in \mathrm{Par}(n,m)} G_m^{\nu} \otimes F_n^{\nu} \right) \\ &\cong \bigoplus_{\substack{\mu \in \mathrm{Par}(n,k) \\ \nu \in \mathrm{Par}(n,m)}} (G_k^{\mu} \otimes G_m^{\nu}) \otimes (F_n^{\mu} \otimes F_n^{\nu}). \end{aligned}$$

Hence $(S_k \times S_m, \mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C}))$ is a dual reductive pair, so we can form seesaw



We embed $S_k \times S_m \hookrightarrow S_{k+m}$ by sending (σ_1, σ_2) to the permuation in S_{k+m} that permutes the first k entires according to σ_1 and permutes the last m entries according to σ_2. We embed $\mathrm{GL}(n,\mathbb{C}) \hookrightarrow \mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$ diagonally by $g \mapsto (g,g)$. It is evident the representations from diagonal dual reductive pairs restrict to a common representation on $(S_k \times S_m) \times \mathrm{GL}(n,\mathbb{C})$. Thus by Theorem 4.9.1, we have

$$[G_{k+m}^{\lambda} : G_k^{\mu} \otimes G_m^{\nu}] = [F_n^{\mu} \otimes F_n^{\nu} : F_n^{\lambda}] = c_{\mu\nu}^{\lambda}.$$

This gives a third interpretation of $c_{\mu\nu}^{\lambda}$ in terms of the branching rule of the symmetric group.

4.11 Stable Branching Rule for $\mathbf{O}(n,\mathbb{C}) \times \mathbf{O}(n,\mathbb{C}) \downarrow \mathbf{O}(n,\mathbb{C})$

In this section, we will make use of the $\mathrm{O}(n,\mathbb{C})$-$\mathfrak{sp}(k,\mathbb{C})$ duality to deduce a branching rule of $\mathrm{O}(n,\mathbb{C}) \times \mathrm{O}(n,\mathbb{C}) \downarrow \mathrm{O}(n,\mathbb{C})$ for a certain range of irreducibles, where we embed $\mathrm{O}(n,\mathbb{C})$ diagonally inside $\mathrm{O}(n,\mathbb{C}) \times \mathrm{O}(n,\mathbb{C})$. This section follows the paper [HTW05].

In Section 10.2 of [GW09], it is shown that all irreducible regular representations of $\mathrm{O}(n,\mathbb{C})$ can be indexed uniquely by a non-negative integer partition ν such that $(\nu')_1+(\nu')_2 \leq n$, where ν' denote the transpose partition of ν. Let E_n^ν denote the irreducible regular representation of $\mathrm{O}(n,\mathbb{C})$ indexed this way.

Our goal of this section is to prove the following stable branching rule.

Theorem 4.11.1 (Theorem 2.1.2 of [HTW05])**.** *Let λ, μ, ν be non-negative integer partitions such that $\ell(\lambda) \leq \lfloor n/2 \rfloor$ and $\ell(\mu) + \ell(\nu) \leq \lfloor n/2 \rfloor$. Then*

$$[E_n^\mu \otimes E_n^\nu : E_n^\lambda] = \sum c_{\alpha\beta}^\lambda c_{\alpha\beta}^\mu c_{\beta\gamma}^\nu,$$

where the sum is over all non-negative integer partitions α, β, γ.

We will assume the following stronger version of $\mathrm{O}(n,\mathbb{C})$-$\mathfrak{sp}(k,\mathbb{C})$ duality (Theorem 4.8.3), which gives the exact decomposition into irreducibles with our index convention.

Theorem 4.11.2 (Theorem 3.2 of [HTW05])**.** *The decomposition in Theorem 4.8.3 is*

$$\mathcal{P}(\mathrm{Mat}_{n,k}(\mathbb{C})) \cong \bigoplus_\lambda E_n^\lambda \otimes \widetilde{E}_k^\lambda,$$

where E_n^λ is the irreducible representation of $\mathrm{O}(n,\mathbb{C})$ indexed by λ, and $\widetilde{E}_k^\lambda$ is an irreducible representation of $\mathfrak{sp}(k,\mathbb{C})$. In the decomposition, λ runs through the set of all integer partitions such that $\ell(\lambda) \leq k$ and $(\lambda')_1 + (\lambda')_2 \leq n$. Moreover, in the stable range $n \geq 2k$, restricting $\widetilde{E}_k^\lambda$ to $\mathfrak{gl}(k,\mathbb{C}) \subset \mathfrak{gl}(k,\mathbb{C}) \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-$ as in the complexified Cartan decomposition of $\mathfrak{sp}(k,\mathbb{C})$, it decomposes as representations of $\mathfrak{gl}(k,\mathbb{C})$ by

$$\widetilde{E}_k^\lambda \cong S(S^2\mathbb{C}^k) \otimes F_k^\lambda,$$

where $S(S^2\mathbb{C}^k)$ is the symmetric algebra of the symmetric 2-tensor of $\mathbb{C}^k$, the standard representation of $\mathfrak{gl}(k,\mathbb{C})$, and F_k^λ is the irreducible representation of $\mathrm{GL}(k,\mathbb{C})$ (induced on its Lie algebra) with highest weight λ.

Remark 4.11.3. The decomposition of $\widetilde{E}_k^\lambda$ as a $\mathfrak{gl}(k,\mathbb{C})$-module is an instance of the universality of holomorphic discrete series; see the discussion before Theorem 3.2 of [HTW05].

Proof of Theorem 4.11.1. Let p, q be positive integers that we will determine in the end. Let $V = \mathrm{Mat}_{n,p+q}(\mathbb{C})$. We first let $\mathrm{O}(n,\mathbb{C}) \times \mathrm{O}(n,\mathbb{C})$ act on $\mathcal{P}(V)$ diagonally through the decomposition $\mathrm{Mat}_{n,p+q}(\mathbb{C}) \cong \mathrm{Mat}_{n,p}(\mathbb{C}) \oplus \mathrm{Mat}_{n,q}(\mathbb{C})$ by

$$(g_1, g_2).f(v_1, v_2) = f(g_1^{-1}v_1, g_2^{-1}v_2).$$

On each summand, this is the same action as in Theorem 4.8.3, so the action of $\mathrm{O}(n,\mathbb{C}) \times \mathrm{O}(n,\mathbb{C})$ commutes with $\mathfrak{sp}(p,\mathbb{C}) \oplus \mathfrak{sp}(q,\mathbb{C})$, and together they have a multiplicity-free decomposition. With respect to the dual reductive pair $(\mathrm{O}(n,\mathbb{C}) \times \mathrm{O}(n,\mathbb{C}), \mathfrak{sp}(p,\mathbb{C}) \oplus \mathfrak{sp}(q,\mathbb{C}))$, by Theorem 4.11.2 to each summand, and after rearranging terms, we have

$$\mathcal{P}(V) \cong \bigoplus_{\mu,\nu} (E_n^\mu \otimes E_n^\nu) \otimes (\widetilde{E}_p^\mu \otimes \widetilde{E}_q^\nu),$$

where the sum is over all μ and ν such that $\ell(\mu) \le p$, $(\mu')_1 + (\mu')_2 \le n$, $\ell(\nu) \le q$, $(\nu')_1 + (\nu')_2 \le n$. On the other hand, let $\mathrm{O}(n,\mathbb{C})$ act on $\mathcal{P}(V)$ by $g.f(v) = f(g^{-1}v)$, which by Theorem 4.8.3, commutes with the action of $\mathfrak{sp}(p+q,\mathbb{C})$. So we have another dual reductive pair $(\mathrm{O}(n,\mathbb{C}), \mathfrak{sp}(p+q,\mathbb{C}))$, which gives decomposition

$$\mathcal{P}(V) \cong \bigoplus_{\lambda} E_n^\lambda \otimes \widetilde{E}_{p+q}^\lambda,$$

where the sum if over all integer partitions λ with $\ell(\lambda) \le p+q$ and $(\lambda')_1 + (\lambda')_2 \le n$. So we can form seesaw

$$\begin{array}{ccc} \mathrm{O}(n,\mathbb{C}) \times \mathrm{O}(n,\mathbb{C}) & & \mathfrak{sp}(p+q,\mathbb{C}) \\ \uparrow & \times & \uparrow \\ \mathrm{O}(n,\mathbb{C}) & & \mathfrak{sp}(p,\mathbb{C}) \oplus \mathfrak{sp}(q,\mathbb{C}) \end{array}$$

where $\mathrm{O}(n,\mathbb{C}) \hookrightarrow \mathrm{O}(n,\mathbb{C}) \times \mathrm{O}(n,\mathbb{C})$ is $g \mapsto (g,g)$ and $\mathfrak{sp}(p,\mathbb{C}) \oplus \mathfrak{sp}(q,\mathbb{C}) \hookrightarrow \mathfrak{sp}(p+q,\mathbb{C})$ is given by $(X,Y) \mapsto \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$. It is clear that the representations from diagonal dual pairs restrict to a common representation at the bottom pair. Hence by Theorem 4.9.1, we have

$$[E_n^\mu \otimes E_n^\nu : E_n^\lambda] = [\widetilde{E}_{p+q}^\lambda : \widetilde{E}_p^\mu \otimes \widetilde{E}_q^\nu].$$

Now the problem becomes determining the branching law for $\mathfrak{sp}(p+q,\mathbb{C}) \downarrow \mathfrak{sp}(p,\mathbb{C}) \oplus \mathfrak{sp}(q,\mathbb{C})$. We assume we are in the stable range $n \ge 2(p+q)$. By the second half of Theorem 4.11.2, as $\mathfrak{gl}(p+q,\mathbb{C})$ representations,

$$\widetilde{E}_{p+q}^\lambda \cong S(S^2\mathbb{C}^{p+q}) \otimes F_{p+q}^\lambda.$$

As $\mathfrak{gl}(p,\mathbb{C}) \oplus \mathfrak{gl}(q,\mathbb{C})$ representations, $\mathbb{C}^{p+q} \cong \mathbb{C}^p \oplus \mathbb{C}^q$, so that

$$S(S^2\mathbb{C}^{p+q}) \cong S(S^2\mathbb{C}^p) \otimes S(S^2\mathbb{C}^q) \otimes S(\mathbb{C}^p \otimes \mathbb{C}^q).$$

Since $n \ge 2(p+q)$ implies $n \ge 2p$ and $n \ge 2q$, again by Theorem 4.11.2, as $\mathfrak{gl}(p,\mathbb{C})$ representations,

$$\widetilde{E}_p^\mu \cong S(S^2\mathbb{C}^p) \otimes F_p^\mu,$$

and as $\mathfrak{gl}(q,\mathbb{C})$ representations,

$$\widetilde{E}_q^\nu \cong S(S^2\mathbb{C}^q)\otimes F_q^\nu.$$

By cancelling out $S(S^2\mathbb{C}^p)\otimes S(S^2\mathbb{C}^q)$ in $[\widetilde{E}_{p+q}^\lambda : \widetilde{E}_p^\mu\otimes\widetilde{E}_q^\nu]$ after plugging in the the decompositions, we have

$$[\widetilde{E}_{p+q}^\lambda : \widetilde{E}_p^\mu\otimes\widetilde{E}_q^\nu] = [S(\mathbb{C}^p\otimes\mathbb{C}^q)\otimes F_{p+q}^\lambda : F_p^\mu\otimes F_q^\nu].$$

By Corollary 4.7.4, we have

$$S(\mathbb{C}^p\otimes\mathbb{C}^q)\cong\bigoplus_\gamma F_p^\gamma\otimes F_q^\gamma,$$

for $\ell(\gamma)\le\min\{p,q\}$. Since $n\ge 2(p+q)$, by the definition of Littlewood-Richardson coefficients, we have

$$F_{p+q}^\lambda\cong\bigoplus_{\alpha,\beta} c_{\alpha\beta}^\lambda F_p^\alpha\otimes F_q^\beta,$$

for $\ell(\alpha)\le p$ and $\ell(\beta)\le q$. Hence

$$[\widetilde{E}_{p+q}^\lambda : \widetilde{E}_p^\mu\otimes\widetilde{E}_q^\nu] = \sum_{\alpha,\beta,\gamma} c_{\alpha\beta}^\lambda[(F_p^\alpha\otimes F_p^\gamma)\otimes(F_q^\beta\otimes F_q^\gamma) : F_p^\mu\otimes F_q^\nu].$$

By the alternative interpretation of Littlewood-Richardson coefficients, we have

$$[F_p^\alpha\otimes F_p^\gamma : F_p^\mu] = c_{\alpha\gamma}^\mu,\quad [F_q^\beta\otimes F_q^\gamma : F_q^\nu] = c_{\beta\gamma}^\nu.$$

Thus

$$[E_n^\mu\otimes E_n^\nu : E_n^\lambda] = \sum_{\alpha,\beta,\gamma} c_{\alpha\beta}^\gamma c_{\alpha\gamma}^\mu c_{\beta\gamma}^\nu,$$

for all α,β,γ such that $\ell(\alpha)\le p$, $\ell(\beta)\le q$, and $\ell(\gamma)\le\min\{p,q\}$. The restrictions for λ,μ,ν are $\ell(\mu)\le p$, $\ell(\nu)\le q$, $\ell(\lambda)\le p+q$, and at the same time to stay in stable range we need $n\ge 2(p+q)$. If λ,μ,ν satisfy $\ell(\lambda)\le\lfloor n/2\rfloor$ and $\ell(\mu)+\ell(\nu)\le\lfloor n/2\rfloor$, then we can choose $p=\ell(\mu)$ and $q=\lfloor n/2\rfloor-p$.

□

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