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| **6.885 Algebra and Computation** | | November 28, 2005 |
| | Lecture 22 | |
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Today we present various algebraic models of computation, and discover a few lower bounds.

# 1 Algebraic models of computation

## 1.1 Considered problems

Let $f : R^n \to R^m$ be a function that maps $n$ elements of a ring $R$ into $m$ elements of the same ring. Given $x_1, x_2, \ldots, x_n \in R$, compute $f(x_1, x_2, \ldots, x_n)$.

Alternatively, for a function $f : R^n \times R^m \to R$, given $x_1, x_2, \ldots, x_n \in R$, determine $y_1, y_2, \ldots, y_n \in R$ such that $f(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) = 0$.

## 1.2 Uniform model of computation

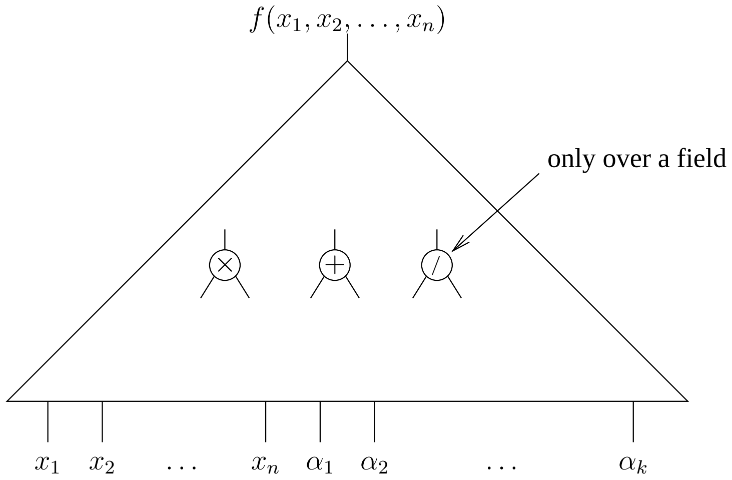
In the late 1980's Blum, Shub and Smale came up with a uniform model model of computation. It was a "Turing machine" over a ring. In this lecture we consider only non-uniform models of computation.

## 1.3 Algebraic circuits (Straight line programs)

An *algebraic circuit* is an acyclic network of gates with the following properties:

- the circuit has $n$ inputs accepting $x_1, x_2, \ldots, x_n \in R$ and an arbitrary number of constants $\alpha_i$ in $R$,
- the circuit has $m$ outputs, $y_1, y_2, \ldots, y_m \in R$, and computes a function $f$, i.e. $f(x_1, \ldots, x_n) = (y_1, \ldots, y_m)$,
- each gate has two inputs and one output, and computes either the sum or product of input values (if $R$ is a field, we allow as well division).

The number of gates is a complexity measure of algebraic circuits.

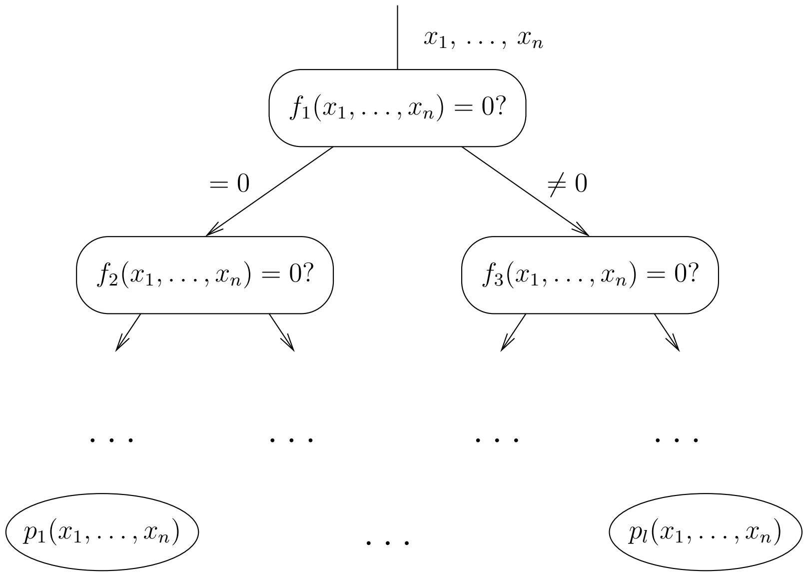



Note that every algebraic circuit is equivalent to a straight line program in which every instruction corresponds to a single gate and has the form "$v_i \leftarrow v_j \diamondsuit v_k$", where $\diamondsuit$ is one of allowed operations.

## 1.4 Algebraic decision trees

An *algebraic decision tree* is a decision tree of the following properties:

- at each internal node we evaluate a polynomial of input elements $x_1, x_2, \ldots, x_n$, and branch, depending on whether the computed value of the polynomial equals 0 or not,
- each leaf contains a polynomial of the input elements which is the required values which we want to compute.

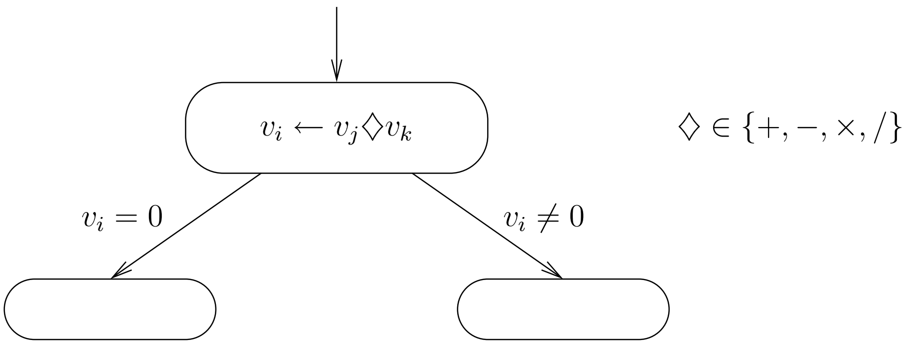



There are two complexity measures:

1. The depth of a tree.
2. The degree of polynomials at internal nodes.

## 1.5 Algebraic computation trees

An *algebraic computation tree* is a tree in which at each internal node we perform a single instruction of the form "$v_i \leftarrow v_j \diamondsuit v_k$", where $\diamondsuit$ is one of basic operations allowed over $R$, and branch, depending on whether the computed value equals 0 or not.



## 2 Ostrowski's conjecture

### 2.1 The problem: univariate polynomial evaluation

Given $a_0, a_1, \ldots, a_n \in R$ and $x \in R$, compute

$$\sum_{i=0}^{n} a_i x^i.$$

### 2.2 Horner's rule

Horner's rule enables us to evaluate a polynomial by $n$ additions and $n$ multiplications in the following way:

$$\begin{array}{rcl} v_1 & \leftarrow & a_n \cdot x + a_{n-1} \\ v_2 & \leftarrow & v_1 \cdot x + a_{n-2} \\ & \ldots & \\ v_i & \leftarrow & v_{i-1} \cdot x + a_{n-i} \\ & \ldots & \\ v_n & \leftarrow & v_{n-1} \cdot x + a_0 \end{array}$$

### 2.3 The conjecture

Ostrowski came up in 1954 with the conjecture that Horner's rule is optimal, i.e. one needs $n$ additions and $n$ multiplications (in the algebraic circuit model). He managed to prove that $n$ additions are necessary, and in 1966 Pan proved that so are $n$ multiplications.

### 2.4 Ostrowski's lower bound

To show that we need $n$ additions, we substitute $x = 1$, and the problem of evaluation of the polynomial reduces to the problem of computing the sum of coefficients.

**Claim 1** *To evaluate the sum of $a_0$ to $a_n$ over a ring at least $n$ additions are necessary in the algebraic circuit model.*

**Proof** The proof goes by induction on $n$. For $n = 1$, all that we can compute, not using additions, is $ca_0^{d_0} a_1^{d_1}$, where $c \in R$, which definitely differs from $a_0 + a_1$. For $n > 1$, the first addition in any straight line program looks like

$$c_1 \prod_{i=1}^{n} a_i^{d_i} + c_2 \prod_{i=1}^{n} a_i^{e_i},$$

and since it does not make sense to add constants as they can be hardcoded, we can assume that one of $d_i$'s or $e_i$'s is nonzero. Without loss of generality $d_n \neq 0$, for $a_n = 0$ the first addend disappears, and by the induction assumption we still need to spend $n - 1$ additions to compute $a_0 + a_1 + \cdots + a_{n-1}$. ■

### 2.5 Pan's lower bound

This time we substitute $a_0 = 0$. Note first that any algebraic circuit computes some polynomial in $R[a_1, a_2, \ldots, a_n, x]$. A multiplication $v_j \cdot v_k$ is *insignificant* if one of the following holds:

1. Both $v_j$ and $v_k$ belong to $R[x]$.

2. One of $v_j$ and $v_k$ belongs to $R$.

Certainly, a multiplication that is not insignificant is *significant*. We will show that the number of significant multiplications is large enough in some more general case.

**Claim 2** *Let $f : R^{n+1} \to R$ be a function of the form*

$$f(a_1, a_2, \ldots, a_n, x) = \sum_{i=1}^{k} l_i(a_1, \ldots, a_n)x^i + r(x) + l_0(a_1, a_2, \ldots, a_n),$$

*where each $l_i$ is a linear function, and $R$ is a field. An algebraic circuit computing $f$ has at least $\operatorname{rank}\{l_1, l_2, \ldots, l_k\}$ significant multiplications.*

**Proof** Look at the first significant multiplication. It has the following form:

$$\left(\sum_i c_i a_i + c_0(x)\right) \cdot \left(\sum_i d_i a_i + d_0(x)\right).$$

Without loss of generality $c_1 \neq 0$, and we restrict $(a_1, \ldots, a_n, x)$ so that the first term equals $c \in R$, achieving

$$\begin{aligned} c &= \sum_i c_i a_i + c_0(x), \\ a_1 &= \frac{c - c_0(x) - \sum_{i=2}^{k} c_i a_i}{c_1} = l(a_2, \ldots, a_n) + p(x) \end{aligned}$$

for some linear function $l$ and polynomial $p$. Now we have a circuit that using one fewer significant multiplication computes

$$\sum_{i=1}^{k} l_i\left(l(a_2, \ldots, a_n) + p(x), a_2, \ldots, a_n\right) x^i + r(x) + l_0(l(a_2, \ldots, a_n) + p(x), a_2, \ldots, a_n)$$

$$= \sum_{i=1}^{k} l_i'(a_2, \ldots, a_n)x^i + r'(x) + l_0'(a_2, \ldots, a_n),$$

where $l_i'(a_2, \ldots, a_n) = l_i(l(a_2, \ldots, a_n), a_2, \ldots, a_n)$, and by basic linear algebra

$$\operatorname{rank}\{l_1', l_2', \ldots, l_n'\} \geq \operatorname{rank}\{l_1, l_2, \ldots, l_n\} - 1.$$

This implies by induction on the number of $a_i$'s that we need at least $\operatorname{rank}\{l_1, l_2, \ldots, l_n\}$ significant multiplications. ∎

# 3 Fixed coefficients

If coefficients of the polynomial are fixed, that is we compute a function $f_{a_0 \ldots a_n} : R \to R$ such that

$$f_{a_0 \ldots a_n}(x) = \sum a_i x^i,$$

it turns out that we need at most $n/2 + 1$ multiplications, and that for most choices of coefficients this number of multiplications is necessary. The main idea is that we can express $f$ as

$$f(x) = q_1(x)(x^2 - b_1) + r_1(x),$$

there exists $b_1$ so that $r_1$ is of degree 0, and both $b_1$ and $r_1$ can be hardwired into a circuit. To show the lower bound we take $a_0$, $a_1$, ..., $a_n$ transcendent over $\widetilde{R}$, and prove that if a program computes $\sum a_i x^i$ with $k$ multiplications, then $(a_1, \ldots, a_n)$ lie in a $2k$-dimensional extension of $\widetilde{R}$.

## 4 Evaluation in $n$ points

Given $a_0, \ldots, a_n, x_0, \ldots, x_n$ in a field $K$, our goal is to compute $z_1$ to $z_n$ such that $z_i = \sum a_j x_i^j$. Using fast Fourier transform, we can achieve this in $O(n \log^{O(1)} n)$ time, and Strassen has proven that we need $\Omega(n \log n)$ operations in any algebraic computation tree. We will cover this topic in the next lecture.