6.841J/18.405J: Advanced Complexity Theory **Wed, Apr 23, 2003**

Lecture 19

Instructor: Prahladh Harsha *Scribe: Megumi Ando*

Today

- Recap of PRGs and deterministic simulation of BPP
- (s, ϵ)-PRG [Nisan-Widgerson]

Definition of PRGs [Blum-Micali-Yao]

From last lecture, we defined a function $G : \{0,1\}^s \longrightarrow \{0,1\}^n$ to be a PRG, if no polynomial size circuit can distinguish its output from a purely ranom output. That is, if $\forall$ polynomial sized circuits C and $\forall$ polynomials p:

$$|Pr_{x \leftarrow U_s}[C(G(x) = 1] = Pr_{y \leftarrow U_n}[C(y) = 1]| \leq \frac{1}{p(n)}$$

and G is efficient, meaning it is constructible in $p(s)$.

Derandomizing BPP

- Given a BPP algorithm, we can run the algorithm on a pseudorandom string $G(x)$, $\forall x$, and take majority vote.
- Notice that OWF exists $\Rightarrow BPP = \bigcap_{\epsilon > 0} DTIME(2^{n^\epsilon})$

Observation

Do we really need such a hard definiton of PRGs for BPP simulation? Using the Blum-Micali-Yao definition of PRG, we can not only derandomize BPP, but we can prove a much stronger statement: PRG exist $\Rightarrow P \neq NP$.

Let's weaken the defintion of PRGs. If our goal is just to derandomize BPP, then it suffices that:

- G runs in $2^{O(s)}$
- G works against circuits of size n or even some simple polynomial n^2

Alternate Definition of PRGs [Nisan-Widgerson]

We define a function $G : \{0,1\}^s \longrightarrow \{0,1\}^n$ to be a (S, ϵ)-PRG, if $\forall$ circuits C of size at most S:

$$|Pr_{x \leftarrow U_s}[C(G(x) = 1] = Pr_{y \leftarrow U_n}[C(y) = 1]| \leq \epsilon$$

Differences from previous definition:

- G fools circuits of size S.
- Probabalistically, the difference between a purely random string and the string generated by G is $\epsilon \neq \frac{1}{p(n)}$.
- G constructible in $2^{O(s)}$.

Observation

If $(n^2, \frac{1}{10})$-PRG $G : \{0,1\}^{O(\log n)} \longrightarrow \{0,1\}^n$ exists $\Rightarrow BPP \subseteq \bigcap_{\epsilon>0} DTIME(2^{n^\epsilon})$. This really implies BPP = P.

Definition of Average Case Hardness

A function $f : \{0,1\}^n \longrightarrow \{0,1\}$ is (S, ϵ)-hard if $\forall$ circuit C of size at most S,

$$|Pr_x[C(x) = f(x)]| \leq \frac{1}{2} + \epsilon$$

Notice that given an (S, ϵ)-hard function f, we can construct a PRG $G : x \mapsto x \circ f(x)$ which is (S, ϵ)-hard. (We will use a similar idea to define designs.)

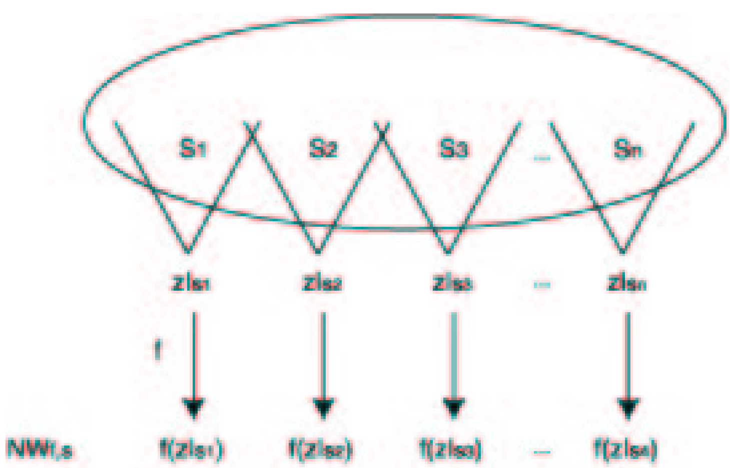
Definition of Designs

A collection of subsets $S = \{S_1, ..., S_n\}$ of a universe $U = [s] = \{1, ..., s\}$ is a (l, a)-design over $[s]$ if:

- $|S_i| = l, \forall i$
- $|S_i \cap S_j| \leq a, \forall i \neq j$

Notation

- Let $S = \{S_1, ..., S_n\}$ be a (l, a)-design over $[s]$.
- Let $f : \{0,1\}^n \longrightarrow \{0,1\}$ be a hard function.
- For any string z, let $z|_S$ denote the substring indexed by the bit positions of S.
  (E.g., if $z = 01000111$, $S = \{1, 3, 5, 7\}$, then $z|_S = 0001$.)
- Define the Nisan-Widgerson PRG $NW_{f,s} : \{0,1\}^l \longrightarrow \{0,1\}^n$ as follows:
  $z \mapsto f(z|S_1) \circ ... \circ f(z|S_n)$



Lemma

For any constant $\gamma > 0$ and any l, n, a, there exists an (l, s)-design over $[s]$ that is constructible in polynomial time in s and n, where:

- $s = O(\frac{l^2}{a})$
- $a = \gamma \log n$

Theorem

There exists a function $f : \{0,1\}^l \longrightarrow \{0,1\}$, computable in DTIME($2^{O(l)}$), which $\forall$ sufficiently large n, when f is restricted to n bits, is $(2^{\epsilon n}, \frac{1}{10n})$-hard
$\Rightarrow$ there exists a function $NW_{f,s} : \{0,1\}^{O(\log n)} \longrightarrow \{0,1\}^n$ which is $(n^2, \frac{1}{10})$-PRG.

Proof

If $NW_{f,s}$ is not a PRG, then there exists a predictor P that predicts bit$_i$ based on $\{\text{bit}_1, ..., \text{bit}_{i-1}\}$ better than a random guess. That is:

$$Pr_z[P(f(z|S_1)...f(z|S_{i-1})) = f(z|S_i)] \geq \frac{1}{2} + \frac{1}{10n}.$$

By averaging, we can determine all the bits of z except for those in S_i and the above statement would still hold. Variables in each of $f(z|S_j)$, $j \neq i$ are at most $a = \gamma \log n$ in number. Thus each of these can be computed using a lookup table T_j of size at most $2^a = n$. So:

$$Pr[P[T_1, ..., T_{i-1}] = f(z|S_i)] \geq \frac{1}{2} + \frac{1}{10n},$$

where P is of size $2^{O(n)} = n^2$, T_i is of size n, and $P[T_1, ..., T_{i-1}]$ is of size $O(n^3)$. Thus, f is approximatable by a circuit of size n^3. Which is a contradiction because f is $(2^{\epsilon l}, \frac{1}{10n})$-hard.

Average Case verses Worse Case

So far, we have shown that average case hardness translates into pseudorandomness. But what can we say about worst case hardness? In the case of a Permanent (See Lecture 15 and Lecture 16), worst case hardness is equivalent to average case harness. Thus, if there exists no circuit of size at most $2^{\epsilon n}$ that computes the Permanent, then BPP = P.

Impagliazzo, Widgerson, Trevisan, Sudan, and Vadhan show how to convert a wosrt case hardness function $f \in$ EXP into another function $f' \in$ EXP which is average case hard, using error-correcting codes. Thus, we conclude that worst-case hardness also translates into pseudorandomness.

How close are we to prove BPP = P?

We have shown that proving circuit lower bound imply derandomization. Similarly, if NEXP has some form of circuit lower bounds, then AM = NP. IKW prove a weaker converse: $AM \neq NEXP \Leftrightarrow NEXP$ is not $\subseteq NP/Poly$

Alternate Proof of BPP $\in \Sigma_2$

The NW paradigm indicate that BPP is no harder than finding a hard function. This gives the following Σ_2 algorithm for BPP:

- Guess a hard function $f : \{0,1\}^l \longrightarrow \{0,1\}$.
- For all circuits C of size at most 2^ϵ, check that C does not approximate f on more than $\frac{1}{2} + \epsilon$ fraction of imputs.
- Use f to construct $NW_{f,s}$ and use this PRG to derandomize the BPP algorithm.