6.S978 Graphs, Linear Algebra, and Optimization – Fall 2015

Problem Set 1

Out: October 11, 2015 *Due: October 28, 2015*

(1) *Submit the solutions as a PDF file (typeset in LaTeX) by emailing it to Aleksander.*

(2) *You can solve the problems in collaboration with one other person, but your writeup has to be prepared independently. Also, please provide the name of the collaborator.*

Problem 1. (**Reducing the directed max flow to the undirected max flow.**) Consider a directed graph $G = (V, E)$ (with unit capacities), a source vertex s and a sink vertex t. Now, let $\widehat{G} = (V, \widehat{E})$ be an *undirected* graph $\widehat{G} = (V, \widehat{E})$ over the same vertex V set as G. The edges set $\widehat{E}$ of $\widehat{G}$ is defined as follows: for each arc $e = (u, v) \in E$ in G, with u being the tail and v being the head of e, the graph $\widehat{G}$ has edges (s, v), (u, v), and (u, t) added to $\widehat{E}$.

(a) Let F^* be the value of the maximum s-t flow in G. Argue that the value $\widehat{F}^*$ of the maximum s-t flow in $\widehat{G}$ is exactly $2F^* + |E|$.
Hint: Use the max-flow min-cut theorem.

(b) (**Extra credit**) Design a nearly-linear time procedure that given a maximum s-t flow $\hat{f}^*$ in $\widehat{G}$ returns a maximum s-t flow in G.
Hint: You might need to use here some advanced data structure result.

Note: This construction extends to arbitrary capacities in a straight-forward manner.

Problem 2. (**Implementing the conjugate gradient method.**) Recall the linear system solving via conjugate gradient method that we discussed in class.

Algorithm 1 Conjugate gradient method.

Compute

$$x_T := \operatorname*{argmin}_{x \in \mathcal{K}_T} g(x), \tag{1}$$

where $\mathcal{K}_T := \text{span}(b, Ab, \ldots, A^{T-1}b)$ is the Krylov's subspace of order T and

$$g(x) := \frac{1}{2}\left(\|e(x)\|_A^2 - \|x^*\|_A^2\right) = \frac{1}{2}\|x\|_A^2 - b^T x.$$

return x_T.

Let $v_1, \ldots, v_T \in \mathbb{R}^n$ be an A-orthogonal basis for $\mathcal{K}_T$. That is, we have that, for each $x \in \mathcal{K}_T$, $x = \sum_{s=1}^{T} \alpha_s v_s$, for some $\alpha_1, \ldots, \alpha_T \in \mathbb{R}$; and $v_i \cdot_A v_j = 0$, if $i \neq j$, where

$$x \cdot_A y := x^T A y$$

is the *inner A-product*.

(a) Show that the optimization problem (1) is equivalent to the following formulation

$$\operatorname*{argmin}_{\alpha_1, \ldots, \alpha_T \in \mathbb{R}} \sum_{s=1}^{T} \left(\frac{\alpha_s^2}{2}\|v_s\|_A^2 - \alpha_s b^T v_s\right). \tag{2}$$

(b) Argue that, given the A-orthogonal basis $v_1, \ldots, v_T$, we can solve problem (2) using only T matrix-vector multiplications of A.

(c) Prove that one can compute the A-orthogonal basis $v_1, \ldots, v_T$ using only $O(T)$ matrix-vector and vector-vector multiplications.
Hint: Proceed in phases. In phase s, given an A-orthogonal basis $v_1, \ldots, v_{s-1}$ for $\mathcal{K}_{s-1}$, extend it to an A-orthogonal basis $v_1, \ldots, v_{s-1}, v_s$ for $\mathcal{K}_s$ by applying the Gram-Schmidt orthogonalization procedure to the vector $v'_s := Av_{s-1}$. (Why $\mathcal{K}_s = span(v_1, \ldots, v_{s-1}, v'_s)$?) What can you say about $v_i \cdot_A v'_s$, for each $i < s-2$?

Problem 3. (**Understanding the lower end of the spectrum of a Laplacian matrix.**) Let us fix an (unweighted) graph $G = (V, E, w)$ and let L be its Laplacian matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

(a) Prove that all the eigenvalues of the Laplacian L are non-negative, i.e., that $\lambda_1 \geq 0$.

(b) Show that $\lambda_1 = 0$ and the all-ones vector $\vec{1} := (1, \ldots, 1)$ is the corresponding eigenvector.

(c) Prove that, for any $k \geq 1$, $\lambda_k = 0$ iff G has at least k connected components.
Note: This means, in particular, that if G is connected then $\lambda_2 > 0$.

Hint: The fact that we mentioned in class that, for any vector $x \in \mathbb{R}^n$, $x^T L x = \sum_{e=(u,v)\in E}(x_u - x_v)^2$ might be useful here.

Problem 4. (**Bipartiteness and the value of λ_n.**) Let $G = (V, E)$ be a bipartite graph and let $\lambda_1 \leq \ldots \leq \lambda_n$ be the eigenvalues of its Laplacian. (A graph is bipartite iff one can partition its vertices into two sets P and Q such that each edge has one endpoint in P and the other one in Q.)

(a) Show that whenever G is d-regular (but not necessarily bipartite) we have that $\lambda_n \leq 2d$. (A graph is d-regular iff each vertex of G has its degree equal to d.)
Note: One can show in a similar way that even when G is not d-regular then $\lambda_n \leq 2d_{\max}$, where $d_{\max}$ is a maximum degree.

(b) Prove that for a d-regular graph G, if G is bipartite then $\lambda_n = 2d$.

(c) (**Extra credit**) Let G be d-regular and connected. Argue that if we have that $\lambda_n = 2d$ then G is bipartite. Does this implication always hold if G is not connected?