Lecture 7

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1 Introduction

In the previous lectures, we looked at several examples of games. The common theme was that all the Nash equilibrium outcomes were very sub-optimal. Today, we would like to quantify this sub-optimality better and, in particular, try to answer the question: how much worse (than the optimum) can the value of Nash equilibrium be. As this question - the way it is stated - is too general to enable a meaningful answer, we will focus on studying it only in the context of a certain class of games called *congestion games.*

2 Setting

For historical reasons, our setup will be slightly different than in the previous lectures. In particular, we will make the players focus not on maximizing their (expected) utility, but on minimizing their (expected) cost (as given by appropriate cost functions). Furthermore, we will require all the cost functions for any strategy to be positive. Note that this altered setting is essentially equivalent to the usual one, we can just take the cost functions to be a negative of the utilities and add a suitably large constant to all of them to ensure positivity – this does not change the Nash equilibrium structure of the game.

3 Price of Anarchy and Price of Stability

We proceed to developing a formal language that will allow us to quantify the sub-optimality of Nash equilibria in games. Clearly, the first thing we need to do here is to choose our notion of optimum. There could be many possible choices (e.g., we could think of the optimal outcome being the one that maximizes *fairness*, i.e., minimizes the maximum cost incurred by any player), but the most canonical one is to look at so-called *social optimum*, i.e., the outcome that minimizes the total cost of all the players. Formally,

$$OPT = \min_{s \in S} \sum_i c_i(s),$$

where $c_i(s)$ is the cost of player i on an outcome s.

Now, we want to measure the quality of the game in terms of the inefficiency (sub-optimality) of its equilibria. One of the most popular measures of that is known as the *Price of Anarchy* (PoA) defined as

$$PoA(G) = \max_{s \in NE(G)} \frac{c(s)}{OPT}$$

where $NE(G)$ is the set of all Nash equilibria of the game G and $c(s) := \sum_i c_i(s)$ is the total cost of the outcome. In other words, the price of anarchy of a given game is the ratio of the cost of the worst Nash equilibrium to the cost of social optimum. (The name comes from the fact that this is the guarantee on the quality of the outcomes when we have absolutely no control over the choice of Nash equilibrium to which the game converges.)

Of course, measuring the quality of games via their price of anarchy has its pros and cons. A definite pro here is that this notion is very robust. That is, if our game has a small PoA, then it doesn't matter to which Nash equilibrium the game converges. They are all good, since the worst one has still its cost within some small factor of the optimum. So, if PoA is small we can be confident that the game always leads to not too sub-optimal outcomes.

On the other hand, however, the problem with PoA notion is that its worst-case nature makes it sensitive to outliers and, in particular, it is not clear how to interpret the case of the PoA of a game being high. After all, it could happen that there is only one Nash equilibrium that is really sub-optimal and there are many others that are very good (and we might be likely to converge to them).
This problem motivates another measure of quality called *Price of Stability* (PoS), which is defined as

$$PoS(G) = \min_{s \in NE(G)} \frac{c(s)}{OPT}.$$

That is, we compare the performance of the best Nash equilibrium to the social optimum. In a sense, we think here of the best game-theoretically stable outcome that we can get, if there is some external way of guiding the choices of the players. Think, for example, about some central entity that can announce some particular Nash equilibrium to all the players, making them believe that this is what everyone else will play and thus it is in their best interest to not deviate from that.
Clearly, as PoA – being a worst-case notion – was most meaningful when it is small, the PoS – being a best-case notion – is most meaningful when its large – as then it really means that the game is bound to give very sub-optimal outcomes. Conversely, when PoS is small it does not necessarily mean that the game will always produce good outcomes (unless we have a way of enforcing the choice of desired Nash equilibrium).
Still, despite these problems, both of these notions are very useful in understanding games, and we proceed to seeing them in action in the context of congestion games.

4 Linear s-t Congestion Games

To keep the discussion more focused, we will not define congestion games in their entire generality here, but start instead with studying their simple variant – *linear s-t congestion games*. Later, we will mention how this variant extends to the general case.
In congestion games, we deal with a directed graph – we can think of this graph as a road network – and a set of n players that want to choose a routing between some two points in this graph. In our simple variant, we assume that all the players are interested in routing between the same two special vertices: a source s and a sink t, and thus the action of each player i is a choice of an s-t path in G.
In our considerations, we want to think of n as being very large, that is, $n \to \infty$ (this is sometimes called a *non-atomic* case). As a result, we prefer to view the set of possible outcomes as a total flow of traffic in which each player controls a minuscule fraction of it. To make this precise, we will view a strategy vector as a convex combination $\{\alpha_p\}_p$ of s-t paths, where α_p is the fraction of players that chose the s-t path p.
Alternatively and equivalently, we can think of a strategy vector as unit flow in G from s to t. Recall that a flow f is a function $f : E \to \mathbb{R}^+$ with two types of constraints:

1. *(Flow conservation constraints)* the total inflow into any vertex v other than s or t is equal to the total outflow of that vertex;

2. *(Flow value constraint)* the total outflow out of s is equal to one.

It is easy to see the equivalence of these two views of strategies. Given a convex combination $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of paths, the corresponding flow over arc e, is given by $f_e = \sum_{p \ni e} \alpha_p$. Conversely, if we have a unit s-t flow, then it can be decomposed into a convex combination of paths $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ (note that this decomposition does not need to be unique!). Also, for future reference, we note the following simple observation

Observation 1 *The set of strategies $\mathcal{S}$ of a congestion game is convex. That is, for any two strategies $f_1, f_2 \in \mathcal{S}$ and $\alpha \in [0, 1]$ we have that $\alpha f_1 + (1 - \alpha) f_2$ is also a valid strategy from $\mathcal{S}$.*

4.1 Costs

Now, once we defined the set of possible strategies/outcomes of a (linear s-t) congestion game, we only need to define the corresponding costs. The cost of a given outcome will be viewed as the cost of the corresponding flow in the setting where cost of flowing one unit of flow through a particular arc e is a (monotonically-increasing) linear function of the total traffic x through that arc. That is, for any arc e, its (marginal) cost (that we can interpret as congestion) $c_e(x)$ of pushing one unit of flow through it, when the total traffic is x, is equal to

$$c_e(x) := a_e x + b_e,$$

where $a_e, b_e > 0$ are two predefined parameters corresponding to arc e.
So, in this situation, the total (social) cost of a given outcome f is just its total costs:

$$c(f) := \sum_e c_e(f_e) \cdot f_e = \sum_e (a_e {f_e}^2 + b_e f_e) \tag{1}$$

and the socially optimal outcome – to which we want to compare to – is the one that minimizes $c(f)$, i.e.,

$$OPT = \min_{f \in S} c(f).$$

(Note that one can view the cost $c(f)$ of a given flow f, as an average delay that the players suffer when following the routing pattern described by f.)
From (1) it is immediately clear that the cost function is convex. Thus, as the set over which we are optimizing this cost function is convex too (cf. Observation 1), we can conclude that

Observation 2 *Any local minimum of the cost function $c(f)$ is also its global minimum.*

4.2 Nash Equilibria

As we will be interested in comparing the cost of the socially optimal outcome to the cost of Nash equilibria, we will find the following characterization of Nash equilibria of a congestion game very useful.

Lemma 3 *An outcome $\bar{\alpha}$ is a Nash equilibrium if and only if for any s-t path p with $\alpha_p > 0$, and for any s-t path p', we have*

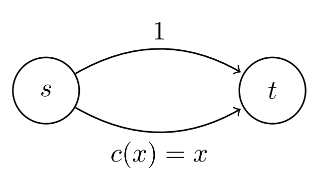
$$c(p) \leq c(p'),$$

where $c(p'')$ is the cost of a given path p'' with respect to cost functions corresponding to $\bar{\alpha}$.

Proof Note that the right-hand-side condition is not true, there is an incentive for a player that routes his/her (minuscule) flow on path p to deviate by moving it to path p' (here we crucially use non-atomic character of our game, i.e., that one-player deviations from a given strategy does not influence the costs of arcs). So, this condition captures exactly the definition of a Nash equilibrium. ■

5 Price of Anarchy of Linear s-t Congestion Games

We proceed now to estimation of the price of anarchy of linear s-t congestion games. To get a feel for it, let us first look at a the following simple example (known as Pigou's example). In this example, we have the following simple graph and cost function.



To calculate the value of OPT here, we need to minimize the total cost $c(f)$ over all the s-t flows of value 1. Note that in this case the total flow $f_{e_1} + f_{e_2}$ on both arcs has to be equal to 1 and thus we can express the cost $c(f)$ of that flow in terms of f_{e_1} alone as

$$c(f) = \sum_{e \in E} c_e(f_e) = f_{e_1}^2 + f_{e_2} = f_{e_1}^2 + (1 - f_{e_1}) \tag{2}$$

Taking a derivative and setting $\frac{dC(f)}{df_{e_1}} = 0$, we get that $f_{e_1}^* = \frac{1}{2}$, i.e., socially optimal solution routes half of the flow on each of the arcs and thus has a total cost of

$$OPT = 1 \cdot \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

Now, the (unique) Nash equilibrium here corresponds to all of the players taking the lower edge, i.e a flow of 1 on bottom arc. To see that, suppose a player chooses the top arc. In this case, there always will be an incentive for him/her to switch to the lower arc, since the cost of the lower arc at this point is strictly less than 1. On the other hand, if the flow on lower arc is 1, then no player has any incentive to switch to the upper arc since its cost is also 1. So, the only Nash equilibrium of this example has cost 1 and thus the price of anarchy PoA of congestion game in this particular graph example is exactly

$$PoA = \max_{s \in NE(G)} \frac{c(S)}{OPT} = \frac{4}{3}.$$

Clearly, the above example shows that the price of anarchy of linear s-t congestion games can be as large as $\frac{4}{3}$. The natural question to ask now is: how much worse can it be if we consider more complicated graphs and arbitrary (but still linear and non-negative) cost functions?
Somewhat surprisingly, the $\frac{4}{3}$ achieved by our simple example turns out to be the worst possible.

Theorem 4 *Price of anarchy of linear s-t congestion games is at most $\frac{4}{3}$. Equivalently, for any s-t congestion game described by a graph G with linear and non-negative cost functions, and any flow f that is its Nash equilibrium, $c(f) \leq \frac{4}{3}OPT(G)$.*

We will prove this theorem today, but before we can do that, we need to establish an interesting relationship between social optimum of a linear s-t congestion game and Nash equilibrium of certain related game.

Lemma 5 *Let G be a graph with cost functions $c_e(x) = a_e x + b_e$, for each of its arcs e. Let G' be the same graph, but with cost functions $c'_e(x) = 2a_e x + b_e$. A flow f is a social optimum in G iff f is a Nash equilibrium in G'.*

Proof Let us first prove the left-to-right implication. To this end, let f be a social optimum for G and let $\bar{\alpha}$ be its (arbitrary) path decomposition. Let us fix any s-t path p with $\alpha_p > 0$ and any s-t path p'. We need to have that

$$\frac{\partial c(\bar{\alpha})}{\partial \alpha_p} \leq \frac{\partial c(\bar{\alpha})}{\partial \alpha_{p'}}, \tag{3}$$

since otherwise, for some sufficiently small ε, diverting ε units of flow from p to p' yields smaller cost, which would contradict the social optimality of f.
Now, the key thing to notice is that

$$\frac{\partial c(\bar{\alpha})}{\partial \alpha_{p''}} = c'(p''),$$

for any path p'' (we defined the cost functions c' to make it satisfy exactly this equality). As a result, (3) implies that we need to have

$$c'(p) \leq c'(p'),$$

which is necessary and sufficient for f to be a Nash equilibrium in G' (cf. Lemma 3).
To see the right-to-left implication, we just need to note that all the reasoning above can be reversed and, in particular, we have that if f is a Nash equilibrium in G' then equation (3) holds for any paths p and p' with $\alpha_p > 0$. This means, however, that f is a local minimum of the function $c(f)$ and – by Observation 2 – it has to be a global one, as well. ∎

We are now ready to prove Theorem 4.

Proof of Theorem 4: Let f be a Nash equilibrium of the game described by the graph G with cost functions $c_e(x) := a_e x + b_e$, for each arc e. Let us define a graph $\widetilde{G}$ with cost functions $\tilde{c}_e(x)$ to be the graph on the same vertex set as G, but in which, for every arc e in G, we have two copies e, $\tilde{e}$ of that arc in $\widetilde{G}$. Now, for each such pair of copies e, $\tilde{e}$, the cost function $\tilde{c}_e(x)$ of the first copy is the same as in G, i.e., $\tilde{c}_e(x) := c_e(x)$, while the cost function $\tilde{c}_{\tilde{e}}(x)$ of the second copy is *constant* and equal to the marginal cost of the arc e in the flow f in G, i.e., $\tilde{c}_{\tilde{e}}(x) := c_e(f_e)$.
We first want to note that f is also a Nash equilibrium in graph $\widetilde{G}$ (with respect to the cost functions $\tilde{c}_e(x)$). This is easy to see as – by definition of our cost functions $\tilde{c}$ – for any s-t path $\tilde{p}$ in $\widetilde{G}$ (that might use some parallel arcs $\tilde{e}$) its cost with respect to f is exactly the same as the corresponding cost of an s-t path p' in G that just substitutes any arcs $\tilde{e}$ of $\tilde{p}$ with their original copy e. So, if there would be an incentive to deviate from f in $\widetilde{G}$ then such incentive had to be already present in the graph G.
Let us consider now, a flow g in $\widetilde{G}$ defined us $g_e := f_e/2$ and $g_{\tilde{e}} := f_e/2$ for each arc e in G. (So, g is just a version of f that splits the flow evenly between parallel arcs.) We claim that g is a social optimum in $\widetilde{G}$ (with respect to the cost functions $\tilde{c}_e(x)$).
To show that we appeal to Lemma 5. More precisely, let $\widetilde{G}'$ be the graph that is related to $\widetilde{G}$ in exactly the same way as G' is related to graph G in the statement of that lemma. Note that, for any arc e in G, we have

$$\tilde{c}'_e(g) = 2a_e f_e/2 + b_e = c_e(f_e)$$
$$\tilde{c}'_{\tilde{e}}(g) = c_e(f_e).$$

Therefore, g has to be a Nash Equilibrium in $\widetilde{G}'$, since f is a Nash Equilibrium in G and any incentive to deviate from g in $\widetilde{G}'$ can be directly translated to an incentive to deviate from f in G. Therefore, by Lemma 5, g is a social optimal in $\widetilde{G}$, as desired.
Now, to finish the proof of the theorem, we just notice that we must have that $OPT(G) \geq OPT(\widetilde{G})$, since adding arcs cannot make the social optimum worse (keep in mind though that this is *not* true in case of Nash equilibrium cost – see Braess paradox from Lecture 5, as well as, Section 7 below). Therefore, we must have that

$$\begin{aligned}
OPT(G) \geq OPT(\tilde{G}) = \tilde{c}(g) &= \sum_{e\in E}\left(\frac{a_e f_e^2}{4} + \frac{b_e f_e}{2} + \frac{c_e(f_e) f_e}{2}\right) \\
&= \sum_{e\in E}\left(\frac{3}{4}a_e f_e^2 + b_e f_e\right) \\
&\geq \frac{3}{4}\sum_{e\in E}\left(a_e f_e^2 + b_e f_e\right) \\
&= \frac{3}{4}c(f),
\end{aligned}$$

which gives us the theorem. ∎

6 Price of Stability of Linear s-t Congestion Games

In our considerations above, we managed to bound the price of anarchy of any linear s-t congestion game to be at most $\frac{3}{4}$. How about its price of stability? How much smaller than the price of anarchy can it be?
Surprisingly, as the following lemma shows, in linear s-t games the cost of each Nash equilibrium is exactly the same and thus the price of stability is equal to price of anarchy.

Lemma 6 *In any linear s-t congestion game, every Nash equilibrium has the same cost. In particular, the price of stability is equal to the price of anarchy.*

Proof Let us consider an arbitrary linear s-t congestion game corresponding to some graph G' and cost functions $c'_e(x) := a'_e x + b'_e$, for each arc e. Let f^1 and f^2 be two Nash equilibria in that game. By Lemma 5, both f_1 and f_2 have to be social optimum in the graph G (that is just identical to G') with cost functions $c_e(x) := a'_e/2 \cdot x + b'_e$, for each e. In particular, it means that $c(f^1) = c(f^2)$. (Note that $c(f^1) = c(f^2)$ does not necessarily mean yet that $c'(f^1) = c'(f^2)$, which is what we want to establish.)
Now, as the set of strategies is convex (cf. Observation 1), a flow $f' := \frac{f^1+f^2}{2}$ has to also be a valid strategy and thus – by convexity of the cost function (cf. (1)) – its cost $c(f')$ cannot be larger than the cost of $c(f^1)$ and $c(f^2)$.
However, looking at the definition (1) of the flow cost, one can immediately see that the only way for $c(f')$ to be equal to $c(f^1) = c(f^2)$ is that $f^1_e = f^2_e$ for every arc e with $a'_e > 0$. But if f^1 and f^2 are the same on all the arcs with positive a'_e, it must be that not only $c(f^1) = c(f^2)$, but also $c'(f^1) = c'(f^2)$, as desired. ■

Note that in the light of the above lemma, for a given linear s-t congestion game described by a graph G, the notion of the cost $NashCost(G)$ of its Nash equilibria is well defined.

7 Braess' Paradox Revisited

Recall the Braess' paradox that we described in Lecture 5. In this "paradox", we have two linear s-t congestion games corresponding to two graphs G and G' – cf. Figure 1 – such that G' is the graph G with one additional zero-cost arc added (the cost functions on the remaining arcs are identical). As we already argued in Lecture 5, the surprising phenomena is that the cost of Nash equilibrium in G' is actually larger than the cost of Nash equilibrium in G, even though we would expect that adding a new arc should only improve the cost of the Nash equilibrium.

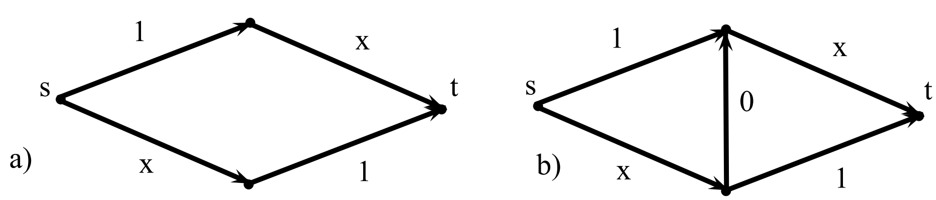


Figure 1: Illustration of Braess' paradox.

Now, the question we want to investigate is: by how much can adding arcs to a graph increase the cost of the corresponding Nash equilibrium? In the example from Figure 1, we see that this increase can be by a factor of at least $\frac{4}{3}$. Can it be larger?
It turns out that with our new understanding of price of anarchy of such games, we can easily show that $\frac{4}{3}$ is again the worst possible.

Lemma 7 *For any two linear s-t congestion games described by graphs G and G' in which G' is obtained from G by adding some arcs to it, we have*

$$NashCost(G') \leq \frac{4}{3} NashCost(G).$$

Proof To prove the lemma, we just need to note that

$$NashCost(G) \geq OPT(G) \geq OPT(G') \geq \frac{3}{4} NashCost(G'),$$

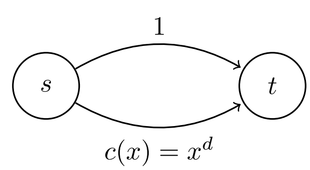
where the last inequality follows from Theorem 4. ■

8 General Congestion Games

So far, we only considered a rather simple variant of the congestion games – the linear s-t congestion games. This variant differs from the general case in two main aspects. First, in general congestion games one allows the cost functions to be non-linear, e.g., to correspond to a class of polynomials of degree d with non-negative coefficients. The second aspect is that, in general, one considers different players looking for routing between different pairs of points.
We now discuss briefly how the price of anarchy and the price of stability behave once we allow these extensions.

8.1 More General Cost Functions

It is not hard to see that if we allow our cost functions to be more general, i.e., not necessarily linear, (but still each of the players is interested in s-t routing) then the price of anarchy can increase significantly. For example, one can consider a Pigou-type example below, in which, again, one arc has a constant cost of 1, and the other arc has a cost $c(x) = x^d$ for some $d \geq 1$. It is not hard to see that as $d \to \infty$ the resulting price of anarchy goes to ∞ too.

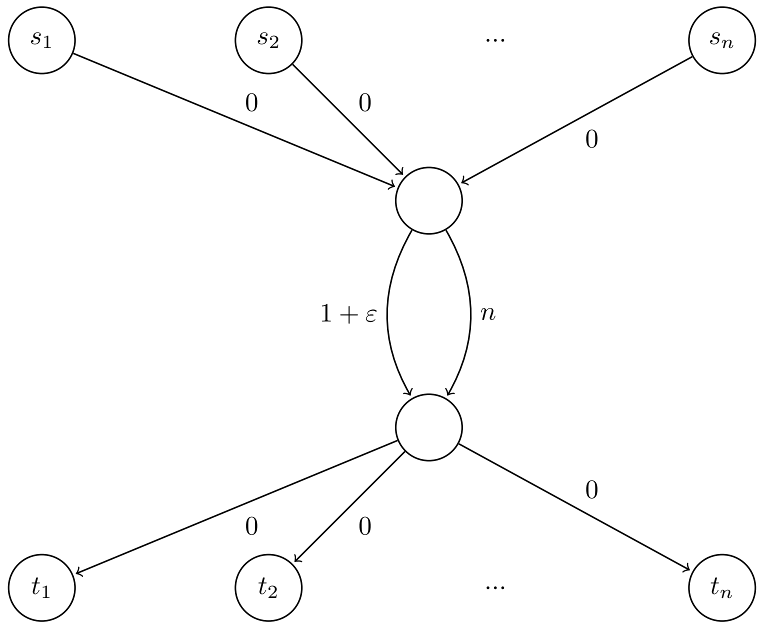


However, it turns out that the approach we applied to linear s-t congestion games can be extended to handle these more general cost functions too (at least, as long as, they are "reasonably" well-behaved). Namely, what one can prove is that it is still true that in such games the price of stability is equal to the price of anarchy and that the worst-case example for the price of anarchy is actually the simple Pigou-type example of the kind we presented above.

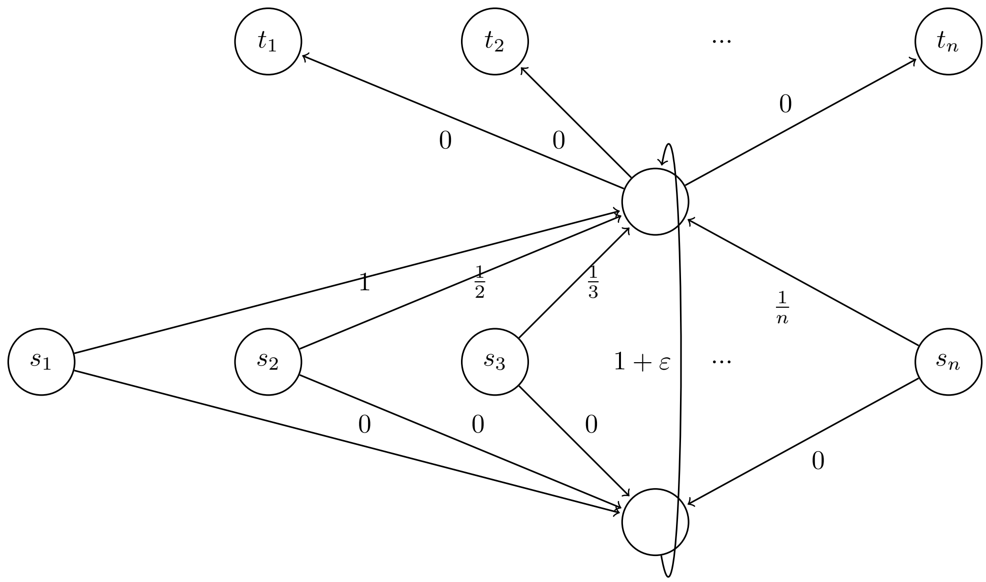
8.2 Allowing Multiple Source-Sinks

Unfortunately, once we allow different players be interested in routing between different vertices of the graph, the approach we developed for linear s-t congestion games does not apply anymore and the structure of price of anarchy and price of stability becomes more complex.
To get a feel for how differently these two quantities can behave, consider a popular variant of (multi source-sink) congestion games, in which we have again a directed graph G and n players, and now each player i wants to choose a routing from his/her source vertex s_i to his/her sink vertex t_i. (We do not

assume here anymore that n is very large, i.e., we consider the atomic case.) The cost function $c_e(l)$ of each arc e (as a function of number l of players that chose a routing that passes through e) is defined as $c_e(l) := \frac{a_e}{l}$, for some $a_e > 0$. That is, the players whose routes pass through given arc share its cost among themselves.
Now, the following example shows that price of anarchy of such games can be $\Omega(n)$ (and one can show that this is the worst possible).



To see that this example has indeed a price of anarchy of $\Omega(n)$, note that one Nash equilibrium here corresponds to all the players sharing the arc of cost n. However, the social optimum (and Nash equilibrium too!) is if all the players use the arc of cost $(1+\varepsilon)$.
The above example shows not only that price of anarchy can be large, but also that it is not always equal to the price of stability anymore. So, how large can the price of stability be in such games? (In example above it was 1.)
As the following graph shows, the price of stability can be $\Omega(\log n)$ (and, again, one can show that this is the worst possible).



To analyze this example, let us note first that the social optimum here is if all the players use the $(1+\varepsilon)$ arc. The cost then is $(1+\varepsilon)$. This outcome is not a Nash equilibrium, however, and the (only) Nash equilibrium here is corresponding to each player i using the arc of the cost $1/i$. The cost of this equilibrium is $\Omega(\log n)$.
To see why this is the only Nash equilibrium, consider any other outcome – it needs to have at least one player using the $(1+\varepsilon)$-cost arc. Let i^* be the player with largest index that is using this arc. Clearly, this player can share the cost of this arc with at most i^*-1 other players. So, his/her marginal cost is at least $\frac{(1+\varepsilon)}{i^*}$. This means, however, that he/she has an incentive to deviate and choose a path that uses the $\frac{1}{i^*}$-cost arc. Thus, indeed, the only Nash equilibrium is the one when no player uses the $(1+\varepsilon)$-cost arc.