CS-621 Theory Gems	October 14, 2012

Problem Set 2

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Problem 1. A three-player game is zero-sum if the utilities of all three players always sum up to zero. Show that one can reduce the problem of finding a Nash equilibrium in a *general* two-player game to the problem of finding a Nash equilibrium in a three-player *zero-sum* one.

Note: In Lecture 6, we saw that one can find an (approximate) Nash equilibrium of any two-player zero-sum games efficiently. As there is strong evidence suggesting that finding Nash equilibrium is computationally hard already for general two-person games, the above reduction shows that finding Nash equilibrium in zero-sum games becomes much more difficult once we move from two to three players.

Problem 2. Consider a two-player game where the sets of possible (pure) actions of both players are the same, i.e., $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}^*$. We say that such game is *symmetric* iff for any $s_1, s_2 \in \mathcal{S}^*$,

$$u_1(s_1, s_2) = u_2(s_2, s_1),$$

i.e., the values of players' utilities swap when the players swap their actions.

(a) Show that if there are only two (pure) actions, i.e., $|\mathcal{S}^*| = 2$, then any symmetric game has to have a *pure* Nash equilibrium.

(b) Consider now the case when there is more than two (pure) actions, i.e., $|\mathcal{S}^*| > 2$. Either prove that each such game has to have a pure Nash equilibrium too, or disprove this claim by exhibiting a counterexample.

Problem 3. Consider a communication network modeled as a directed graph $G = (V, E)$ in which every arc $e \in E$ is owned by different player. Our goal is to choose a communication path between two specified nodes s and t in G and we want to do it in a way that maximizes the social welfare. Each player has a *private* utility function u_e that is 0 if his/her arc is not a part of this s-t path, and equal to $-c_e$ otherwise. (We assume here that all c_e are non-negative.) Design an incentive compatible mechanism that will choose this path in a way that maximizes social welfare. Be explicit regarding what the outcome choice is and what are the corresponding payments.

Problem 4. Let us say that a two-person game is a *conflicting interests* game if for any two outcomes $s, s' \in \overline{\mathcal{S}}$, we have

$$u_1(s) > u_1(s') \text{ iff } u_2(s) < u_2(s').$$

(Note that any zero-sum game is a conflicting interests game, but the reverse is not necessarily true.) Let us fix some conflicting interests game and let us define

$$s_1^* := \arg\max_{s_1 \in \overline{\mathcal{S}}_1} \min_{s_2 \in \overline{\mathcal{S}}_2} u_1(s_1, s_2) \text{ and } s_2^* := \arg\max_{s_2 \in \overline{\mathcal{S}}_2} \min_{s_1 \in \overline{\mathcal{S}}_1} u_2(s_1, s_2).$$

Show that (s_1^*, s_2^*) is a Nash equilibrium.

Note: MinMax theorem proves this for the case of zero-sum games.