CS-621 Theory Gems November 4, 2012

Problem Set 3

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Problem 1. We say that a graph $G = (V, E)$ is *bipartite* if we can partition its vertices into two sets P and Q such that each edge $e \in E$ has one of its endpoints in P and another one in Q. (In other words, we have that $P \cap Q = \emptyset$, $P \cup Q = V$, and $E \subseteq P \times Q$.)

(a) Show that if a graph G is bipartite then its walk matrix W has one of its eigenvalues equal to -1.

(b) Show that if a graph G is connected and its walk matrix W has one of its eigenvalues equal to -1 then G is bipartite.

Note: This, in particular, means that the only connected graphs on which the random walks do not mix are the bipartite ones.

Problem 2. Let $n = 2^k - 1$, for some $k \geq 1$, and let T_n be a full binary tree graph on n vertices. Show that $\lambda_2(T_n) = \Omega(1/n)$.

Hint: It might be helpful to prove and use the following inequality (that is a slight generalization of the path inequality we have proved in the class). For any weights $w_1, \ldots, w_{n-1} > 0$,

$$w \cdot \left(\sum_{i=1}^{n-1} w_i \cdot L^{(i,i+1)} \right) \succeq L^{(1,n)},$$

where $L^{(i,j)}$ is a Laplacian of a graph consisting of only one edge (i, j) and $w := \sum_{i=1}^{n-1} \frac{1}{w_i}$.

Note: In the class, we proved a weaker lowerbound of $\lambda_2(T_n) = \Omega(1/n \log n)$.

Problem 3. Let $n = 2^k$, for some $k \geq 1$, and let H_n be a graph on n vertices whose vertex set is $\{0, 1\}^k$ and two vertices are connected in it iff they differ at exactly one coordinate.

(a) Verify that for any $a \in \{0, 1\}^k$, the vector v^a with coordinates given by $v^a_i := (-1)^{a \cdot i}$ is an eigenvector of H_n.[1] What is the eigenvalue that v^a corresponds to?

(b) What is the conductance Φ_{H_n} of this graph (as a function of n)? (Provide an upper and lower bound that are as close to each other as you can make it.)

Problem 4 (Extra credit). Let $\phi(x_1, \ldots, x_n)$ be a $2-CNF$ formula in n Boolean variables $x_1, \ldots, x_n$, i.e., $\phi(x_1, \ldots, x_n) = \bigwedge_{j=1}^{k} \psi_j(x_1, \ldots, x_n)$ is a conjunction of k clauses $\psi_j(x_1, \ldots, x_n)$, where each of these clauses is an alternative of only two literals. (A literal is some variable x_i or its negation $\overline{x_i}$).

Assume that we know that $\phi(x_1, \ldots, x_n)$ is satisfiable, i.e., that there exists (at least one) assignment σ^* of Boolean values to the variables, that makes all the clauses satisfied simultaneously. Consider the following randomized algorithm for finding a satisfying assignment for $\phi(x_1, \ldots, x_n)$:

1. Start with an arbitrary assignment σ.
2. As long as, σ does not satisfy the formula $\phi(x_1, \ldots, x_n)$:
   Pick a clause that is not satisfied by σ and one of the two literals in it *at random*. Modify σ by flipping the value of the variable corresponding to the chosen literal.

Show that this algorithm terminates with satisfying assignment in expected $O(n^2)$ iterations.

[1] Note that coordinates of v^a are indexed by vertices of H_n and these are binary k-dimensional vectors. So, the inner product $a \cdot i$ in the definition of i-th coordinate of the vector v^a is well-defined.