MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems—Fall 2002

TUTORIAL FOR THE WEEK OF OCTOBER 7TH - OCTOBER 11TH

Office Hours
Monday 3-5pm
By Appointment

Important Due Dates:

- NO PROBLEM SET THIS WEEK!!!
- Quiz 1 will be held on Thursday, October 17, from 7:30 pm to 9:30 pm in Walker Memorial. Memorial.
- Quiz 1 review sessions are tentatively scheduled for Wednesday, October 9, from 7-9pm, and Thursday, October 10, from 7-9pm. Watch your e-mail for schedule changes and room announcements.

Today: Part I (Review)

1. CT and DT Fourier Series
2. Frequency Response
3. Convolution
4. System properties

Today: Part II (Looking forward)

1. Continuous Time Fourier Transform and related concepts (New stuff)

Continuous Time Fourier Series

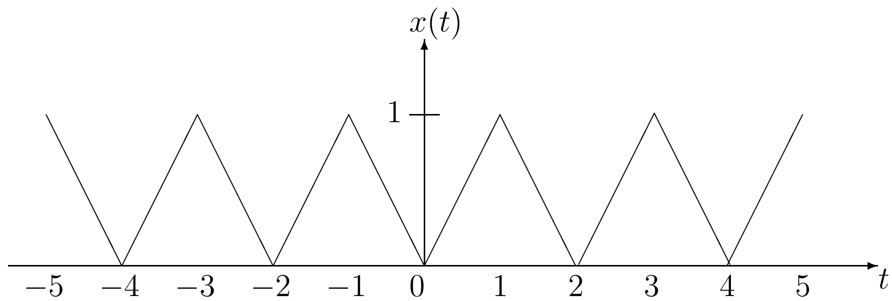
Recall:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_o t}dt \text{ , where } \omega_o = \frac{2\pi}{T}.$$

[**Example**]

1. The periodic waveform, $x(t)$ shown below, has Fourier coefficients a_k.



   (a) Find a_0.

   (b) Find the Fourier series for $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$.

   (c) Use the result of (b) and properties of the continuous-time Fourier series to help determine the Fourier coefficients of $x(t)$.

Discrete Time Fourier Series

Recall:

$$\begin{aligned} x[n] &= \sum_{k=\langle N\rangle} a_k e^{jk\omega_o n} \\ a_k &= \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jn\omega_o k} \text{ , where } \omega_o = \frac{2\pi}{N}. \end{aligned}$$

[Example]

1. Suppose we are given $x[n]$, $y[n]$, and $z[n]$ as follows

$$\begin{aligned} x[n] &= 1+\cos(\frac{7\pi}{3}n) \\ y[n] &= \sin(\frac{\pi}{3}n+\frac{\pi}{4}) \\ z[n] &= x[n]y[n] \\ g[n] &= x[n]*y[n] \end{aligned}$$

   Find the Fourier series coefficients for $z[n]$ and $g[n]$.

Frequency Response

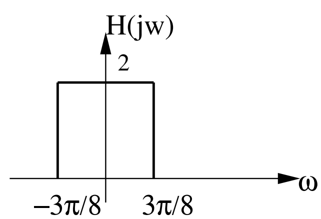
Recall: If we put an input $x(t) = e^{j\omega t}$ into an LTI system we get out:

$$\begin{aligned} y(t) &= \int_{\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{\infty}^{\infty} h(\tau)e^{j\omega t}e^{-j\omega\tau}d\tau \\ &= e^{j\omega t}\int_{\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau \\ &= e^{j\omega t}H(j\omega) \end{aligned}$$

where $H(j\omega) = \int_{\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$. We call $H(j\omega)$ is the *frequency response* of the system.

[**Example**]

1. An LTI system has a frequency response depicted below:



   If the input, $x(t) = \cos(\frac{\pi}{2}t) + \cos(\frac{\pi}{4}t)$, what will $y(t)$, the output be?

2. Plot the magnitude of the following frequency response:

$$H(e^{j\omega}) = 1 - e^{-j\omega}$$

Work Space

Convolution and System Properties

Recall: When we put a signal $x(t)$ (or $x[n]$) into a system characterized by an impulse response of $h(t)$ (or $h[n]$), then the output will be $y(t)$ (or $y[n]$), which is given by

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ y[n] &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \end{aligned}$$

Use Flip and Shift method to determine bounds on t or n (ie, regions where $h[n]$ and $x[n]$ overlap). Then go through the motions of actually calculating the output.

For system properties, there are

- Linearity - Check whether the output of $ax_1(t) + bx_2(t)$ is equal to $ay_1(t) + by_2(t)$, where $y_1(t)$ and $y_2(t)$ are the results of inputting $x_1(t)$ and $x_2(t)$, respectively.
- Time Invariance - Check whether the output of $x_1(t - t_0)$ is equal to $y_1(t - t_0)$, where $y_1(t)$ is the result of inputting $x_1(t)$.
- Stability - Check whether the impulse response, $h(t)$ (or $h[n]$), is absolutely integrable (summable)
$$\int_{-\infty}^{\infty} |h(t)|dt < \infty$$
If it **is** absolutely integrable, then the system is stable.
- Causality - Check whether the impulse response, , $h(t)$ (or $h[n]$), is non-zero for negative values of t (or n).
$$h(t) \neq 0 \qquad \text{for} \;\; t < 0$$
If the above condition is true, then the system is **non**-causal.

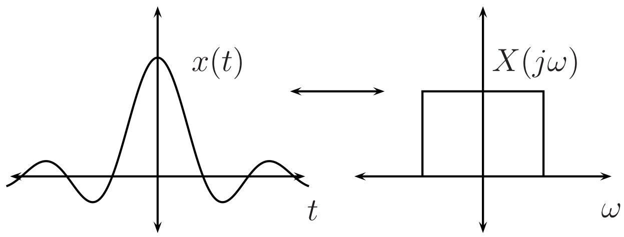
[**Example**]

1. Suppose a system takes an input $x(t)$ and produces the output
$$y(t) = x(-t + 5)$$
Is the system time invariant?

Continous Time Fourier Transform

Today, we'll focus on exploring properties of the CTFT via the sinc-box transform pair.

A function of the form $x(t) = \frac{\sin(W\pi t)}{\pi t}$ is known as a *sinc*. This function has zero crossings whenever $\sin(W\pi t)$ has zero crossings, except at $t = 0$. Therefore, it has zero crossings at



The Fourier Transform of any $x(t)$ of the form $\frac{\sin(At)}{Bt}$ will be a box in frequency. That is, it will be of the form

$$X(j\omega) = \begin{cases} C & \text{for } \ |\omega| \leq D \\ 0 & \text{otherwise} \end{cases}$$

Likewise, a box in time, will have a Fourier Transform which is a sinc in frequency. All you have to really remember is

$$\text{box} \longleftrightarrow \text{sinc}$$

Everything else works out by using CTFT properties

[Example]

Find the Fourier Transform of the function $x(t) = \frac{\sin(At)}{Bt}$.

Solution

First, let's characterize this function.

- Zero crossings - We know that $x(t)$ has zero crossings whenever At is a multiple of π. Therefore, the zero crossings occur at $\frac{k\pi}{A}$ for all non-zero integers k.

- $x(0)$ - We can do this by limits:

$$\begin{aligned} \lim_{t \to 0} x(t) &= \frac{\lim_{t \to 0} \sin(At)}{\lim_{t \to 0} Bt} \\ &= \frac{\lim_{t \to 0} A\cos(At)}{\lim_{t \to 0} B} \\ \Rightarrow x(0) &= \frac{A}{B} \end{aligned}$$

Next, let's figure out what it's Fourier Transform is going to be. We're going to make use of the following important properties:

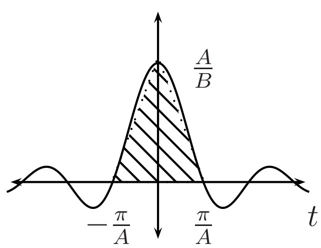
$$\begin{aligned} X(j0) &= \int_{-\infty}^{\infty} x(t)dt \\ \int_{-\infty}^{\infty} X(j\omega)d\omega &= 2\pi x(0) \end{aligned}$$

Since we know that $X(j\omega)$ is of the form

$$X(j\omega) = \begin{cases} C & \text{for } |\omega| \leq D \\ 0 & \text{otherwise} \end{cases}$$

we just need to determine C and D.

- C - We know that $C = X(j0)$. This is just the area under $x(t)$. Note that the area under the sinc is just given by the area of the inscribed triangle



$$\int_{-\infty}^{\infty} x(t)dt = \frac{1}{2} \times \left(\frac{A}{B}\right) \times \left(\frac{2\pi}{A}\right)$$

  Therefore, $C = X(j0) = \frac{\pi}{B}$.

- D - This is the width of the box. Since we know that the area under $X(j\omega)$ is equal to $C \times (2D)$, we can use

$$\begin{aligned} \int_{-\infty}^{\infty} X(j\omega)d\omega &= 2\pi x(0) \\ C \times 2D &= 2\pi \times \frac{A}{B} \\ \frac{\pi}{B} \times D &= \pi \times \frac{A}{B} \\ \Rightarrow D &= A \end{aligned}$$

Therefore,

$$X(j\omega) = \begin{cases} \frac{\pi}{B} & \text{for } |\omega| \leq A \\ 0 & \text{otherwise} \end{cases}$$

Eigenfunctions of LTI Systems

Complex exponentials are eigenfunctions of LTI systems for both DT and CT.

$$x(t) = \mathrm{e}^{j\omega_0 t} \longrightarrow \boxed{\begin{array}{c} h(t) \\ H(j\omega) \end{array}} \longrightarrow y(t) = H(j\omega_0)\mathrm{e}^{j\omega_0 t}$$

$$x[n] = \alpha^n \longrightarrow \boxed{\begin{array}{c} h[n] \\ H(\mathrm{e}^{j\Omega}) \end{array}} \longrightarrow y[n] = H(\alpha)\alpha^n$$

This is especially nice if the inputs happen to be periodic signals, because then we can represent them as a linear combination of exponentials. The output can then be represented in terms of the sum of the responses to each of these exponentials.

For example, if $x(t)$ is presented to a system with a frequency response $H(j\omega)$, and $x(t)$ is periodic, then

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{jk\omega_0 t}$$

and the output, $y(t)$ will be given by

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)\mathrm{e}^{jk\omega_0 t}$$

That is, $y(t)$ will have Fourier coefficients $a_k H(jk\omega_0)$.

This tells us *meaning* of the frequency response, $H(j\omega)$:

> The frequency response, $H(j\omega)$, essentially "magnifies" the frequency component of $x(t)$ at ω_0 by the amount $H(j\omega_0)$.

Examples

1. Suppose we have a CT system whose frequency response is as follows:

$$H(j\omega) = \begin{cases} 1 & \text{if } \ |\omega| = \omega_0 \\ \frac{1}{2} & \text{if } \ |\omega| = 2\omega_0 \\ 0 & \text{otherwise} \end{cases}$$

Find the output $y(t)$ if the input signal is

$$x(t) = \cos(\omega_0 t) + \cos(2\omega_0 t) + \cos(3\omega_0 t)$$

Work Page

Convolution Review

DT convolution

1. Evaluate the following DT convolution sum:

$$y[n] = (2^n u[n]) * (3^{-n} u[n])$$

CT convolution

1. The impulse response of a system is $h(t) = \mathrm{e}^{-at} u(t)$, $a > 0$. Calculate the step response $s(t)$ of the system.

2. Evaluate the following CT convolution integral:

$$y(t) = [u(10 + t) - u(3 + t)] * \mathrm{e}^{-t}.$$

Work Space

Extra Problems

1. $x[n]$ is a periodic discrete signal with the following properties:

   (a) It is real.

   (b) It is even.

   (c) Its fundamental period is $N = 5$.

   (d) $\sum_{n=-2}^{n=2} x[n] = 0$.

   (e) Its Fourier coefficient $a_6 = 1$.

   (f) $\frac{1}{5}\sum_{n=0}^{n=4} |x[n]|^2 = 2$.

   Find an expression that describe $x[n]$ completely.

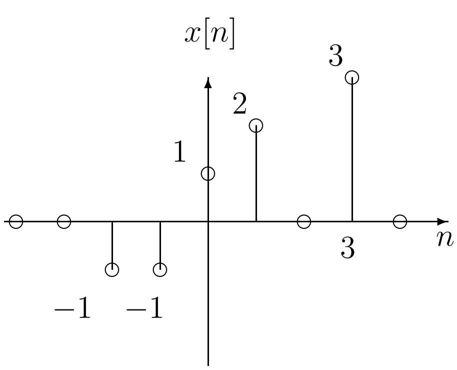
2. Suppose $x[n]$ is a discrete time periodic signal with period 4, such that

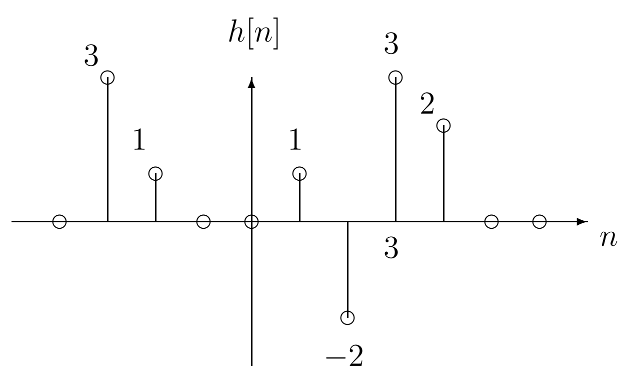
$$x[n] = \begin{cases} 2, & n = 0 \\ 1, & n = 1 \\ 0, & n = 2 \\ 1, & n = 3 \end{cases}.$$

   (a) Let $x_a[n] = x[n]$, $x_b[n] = (x[n])^2$, and $x_c[n] = x[-n]$. Find the discrete time Fourier series of each of these signals.

   (b) In addition, if each of these signals is the input to an LTI system, with frequency response $H(e^{j\Omega}) = 3e^{-2j\Omega}$, find the Fourier series of the output signal in each case.

3. Compute the convolution $y[n] = x[n] * h[n]$:





Work Space