Differential Geometry: Review

Ramesh Sridharan and Matthew Johnson

Quick Reference

$f : U \to \mathbb{R}^{n+1}$, where U is an open set in $\mathbb{R}^n$, parametrizes an n-dimensional submanifold in $n+1$ dimensions.

The **first fundamental form** is

$$\begin{aligned} I(X,Y) &= \langle X, Y \rangle_{\mathbb{R}^{n+1}} \text{ for } X, Y \in T_u f \\ (V,W) &\mapsto \langle Df|_u(V), Df|_u(W) \rangle_{\mathbb{R}^{n+1}} \text{ for } V, W \in T_u U \\ &= \langle V, (g_{ij})W \rangle_U \\ g_{ij} &= \left\langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle \end{aligned}$$

The **Gauss map** $\nu : U \to S^n$ gives the unit normal perpendicular to $T_u f$.
The **shape operator** maps $T_u f$ to itself. It is given by

$$L_u \triangleq (D\nu|_u) \circ (Df|_u)^{-1}$$

The **second fundamental form** is

$$\begin{aligned} II(X,Y) &= \langle L_u X, Y \rangle_{\mathbb{R}^{n+1}} \text{ for } X, Y \in T_u f \\ (V,W) &\mapsto \langle V, (h_{ij})W \rangle_U \text{ for } V, W \in T_u U \\ h_{ij} &= \left\langle \nu, \frac{\partial^2 f}{\partial u_i \partial u_j} \right\rangle \\ &= -\left\langle \frac{\partial \nu}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle \end{aligned}$$

The **principal curvatures** κ_i are the eigenvalues of L_u. They are also the extrema of $II(X,X)$ subject to $I(X,X) = 1$ for $X \in T_u f$. The **Gaussian curvature** K is their product, and the **mean curvature** H is their arithmetic mean.

$$\begin{aligned} \{\kappa_i\} &= \operatorname{eig}(L_u) \\ K &= \det(L_u) \\ H &= \frac{1}{n} \operatorname{tr}(L_u) \end{aligned}$$

By doing algebraic manipulations, we can also find

$$K = \det(L_u) = \frac{\det(h_{ij})}{\det(g_{ij})}$$

Intuition to Remember

First Fundamental Form

Our manifold is parametrized by a function $f : U \to \mathbb{R}^{n+1}$, where U is an open set in $\mathbb{R}^n$ (it is often referred to as the *parameter space*). The **first fundamental form** is defined as $I(X, Y) = \langle X, Y \rangle_{\mathbb{R}^{n+1}}$ for $X, Y \in T_u f$; that is, at a particular point on the manifold, it restricts the standard inner product to that point's tangent hyperplane. We can also consider the corresponding inner product in the parameter space U. For $(v, w) \in T_u U \times T_u U$, we say that $(V, W) \mapsto \langle Df|_u(V), Df|_u(W) \rangle_{\mathbb{R}^{n+1}} = \langle V, (g_{ij}) W \rangle_U$, where (g_{ij}) is the first fundamental form matrix: $(g_{ij}) = \left(\left\langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle \right)$. Note that G is $(Df|_u)^T (Df|_u)$; in some sense it is the square of the Jacobian. For 2-dimensional submanifolds in 3 dimensions, surface integrals are given by $\iint_Q \alpha(\cdot) dA = \iint_Q (\alpha \circ f)(u, v) \sqrt{\det(g_{ij})}\, du\, dv$.

Gauss Map and Shape Operator

The Gauss map is defined as $\nu : U \to S^n$. It maps points u in our parameter space to the unit normal vector to the manifold at $f(u)$.

The shape operator is defined as $L_u \triangleq -(D\nu|_u) \circ (Df|_u)^{-1}$, where in order to take the inverse of $Df|_u$, we restrict ourselves to the image of $Df|_u$. It is a map from $T_u f$ to $T_u f$. In particular, given a vector in the tangent space, it maps that vector to the corresponding differential change in the normal vector while moving in that direction.

We can see this by considering $L_u X$ for some $X \in T_u f$. Applying $(Df|_u)^{-1}$ maps X to its corresponding preimage in $T_u U$; call this vector V. We then act on V with $(D\nu|_u)$, which maps V to the tangent space of ν. Since the tangent space of ν at u is parallel to the tangent space of f at u, these spaces can be thought of as the same.

Second Fundamental Form and Curvature

The **second fundamental form** is defined as $II(X, Y) = II(L_u X, Y)$ for $X, Y \in T_u f$. Again, we consider the corresponding vectors in our parameter space, and as with G for the first fundamental form above, we define H for the second: $(V, W) \mapsto \langle L_u Df|_u(V), Df|_u(W) \rangle_{\mathbb{R}^{n+1}} = \langle V, (h_{ij} W \rangle_U$. Therefore, $(h_{ij}) = \left(\left\langle \nu, \frac{\partial^2 f}{\partial u_i \partial u_j} \right\rangle \right) = \left(-\left\langle \frac{\partial \nu}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle \right)$. Note that $II(X, X)$ is the inner product of a tangent vector X with the corresponding change it induces in ν by moving in the direction of X.

The **principal curvatures** $\{\kappa_i\}$ at a point on a submanifold are the local extrema of $II(X, X)$ subject to $I(X, X) = 1$. They are also the eigenvalues of the shape operator L_u. The **Gaussian curvature** is their product, or alternately the determinant of L_u. The **average or mean curvature** is the trace of L_u scaled by $1/n$ (n is the dimension of the submanifold), or equivalently the arithmetic mean of the principal curvatures.