Solutions to the Final Examination

Problem 1 (6 points). **Simple Graphs**

(a) (2 points) Prove that the average degree of a tree is less than 2. *Hint:* Handshaking.

Solution. *Proof.* A tree T with v vertices has $v-1$ edges. By the Handshaking Lemma,

$$\sum_{x \in T} \deg(x) = 2(v-1) < 2v.$$

Dividing both sides by v proves the claim. □

■

(b) (4 points) The following parts refer to the graph, H, in Figure 1 below.

1. How many isomorphisms are there from H to itself? _______

Solution. 48.

H is isomorphic the 3-cube and so there are 8 possible vertices that vertex 1 could map to. For each different choice for the image of 1, there are 3! possible choices for the images of the vertices adjacent to 1. The images of the remaining vertices are uniquely defined. ■

2. What is the largest k such that H remains connected as long as fewer than k edges are deleted? _______

Solution. 3.

As shown in Problem Set 5, the 3-cube is 3-routed and is thus 3-edge-connected. Deleting the 3 edges incident to a single vertex will disconnect the graph and so it is not 4-edge-connected. ■

3. What is the chromatic number of H? _______

Solution. 2.

It's easy to see that the graph has no odd length cycles, and so is bipartite. ■

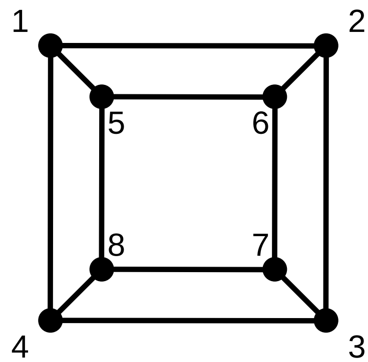


Figure 1: *The graph, H.*

Problem 2 (6 points)**. Graph Coloring**

Recall that a ***coloring*** of a graph is an assignment of a color to each vertex such that no two adjacent vertices have the same color. A k***-coloring*** is a coloring that uses at most k colors.

False Claim. *Let G be a graph whose vertex degrees are all $\leq k$. If G has a vertex of degree strictly less than k, then G is k-colorable.*

(a) (1 point) Give a counterexample to the False Claim when $k = 2$.

Solution. One node by itself, and a separate triangle (K_3). The graph has max degree 2, and a node of degree zero, but is not 2-colorable. ■

(b) (3 points) Underline the exact sentence or part of a sentence where the following proof of the False Claim first goes wrong:

False proof. Proof by induction on the number n of vertices:

Induction hypothesis:

$P(n)$::= "Let G be an n-vertex graph whose vertex degrees are all $\leq k$. If G also has a vertex of degree strictly less than k, then G is k-colorable."

Base case: ($n = 1$) G has one vertex, the degree of which is 0. Since G is 1-colorable, $P(1)$ holds.

Inductive step:

We may assume $P(n)$. To prove $P(n+1)$, let G_{n+1} be a graph with $n+1$ vertices whose vertex degrees are all k or less. Also, suppose G_{n+1} has a vertex, v, of degree strictly less than k. Now we only need to prove that G_{n+1} is k-colorable.

To do this, first remove the vertex v to produce a graph, G_n, with n vertices. Let u be a vertex that is adjacent to v in G_{n+1}. Removing v reduces the degree of u by 1. So in G_n, vertex u has degree strictly less than k. Since no edges were added, the vertex degrees of G_n remain $\leq k$. So G_n satisfies the conditions of the induction hypothesis, $P(n)$, and so we conclude that G_n is k-colorable.

Now a k-coloring of G_n gives a coloring of all the vertices of G_{n+1}, except for v. Since v has degree less than k, there will be fewer than k colors assigned to the nodes adjacent to v. So among the k possible colors, there will be a color not used to color these adjacent nodes, and this color can be assigned to v to form a k-coloring of G_{n+1}. □

Solution. The flaw is that if v has degree 0, then no such u exists. In such a case, removing v will not reduce the degree of any vertex, and so there may not be any vertex of degree less than k in G_n, as in the counterexample of part (a).

So the mistaken sentence is "Let u be a vertex that is adjacent to v in G_{n+1}."

Alternatively, you could say that it's OK to reason about a nonexistent u, and the only mistake is the claim that u exists. This claim is hidden in the phrase "So G_n satisfies the conditions of the induction hypothesis, $P(n)$". ■

(c) (2 points) With a slightly strengthened condition, the preceding proof of the False Claim could be revised into a sound proof of the following Claim:

Claim. *Let G be a graph whose vertex degrees are all $\leq k$. If* $\langle$statement inserted from below$\rangle$ *has a vertex of degree strictly less than k, then G is k-colorable.*

Circle each of the statements below that could be inserted to make the Claim true.

- G is connected and
- G has no vertex of degree zero and
- G does not contain a complete graph on k vertices and
- every connected component of G
- some connected component of G

Solution. Either the first statement "G is connected and" or the fourth statement "every connected component of G" will work. ■

Problem 3 (4 points)**. Probabilistic Land of Paradox**

Recall the land of Paradox where everyone is either a liar who *always* claims a statement is true iff it is actually false, or else is a truth-teller who *always* does the opposite. Let L be the event that a randomly chosen Paradoxian, call him "Bill," is a liar.

The only way out of Paradox at any time is through the North gate or the South gate, and exactly one of these gates is kept open. Which gate is open is determined by independently flipping a biased coin. Let N be the event that the way out is through the North gate. Finally, let R be Bill's response to the statement "L iff $\overline{N}$."

(a) (2 points) Draw a tree diagram that describes a sample space for events L and N. Conclude that $R = N$.

Solution. TBA ■

(b) (2 points) Suppose $\Pr\{N\} = 1/4$, $\Pr\{L\} = 1/3$. Given that Bill's response, R, is "true," what is the probability that Bill is a liar?

Solution. Since $R = N$, and N and L are defined to be independent, $\Pr\{L \mid R\} = \Pr\{L \mid N\} = \Pr\{L\} = 1/3$. ■

Problem 4 (7 points)**. Counting & Probability**

For the following problem parts, you do not need to simplify your answers.

(a) (2 points) Each week Bob purchases exactly 10 packages of meat and 20 packages of vegetables for his fraternity. There are three kinds of meat and five kinds of vegetables he can choose from. How many different collections of meat and vegetables can Bob form?

Solution. The total number of ways of selecting 10 meats is $\binom{12}{2}$ by the donut-selection rule. The total number of ways of selecting 20 vegetables is $\binom{24}{4}$. The total number of collections is equal to product of these terms, which is:

$$\boxed{\binom{12}{2} \cdot \binom{24}{4}}$$

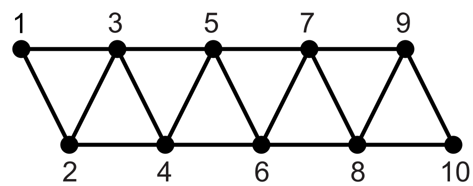
Partial credit was given to people who noticed this was a donut-selection problem but had small errors (like off-by-1 errors). ■

(b) (2 points) You are playing TextTwist®, and you've gotten the letters ADEEHR. The two winning 6-letter words are ADHERE and HEADER. You press "Twist," which randomly scrambles the letters. What is the probability that your twist returns one of the winning words?

Solution. There are $\binom{6}{1,1,2,1,1}$ distinct words from these letters, and each is equally likely, so the probability of returning either of the two target words is the sum of the probabilities of each, which is:

$$\boxed{\frac{2}{6!} + \frac{2}{6!} = \frac{4}{6!}}$$

■



(c) (3 points) An A,B,C,D-labeling assigns letters A,B,C, or D to each vertex of a graph such that adjacent vertices have different labels. How many A,B,C,D-labelings are there of the above graph?

Solution. There are 4 ways to label vertex 1, 3 ways to label vertex 2, and 2 ways to label each of the remaining vertices. By the general product rule, the total number of colorings is:

$$\boxed{4 \cdot 3 \cdot 2^8}$$

Partial credit was given to people who observed that the first triangle (vertices 1, 2, and 3) can be labeled in $4 \cdot 3 \cdot 2 = 24$ ways. ■

Problem 5 (6 points)**. DAG's**

Sauron wants to conquer Middle Earth. This project involves n tasks. Some of these tasks must be completed before others are begun. For example, the task of *locating* the One Ring must precede the task of *seizing* the ring. Each task can be completed by a horrible creature called a *Ringwraith* in exactly one week. A Ringwraith can complete multiple tasks, but can only work on one at a time.

(a) (1 point) Sauron would like to model this scheduling problem as a directed acyclic graph. What should vertices represent? When should there be a directed edge between two vertices?

Solution. Each vertex corresponds to a task. There is a directed edge from vertex a to vertex b if task a is a *direct* (or immediate) prerequisite of task b. Full credit was given for directed edges from a to b when a is an indirect (or implicit) prerequisite of b. ■

(b) (2 points) Sauron is trying to describe various features of his scheduling problem using standard terminology. Next to each feature below, write the number of the corresponding term.

Standard Terminology

1. Positive path relation
2. Topological sort
3. Chain
4. Antichain
5. Size of the largest antichain
6. Length of the longest chain
7. Size of the smallest antichain
8. Length of the shortest chain

A set of tasks that can be worked on simultaneously. _______

A possible order in which all the tasks could be completed, if only one Ringwraith were available. _______

The minimum number of weeks required to complete all tasks, if an unlimited number of Ringwraiths were available. _______

Solution. 4: Antichain; 2: Topological Sort; 6: Length of longest chain ■

(c) (3 points) Sauron's chief intelligence officer analyzes the dependencies between tasks and reports that Sauron could conquer Middle Earth in t weeks, if he had enough Ringwraiths. Based on this, Sauron realizes he might have to recruit as many as $n - (t - 1)$ Ringwraiths to complete all n tasks in t weeks. Describe a possible dependency graph on the n tasks that would, in fact, require Sauron to use this many Ringwraiths.

Solution. There are many answers, so here is one example of a graph. There are $t - 1$ consecutive tasks, each depending on the previous one, forming a chain. The remaining $n - (t - 1)$ tasks each depend on the last of the $t - 1$ tasks. So Sauron can only complete one task per week for the first $t - 1$ weeks, leaving him with only 1 week to complete the remaining tasks, for which he will need as many Ringwraiths as the number of remaining tasks.

Full credit was given to pictures of the described graph. To receive full credit, the antichain of $n - (t - 1)$ had to be restricted to a single week, which is most easily done by requiring the antichain to be done before or after every task. Pictures without edges from the antichain to other tasks received at best partial credit. ■

Problem 6 (8 points)**. Number Theory**

(a) (3 points) Prove that $5^{168n} \equiv 110^{168n} \pmod{245}$ for all positive integers, n.

Solution. Factoring $245 = 5 \cdot 7^2$, we get $\phi(245) = (5-1)(7^2-7) = 168$. Since 5 is not relatively prime to 245, we cannot use Euler's Theorem to simplify 5^{168n} to 1^n. (In fact, $5^{168} \equiv 50 \pmod{245}$.) But since $110 = 2 \cdot 5 \cdot 11$, we can now use Euler's theorem to simplify the right hand side of the congruence:

$$\begin{aligned}
110^{168n} &= (2 \cdot 5 \cdot 11)^{168n} \\
&= 5^{168n} \left((2 \cdot 11)^{168}\right)^n \\
&\equiv 5^{168n} \cdot 1^n \pmod{245} && \text{(by Euler since 2,11 rel. prime to 245)} \\
&= 5^{168n}.
\end{aligned}$$

■

(b) (2 points) Use Fermat's Little Theorem to calculate $\text{rem}(8^{29}, 5)$.

Solution. $\text{rem}(8^{29}, 5) = 3$:

$$\begin{aligned}
8^{29} &= \left(8^4\right)^7 \cdot 8 \\
&\equiv 1^7 \cdot 8 \pmod{5} && \text{(by Fermat's)} \\
&\equiv 3 \pmod{5}.
\end{aligned}$$

■

(c) (3 points) Part (b) implies that $\text{rem}(8^{29}, 5) \neq 1$. It follows that there must be a mistake in the following proof:

$$\begin{aligned}
1 &\not\equiv \text{rem}(8^{29}, 5) \pmod{5} && \text{(by part (b), since } 0 \leq \text{rem}(a,5) < 5) \\
&\equiv 8^{29} \pmod{5} && (a \equiv \text{rem}(a,n) \pmod{n}) \\
&\equiv 3^{29} \pmod{5} && \text{(since } 8 \equiv 3 \pmod{5}) \\
&\equiv 3^{4} \pmod{5} && \text{(since } 29 \equiv 4 \pmod{5}) \\
&\equiv 1 \pmod{5} && \text{(by Fermat's)}
\end{aligned}$$

Explain the mistake in the reasoning of this proof.

Solution. The mistake occurs on the fourth line. Exponents can be replaced by their remainders on division by $\phi(5) = 4$, not on division by 5. So the "explanation" that $9 \equiv 4 \pmod{5}$ on the fourth line is true, but does not justify that mistaken step. ■

Problem 7 (12 points)**. Combinatorics and Counting**

(a) (10 points) Here are the solutions to the next 10 counting questions, in no particular order:

$$n^m \qquad m^n \qquad \frac{n!}{(n-m)!} \qquad \binom{n-1+m}{m} \qquad \binom{n-1+m}{n} \qquad 2^{mn}$$

1.	How many ways are there to put m indistinguishable balls into n distinguishable urns?	$\binom{n-1+m}{m}$
2.	How many ways are there to put m distinguishable balls into n distinguishable urns?	n^m
3.	How many solutions over the nonnegative integers are there to the equation $x_1 + x_2 + \ldots + x_m = n$?	$\binom{n-1+m}{n}$
4.	How many $\sqrt{n} \times \sqrt{n}$ matrices are there with entries drawn from $\{1, 2, \ldots, m\}$, if entries can be reused?	m^n
5.	How many different subsets of the set $A \times B$ are there, if $\|A\| = m$ and $\|B\| = n$?	2^{mn}
6.	How many total functions are there from set A to set B, if $\|A\| = n$ and $\|B\| = m$?	m^n
7.	How many m-letter words can be formed from an n-letter alphabet, if no letter is used more than once? Assume $n \geq m$.	$\frac{n!}{(n-m)!}$
8.	How many m-letter words can be formed from an n-letter alphabet, if letters can be reused?	n^m
9.	How many relations are there from set A to set B, where $\|A\| = m$ and $\|B\| = n$?	2^{mn}
10.	How many total injective functions are there from set A to set B, where $\|A\| = m$ and $\|B\| = n \geq m$?	$\frac{n!}{(n-m)!}$

(b) (2 points) Here is a combinatorial proof of an equation giving a closed form for a sum from 0 to n.

There are n fire hydrants, each of which is to be painted red, green, blue, black, or white. One way to paint them is choose one of the five colors for each hydrant successively.

An alternative way to assign colors to the hydrants, is to

- choose a number, i, between 0 and n,

- choose a set, S, of i hydrants,
- successively paint the hydrants in S red, green, or blue,
- successively paint the hydrants not in S black or white.

What is the equation?

Solution.

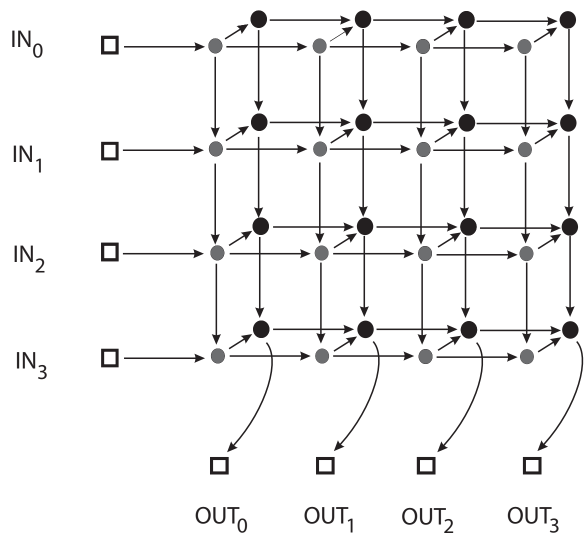
$$\boxed{5^n = \sum_{i=0}^{n} \binom{n}{i} 3^i 2^{n-i}}$$

There are 5 choices of colors for each hydrant, so there are 5^n ways to assign one of these colors to each hydrant.

On the other hand, for each i, there is $\binom{n}{i}$ ways to choose a set, S of size i; then, there are 3^i ways to successively color the hydrants in S. Similarly, there are 2^{n-i} ways to successively color the hydrants not in S. Overall, there are $\binom{n}{i}3^i 2^{n-i}$ ways to paint the hydrants with exactly i of them either red, green, or blue. But i can be anything from 0 to n. This shows that the total number of ways to paint the hydrants is the above sum. p ■

Problem 8 (6 points)**. Communication Networks**

The notes described the n-input *grid* network and proved it had congestion 2 (see the Appendix). In this problem we consider a network called an n-input *2-layer-grid* consisting of two n-input grids connected as pictured below for $n = 4$.



In general, an n-input 2-layer-grid has two layers of switches, with each layer connected like an n-input grid. There is also an edge from each switch in the first layer to the corresponding switch in the second layer. The inputs of the 2-layer-grid enter the left side of the first layer, and the n outputs leave the bottom of the second layer.

(a) (2 points) What is the latency of an n-input 2-layer-grid? ______

Solution. $2n + 1$ ■

(b) (4 points) For any given input-output permutation, there is a way to route packets that achieves congestion 1. Describe how to route the packets in this way.

Solution. To route a packet from input i to output j, use the path from input i to the right along row i of the input layer until column j. Then take the edge to the corresponding switch in column j of the output layer, and continue the path downward along column j in the output layer to output j. Now each packet moves to the right in its own row of the input layer and down in its own column in the output layer, and so packets never cross. That is, congestion is 1. ■

Problem 9 (7 points)**. Structural Induction**

The rational functions of a single variable, x, are defined recursively as follows:

Base cases:

- The identity function, $\text{id}(x) ::= x$, and
- any constant function

are rational functions of x.

Constructor cases:

If f, g are rational functions of x, then so are

$$f + g, \quad f \cdot g, \quad \text{and} \quad \frac{f}{g}.$$

In this problem you will use this definition of rational functions to prove the following Lemma by structural induction:

Lemma. *The derivative, h', of a rational function, $h(x)$, is also a rational function of x.*

The induction hypothesis will be $P(h) ::= [h'$ is a rational function$]$.

(a) (2 points) Prove the base cases of the structural induction.

Solution. *Proof.* We must show $P(\text{id}(x))$ and $P(\text{constant-function})$. But id' is the constant function 1, and the derivative of a constant function is the constant function 0, and these are rational functions of x by definition.

This proves that the induction hypothesis holds in the Base cases. □

■

(b) (5 points) Prove the constructor cases of the structural induction.

Solution. Assuming $P(f)$ and $P(g)$, we must prove $P(h)$ where

Case $h = f + g$: TBA

Case $h = f \cdot g$: TBA

Case $h = \dfrac{f}{g}$: TBA ■

Problem 10 (6 points)**. Generating Functions for kth Powers**

Let g_k be the generating function for the kth powers of the nonnegative integers, that is,

$$g_k(x) ::= 0^k + 1^k x + 2^k x^2 + \cdots + n^k x^n + \cdots,$$

(where $0^0 ::= 1$ by convention).

In this problem you will prove by induction that $g_k(x)$ is a rational function of x. You may assume the results of problem 9.

(a) (1 point) Give a simple formula defining a rational function of x equal to $g_0(x)$.

Solution.

$$g_0(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

■

(b) (2 points) Write a simple expression for g_{k+1} in terms of the derivative of g_k.

Solution.

$$g_{k+1} = x g_k'.$$

■

(c) (3 points) Define

$$P(k) ::= [g_k(x) \text{ is a rational function of } x].$$

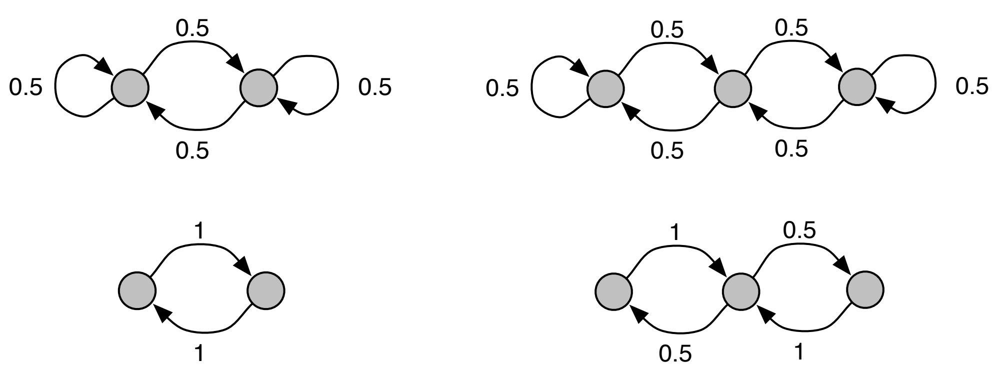
Prove that $\forall k \in \mathbb{N}.P(k)$ by induction.

You should carefully state the induction hypothesis, the induction variable, the base case and the induction step. You may assume all the results of Problem 9 and the previous parts of this problem.

Solution. TBA ■

Problem 11 (4 points)**. Stationary Distributions**

For which of the following graphs is the uniform distribution over nodes a stationary distribution? The edges are labeled with transition probabilities. Circle all that apply.



Solution. All except the last one (bottom right).

One way of approaching this problem is by performing a single update step according to the rule

$$\widehat{d}(v) = \sum_{u \text{ s.t. } u \to v} d(u)p(u,v),$$

where d is the stationary distribution ($1/2$ for all vertices on the left graphs, $1/3$ for all vertices on the right), $\widehat{d}$ is the distribution after one step, and $p(u, v)$ is the edge probability. If $\widehat{d} = d$, then by definition, the uniform distribution is stationary.

Alternatively, you could observe that the uniform distribution is stationary if and only if $\widehat{d}(v) = d(v)$, and hence dividing both sides by probability of being at each vertex, we get

$$1 = \sum_{u \text{ s.t. } u \to v} p(u, v).$$

In other words, the uniform distribution is stationary if and only if the incoming-edge probabilities sum to 1. ■

Problem 12 (11 points)**. Graphs, Logic & Probability**

Let G be an undirected simple graph with $n > 3$ vertices. Let $E(x, y)$ mean that G has an edge between vertices x and y, and let $P(x, y)$ mean that there is a length 2 path in G between x and y.

(a) (1 point) Explain why $E(x, y)$ implies $P(x, x)$.

Solution. Going back and forth on edge x—y is a length two path from x to x. ■

(b) (1 point) Circle the mathematical formula that best expresses the definition of $P(x, y)$.

- $P(x, y) ::= \exists z.\ E(x, z) \wedge E(y, z)$
- $P(x, y) ::= x \neq y \wedge \exists z.\ E(x, z) \wedge E(y, z)$
- $P(x, y) ::= \forall z.\ E(x, z) \vee E(y, z)$
- $P(x, y) ::= \forall z.\ x \neq y \longrightarrow [E(x, z) \vee E(y, z)]$

Solution. $P(x, y) ::= \exists z.\ E(x, z) \wedge E(z, y)$ ■

For the following parts (c)–(e), let V be a fixed set of $n > 3$ vertices, and let G be a graph with these vertices constructed randomly as follows: for all distinct vertices $x, y \in V$, independently include edge x—y as an edge of G with probability p. In particular, $\Pr\{E(x, y)\} = p$ for all $x \neq y$.

(c) (4 points) For distinct vertices w, x, y and z in V, circle the event pairs that are independent.

1. $E(w, x)$ versus $E(x, y)$ **TRUE**

2. $(E(w, x) \wedge E(w, y))$ versus $(E(z, x) \wedge E(z, y))$ **TRUE**

3. $E(x, y)$ versus $P(x, y)$ **TRUE**

4. $P(w, x)$ versus $P(x, y)$ **FALSE**

5. $P(w, x)$ versus $P(y, z)$ **FALSE**

(d) (3 points) Write a simple formula in terms of n and p for $\Pr\{\text{not } P(x, y)\}$, for distinct vertices x and y in V.

Hint: Use part (c), item 2.

Solution. Let $Z ::= V - \{x, y\}$ be the set of all the vertices other than x and y.

$$\begin{aligned}
\Pr\{\text{not } P(x,y)\} &= \Pr\left\{\bigwedge_{z \in Z} \overline{E(x,z) \wedge E(y,z)}\right\} \\
&= \prod_{z \in Z} \Pr\left\{\overline{E(x,z) \wedge E(y,z)}\right\} && \text{(indep. from item (c) 2.)} \\
&= \prod_{z \in Z} (1 - \Pr\{E(x,z)\} \cdot \Pr\{E(y,z)\}) && \text{(indep. from item (c) 1.)} \\
&= \prod_{z \in Z} (1 - p^2) \\
&= (1 - p^2)^{n-2}
\end{aligned}$$

■

(e) (2 points) What is the probability that two distinct vertices x and y lie on a three-cycle in G? Answer with a simple expression in terms of p and r, where $r ::= \Pr\{\text{not } P(x, y)\}$ is the correct answer to part (d).

Hint: Express x and y being on a three-cycle as a simple formula involving $E(x, y)$ and $P(x, y)$.

Solution. x and y lie on a three-cycle iff $E(x, y) \wedge P(x, y)$.

Since $E(x, y)$ and $P(x, y)$ are independent,

$$\begin{aligned}
\Pr\{E(x,y) \wedge P(x,y)\} &= \Pr\{E(x,y)\} \cdot \Pr\{P(x,y)\} \\
&= p(1 - r).
\end{aligned}$$

Substituting in for r (not asked), we get

$$\Pr\{E(x,y) \wedge P(x,y)\} = p(1 - (1 - p^2)^{n-2}).$$

■

Problem 13 (8 points)**. Sampling Concepts**

Yesterday, the programmers at a local company wrote a large program. To estimate the fraction, b, of lines of code in this program that are buggy, the QA team will take a small sample of lines chosen randomly and independently (so it is possible, though unlikely, that the same line of code might be chosen more than once). For each line chosen, they can run tests that determine whether that line of code is buggy, after which they will use the fraction of buggy lines in their sample as their estimate of the fraction b.

The company statistician can use estimates of a binomial distribution to calculate a value, s, for a number of lines of code to sample which ensures that with 97% confidence, the fraction of buggy lines in the sample will be within 0.006 of the actual fraction, b, of buggy lines in the program.

Mathematically, the *program* is an actual outcome that already happened. The *sample* is a random variable defined by the process for randomly choosing s lines from the program. The justification for the statistician's confidence depends on some properties of the program and how the sample of s lines of code from the program are chosen. These properties are described in some of the statements below. Indicate which of these statements are true:

- The probability that the ninth line of code in the *program* is buggy is b.

  Solution. False.

  The program has already been written, so there's nothing probabilistic about the buggyness of the ninth (or any other) line of the program: either it is or it isn't buggy, though we don't know which. You could argue that this means it is buggy with probability zero or one, but in any case, it certainly isn't b. ■

- The probability that the ninth line of code in the *sample* is defective is b.

  Solution. True.

  The ninth line in the sample is equally likely to be any line of the program, so the probability it is buggy is the same as the fraction, b, of buggy lines in the program. ■

- All lines of code in the program are equally likely to be the third line chosen in the *sample*.

  Solution. True.

  The meaning of "random choices of lines from the program" is precisely that at each of the s choices in the sample, in particular at the third choice, each line in the program is equally likely to be chosen. ■

- Given that the first line chosen in the *sample* is buggy, the probability that the second line chosen will also be buggy is greater than b.

  Solution. False.

  The meaning of "*independent* random choices of lines from the program" is precisely that at each of the s choices in the sample, in particular at the second choice, each line in the program is equally likely to be chosen, independent of what the first or any other choice happened to be. ■

- Given that the last line in the *program* is buggy, the probability that the next-to-last line in the program will also be buggy is greater than b.

  Solution. False.

  As noted above, it's zero or one. ■

- The expectation of the indicator variable for the last line in the *sample* being buggy is b.

  Solution. True.

  The expectation of the indicator variable is the same as the probability that it is 1, namely, it is the probability that the sth line chosen is buggy, which is b, by the reasoning above. ■

- Given that the first two lines of code selected in the *sample* are the same kind of statement (such as an assignment statement, a conditional statement, or a loop statement), the probability that the first line is buggy may be greater than b.

  Solution. True.

  We don't know how prone to buggyness different kinds of statements may be. It could be for example, that conditionals are more prone to buggyness than other kinds of statements, and that more conditionals lines than any other kind of line in the program. Then given that two randomly chosen lines in the sample are the same kind, they are more likely to be conditionals, which makes them more prone to buggyness. That is, the conditional probability that they will be buggy would be greater than b. ■

- There is zero probability that all the lines in the *sample* will be different.

  Solution. False.

  We know the length, r, of the program is larger than the "small" sample size, s, in which case the probability that all the lines in the sample are different is

$$\frac{r}{r} \cdot \frac{r-1}{r} \cdot \frac{r-2}{r} \cdots \frac{r-(s-1)}{r} = \frac{r!}{(r-s)!\, r^s} > 0.$$

■

Problem 14 (9 points)**. Linear Recurrence & Expectation**

Robbie the robot jumps around randomly on the integer line according to a peculiar set of rules. Letting X_{n-1} be his position after $n-1$ flips, he again flips an unbiased coin to determine his next position, X_n. If the coin comes up heads, then $X_n = 3X_{n-1} - 2n$, and otherwise $X_n = X_{n-1} + 2$. Let $e_n ::= \mathrm{E}[X_n]$ be Robbie's expected position after n steps.

(a) (2 points) Derive the equation

$$e_n = 2e_{n-1} - (n-1), \tag{1}$$

for $n > 0$, explicitly indicating where your derivation uses such properties as linearity of expectation and/or the Law of Total Expectation.

Solution. By the Law of Total Expectation,

$$\begin{aligned}
e_n &= \mathrm{E}\left[X_n \mid n\text{th flip is heads}\right] \cdot \Pr\left\{\text{heads}\right\} + \mathrm{E}\left[X_n \mid n\text{th flip is tails}\right] \cdot \Pr\left\{\text{tails}\right\} \\
&= (3e_{n-1} - 2n) \cdot \Pr\left\{\text{heads}\right\} + (e_{n-1} + 2) \cdot \Pr\left\{\text{tails}\right\} && \text{(linearity)} \\
&= \frac{3e_{n-1} - 2n}{2} + \frac{e_{n-1} + 2}{2} \\
&= 2e_{n-1} - (n-1)
\end{aligned}$$

■

(b) (2 points) Show that

$$\frac{2x-1}{(1-x)^2} = \sum_{n=0}^{\infty} (n-1)x^n.$$

Solution. There are many good ways to derive the nth coefficient.

1. Start with the generating function
$$\frac{2x-1}{(1-x)^2} = 2\frac{x}{(1-x)^2} - \frac{1}{(1-x)^2}.$$
The coefficient of x^n in $x/(1-x)^2$ is n and the coefficient of x^n in $1/(1-x)^2$ is $n+1$. $2n - (n+1) = n - 1$.
2. Start with the sequences whose nth elements is n and -1.
$$\begin{aligned}
\frac{x}{(1-x)^2} &\longleftrightarrow \langle 0, 1, 2, 3, 4, \ldots \rangle \\
-\frac{1}{1-x} &\longleftrightarrow \langle -1, -1, -1, -1, -1, \ldots \rangle
\end{aligned}$$
Adding these two together gives the generating function whose nth coefficient is $n - 1$.
$$\frac{x}{(1-x)^2} - \frac{1}{1-x} = \frac{x - (1-x)}{(1-x)^2}$$
3. Since $\langle 0, 0, 1, 2, 3, \ldots \rangle \longleftrightarrow x^2/(1-x)^2$ only differs in the first position,
$$\begin{aligned}
\langle -1, 0, 1, 2, 3, \ldots \rangle \longleftrightarrow \left(\frac{x^2}{(1-x)^2} - 1\right) &= \frac{x^2 - (1-x)^2}{(1-x)^2} \\
&= \frac{2x-1}{(1-x)^2}
\end{aligned}$$

■

(c) (2 points) Let $E(x) = e_0 + e_1x + e_2x^2 + \cdots$ be the generating function for Robbie's expected position after n steps. Showing your derivation clearly, find the constant k such that

$$(1-2x)E(x) - \frac{1-2x}{(1-x)^2} = e_0 + k. \tag{2}$$

Solution. $k = -1$

Using (1),

$$
\begin{array}{rcccccccccc}
E(x) & = & e_0 & + & e_1 x & + & e_2 x^2 & + & e_3 x^3 & + \cdots \\
2xE(x) & = & & & 2e_0 x & + & 2e_1 x^2 & + & 2e_2 x^3 & + \cdots \\
\dfrac{2x-1}{(1-x)^2} & = & -1 & + & 0x & + & 1x^2 & + & 2x^3 & + \cdots \\
\hline
E(x) - \left(2xE(x) - \dfrac{2x-1}{(1-x)^2}\right) & = & e_0 - 1 & + & 0x & + & 0x^2 & + & 0x^3 & + \cdots \\
(1-2x)E(x) - \dfrac{1-2x}{(1-x)^2} & = & e_0 - 1. & & & & & & &
\end{array}
$$

■

(d) (3 points) Circle the statement that best describes Robbie's expected behavior. Show your work to receive partial credit.

- $|e_n| = \Theta(n)$
- $|e_n| = \Theta(n)$ if $X_0 < k$ and $|e_n| = \Theta(n^2)$ otherwise
- $|e_n| = \Theta(n^2)$
- $\boxed{|e_n| = \Theta(2^n) \text{ if } X_0 \neq -k \text{ and } |e_n| = \Theta(n) \text{ otherwise}}$
- $|e_n| = \Theta(2^n)$
- $|e_n| = \Theta(2^n)$ if $X_0 < k$ and $|e_n| = \Theta(n)$ otherwise
- $|e_n| = \Theta(2^n)$ if $X_0 \neq -k$ and $|e_n| = \Theta(n^2)$ otherwise

Solution. From (2) we have

$$E(x) = (e_0 - 1)\frac{1}{1-2x} + \frac{1}{(1-x)^2}.$$

The nth coefficient of $1/(1-2x)$ is 2^n and the nth coefficient of $1/(1-x)^2$ is $n+1$. So

$$e_n = (e_0 - 1)2^n + n + 1.$$

But $e_0 = X_0$, so if $X_0 = 1$, then $e_n = n + 1$, and otherwise the exponential term dominates the linear term, that is, $|e_n| = \Theta(2^n)$. ■