Part I

Proofs

Chapter 2

Patterns of Proof

2.1 The Axiomatic Method

The standard procedure for establishing truth in mathematics was invented by Euclid, a mathematician working in Alexandria, Egypt around 300 BC. His idea was to begin with five *assumptions* about geometry, which seemed undeniable based on direct experience. For example, one of the assumptions was "There is a straight line segment between every pair of points." Propositions like these that are simply accepted as true are called *axioms*.

Starting from these axioms, Euclid established the truth of many additional propositions by providing "proofs". A *proof* is a sequence of logical deductions from axioms and previously-proved statements that concludes with the proposition in question. You probably wrote many proofs in high school geometry class, and you'll see a lot more in this course.

There are several common terms for a proposition that has been proved. The different terms hint at the role of the proposition within a larger body of work.

- Important propositions are called *theorems*.
- A *lemma* is a preliminary proposition useful for proving later propositions.
- A *corollary* is a proposition that follows in just a few logical steps from a lemma or a theorem.

The definitions are not precise. In fact, sometimes a good lemma turns out to be far more important than the theorem it was originally used to prove.

Euclid's axiom-and-proof approach, now called the *axiomatic method*, is the foundation for mathematics today. In fact, just a handful of axioms, collectively called Zermelo-Frankel Set Theory with Choice (*ZFC*), together with a few logical deduction rules, appear to be sufficient to derive essentially all of mathematics.

Our Axioms

The ZFC axioms are important in studying and justifying the foundations of mathematics, but for practical purposes, they are much too primitive. Proving theorems in ZFC is a little like writing programs in byte code instead of a full-fledged programming language—by one reckoning, a formal proof in ZFC that $2 + 2 = 4$ requires more than 20,000 steps! So instead of starting with ZFC, we're going to take a *huge* set of axioms as our foundation: we'll accept all familiar facts from high school math!

This will give us a quick launch, but you may find this imprecise specification of the axioms troubling at times. For example, in the midst of a proof, you may find yourself wondering, "Must I prove this little fact or can I take it as an axiom?" Feel free to ask for guidance, but really there is no absolute answer. Just be up front about what you're assuming, and don't try to evade homework and exam problems by declaring everything an axiom!

Logical Deductions

Logical deductions or *inference rules* are used to prove new propositions using previously proved ones.

A fundamental inference rule is *modus ponens*. This rule says that a proof of P together with a proof that P IMPLIES Q is a proof of Q.

Inference rules are sometimes written in a funny notation. For example, *modus ponens* is written:

Rule.

$$\frac{P, \quad P \text{ IMPLIES } Q}{Q}$$

When the statements above the line, called the *antecedents*, are proved, then we can consider the statement below the line, called the *conclusion* or *consequent*, to also be proved.

A key requirement of an inference rule is that it must be *sound*: any assignment of truth values that makes all the antecedents true must also make the consequent true. So if we start off with true axioms and apply sound inference rules, everything we prove will also be true.

You can see why modus ponens is a sound inference rule by checking the truth table of P IMPLIES Q. There is only one case where P and P IMPLIES Q are both true, and in that case Q is also true.

P	Q	$P \longrightarrow Q$
F	**F**	**T**
F	**T**	**T**
T	**F**	**F**
T	**T**	**T**

There are many other natural, sound inference rules, for example:

Rule.

$$\frac{P \text{ IMPLIES } Q, \quad Q \text{ IMPLIES } R}{P \text{ IMPLIES } R}$$

EDITING NOTE:

Rule.

$$\frac{\text{NOT}(P) \text{ IMPLIES } Q, \quad \text{NOT}(Q)}{P}$$

■

Rule.

$$\frac{\text{NOT}(P) \text{ IMPLIES NOT}(Q)}{Q \text{ IMPLIES } P}$$

On the other hand,

Rule.

$$\frac{\text{NOT}(P) \text{ IMPLIES NOT}(Q)}{P \text{ IMPLIES } Q}$$

is not sound: if P is assigned **T** and Q is assigned **F**, then the antecedent is true and the consequent is not.

Note that a propositional inference rule is sound precisely when the conjunction (AND) of all its antecedents implies its consequent.

As with axioms, we will not be too formal about the set of legal inference rules. Each step in a proof should be clear and "logical"; in particular, you should state what previously proved facts are used to derive each new conclusion.

2.2 Proof Templates

In principle, a proof can be *any* sequence of logical deductions from axioms and previously proved statements that concludes with the proposition in question. This freedom in constructing a proof can seem overwhelming at first. How do you even *start* a proof?

Here's the good news: many proofs follow one of a handful of standard templates. Each proof has it own details, of course, but these templates at least provide you with an outline to fill in. In the remainder of this chapter, we'll through several of these standard patterns, pointing out the basic idea and common pitfalls and giving some examples. Many of these templates fit together; one may give you a top-level outline while others help you at the next level of detail. And we'll show you other, more sophisticated proof techniques in Chapter 3.

The recipes that follow are very specific at times, telling you exactly which words to write down on your piece of paper. You're certainly free to say things your own way instead; we're just giving you something you *could* say so that you're never at a complete loss.

2.2.1 Proof by Cases

Breaking a complicated proof into cases and proving each case separately is a useful and common proof strategy. In fact, we have already implicitly used this strategy when we used truth tables to show that certain propositions were true or valid. For example, in section 1.1.5, we showed that an implication $P \longrightarrow Q$ is equivalent to its contrapositive $\neg Q \longrightarrow P$ by considering all 4 possible assignments of **T** or **F** to P and Q. In each of the four cases, we showed that $P \longrightarrow Q$ was true if and only if $\neg Q \longrightarrow P$ was true. (For example, if $P = \mathbf{T}$ and $Q = \mathbf{F}$, then both $P \longrightarrow Q$ and $\neg Q \longrightarrow P$ are false, thereby establishing that $(P \longrightarrow Q) \longleftrightarrow (\neg Q \longrightarrow P)$ is true in for this case.) Hence we could conclude that $P \longrightarrow Q$ was true if and only if $\neg Q \longrightarrow P$ are equivalent.

Proof by cases works in much more general environments than propositions involving Boolean variables. In what follows, we will use this approach to prove a simple fact about acquaintances. As background, we will assume that for any pair of people, either they have met or not. If every pair of people in a group has met, we'll call the group a *club*. If every pair of people in a group has not met, we'll call it a group of *strangers*.

Theorem. *Every collection of 6 people includes a club of 3 people or a group of 3 strangers.*

Proof. The proof is by case analysis[1]. Let x denote one of the six people. There are two cases:

1. Among the other 5 people besides x, at least 3 have met x.

2. Among the other 5 people, at least 3 have not met x.

Now we have to be sure that at least one of these two cases must hold,[2] but that's easy: we've split the 5 people into two groups, those who have shaken hands

[1]Describing your approach at the outset helps orient the reader. Try to remember to always do this.

[2]Part of a case analysis argument is showing that you've covered all the cases. Often this is obvious, because the two cases are of the form "P" and "not P". However, the situation above is not stated quite so simply.

with x and those who have not, so one of the groups must have at least half the people.

Case 1: Suppose that at least 3 people have met x.

This case splits into two subcases:

Case 1.1: Among the people who have met x, none have met each other. Then the people who have met x are a group of at least 3 strangers. So the Theorem holds in this subcase.

Case 1.2: Among the people who have met x, some pair have met each other. Then that pair, together with x, form a club of 3 people. So the Theorem holds in this subcase.

This implies that the Theorem holds in Case 1.

Case 2: Suppose that at least 3 people have not met x.

This case also splits into two subcases:

Case 2.1: Among the people who have not met x, every pair has met each other. Then the people who have not met x are a club of at least 3 people. So the Theorem holds in this subcase.

Case 2.2: Among the people who have not met x, some pair have not met each other. Then that pair, together with x, form a group of at least 3 strangers. So the Theorem holds in this subcase.

This implies that the Theorem also holds in Case 2, and therefore holds in all cases.

■

2.2.2 Proving an Implication

Propositions of the form "If P, then Q" are called *implications*. This implication is often rephrased as "P IMPLIES Q" or "$P \longrightarrow Q$".

Here are some examples of implications:

- (Quadratic Formula) If $ax^2 + bx + c = 0$ and $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- (Goldbach's Conjecture) If n is an even integer greater than 2, then n is a sum of two primes.

- If $0 \leq x \leq 2$, then $-x^3 + 4x + 1 > 0$.

There are a couple of standard methods for proving an implication.

Method #1: Assume P is true

This method is really an example of proof by cases in disguise. In particular, when proving P IMPLIES Q, there are two cases to consider: P is true and P is false. The case when P is false is easy since, by definition, **T** IMPLIES Q is true no matter what Q is. This case is so easy that we usually just forget about it and start right off by assuming that P is true when proving an implication, since this is the only case that is interesting. Hence, in order to prove that P IMPLIES Q:

1. Write, "Assume P."
2. Show that Q logically follows.

For example, we will use this method to prove

Theorem 2.2.1. *If $0 \leq x \leq 2$, then $-x^3 + 4x + 1 > 0$.*

Before we write a proof of this theorem, we have to do some scratchwork to figure out why it is true.

The inequality certainly holds for $x = 0$; then the left side is equal to 1 and $1 > 0$. As x grows, the $4x$ term (which is positive) initially seems to have greater magnitude than $-x^3$ (which is negative). For example, when $x = 1$, we have $4x = 4$, but $-x^3 = -1$. In fact, it looks like $-x^3$ doesn't begin to dominate $4x$ until $x > 2$. So it seems the $-x^3 + 4x$ part should be nonnegative for all x between 0 and 2, which would imply that $-x^3 + 4x + 1$ is positive.

So far, so good. But we still have to replace all those "seems like" phrases with solid, logical arguments. We can get a better handle on the critical $-x^3 + 4x$ part by factoring it, which is not too hard:

$$-x^3 + 4x = x(2 - x)(2 + x)$$

Aha! For x between 0 and 2, all of the terms on the right side are nonnegative. And a product of nonnegative terms is also nonnegative. Let's organize this blizzard of observations into a clean proof.

Proof. Assume $0 \leq x \leq 2$. Then x, $2 - x$, and $2 + x$ are all nonnegative. Therefore, the product of these terms is also nonnegative. Adding 1 to this product gives a positive number, so:

$$x(2 - x)(2 + x) + 1 > 0$$

Multiplying out on the left side proves that

$$-x^3 + 4x + 1 > 0$$

as claimed. ■

There are a couple points here that apply to all proofs:

- You'll often need to do some scratchwork while you're trying to figure out the logical steps of a proof. Your scratchwork can be as disorganized as you like—full of dead-ends, strange diagrams, obscene words, whatever. But keep your scratchwork separate from your final proof, which should be clear and concise.

- Proofs typically begin with the word "Proof" and end with some sort of doohickey like $\square$ or $\blacksquare$ or "q.e.d". The only purpose for these conventions is to clarify where proofs begin and end.

Pitfall

For the purpose of proving an implication P IMPLIES Q, it's OK, and typical, to begin by assuming P. But when the proof is over, it's no longer OK to assume that P holds! For example, Theorem 2.2.1 has the form "if P, then Q" with P being "$0 \leq x \leq 2$" and Q being "$-x^3 + 4x + 1 > 0$," and its proof began by assuming that $0 \leq x \leq 2$. But of course this assumption does not always hold. Indeed, if you were going to prove another result using the variable x, it could be disastrous to have a step where you assume that $0 \leq x \leq 2$ just because you assumed it as part of the proof of Theorem 2.2.1.

Method #2: Prove the Contrapositive

We have already seen that an implication "P IMPLIES Q" is logically equivalent to its *contrapositive*

$$\text{NOT}(Q) \text{ IMPLIES NOT}(P).$$

Proving one is as good as proving the other, and proving the contrapositive is sometimes easier than proving the original statement. Hence, you can proceed as follows:

1. Write, "We prove the contrapositive:" and then state the contrapositive.

2. Proceed as in Method #1.

For example, we can use this approach to prove

Theorem 2.2.2. *If r is irrational, then $\sqrt{r}$ is also irrational.*

Recall that rational numbers are equal to a ratio of integers and irrational numbers are not. So we must show that if r is *not* a ratio of integers, then $\sqrt{r}$ is also *not* a ratio of integers. That's pretty convoluted! We can eliminate both *not*'s and make the proof straightforward by considering the contrapositive instead.

Proof. We prove the contrapositive: if $\sqrt{r}$ is rational, then r is rational.

Assume that $\sqrt{r}$ is rational. Then there exist integers a and b such that:

$$\sqrt{r} = \frac{a}{b}$$

Squaring both sides gives:

$$r = \frac{a^2}{b^2}$$

Since a^2 and b^2 are integers, r is also rational. ■

2.2.3 Proving an "If and Only If"

Many mathematical theorems assert that two statements are logically equivalent; that is, one holds if and only if the other does. Here is an example that has been known for several thousand years:

> Two triangles have the same side lengths if and only if two side lengths and the angle between those sides are the same in each triangle.

The phrase "if and only if" comes up so often that it is often abbreviated "iff".

Method #1: Prove Each Statement Implies the Other

The statement "P IFF Q" is equivalent to the two statements "P IMPLIES Q" and "Q IMPLIES P". So you can prove an "iff" by proving *two* implications:

1. Write, "We prove P implies Q and vice-versa."
2. Write, "First, we show P implies Q." Do this by one of the methods in Section 2.2.2.
3. Write, "Now, we show Q implies P." Again, do this by one of the methods in Section 2.2.2.

Method #2: Construct a Chain of IFFs

In order to prove that P is true iff Q is true:

1. Write, "We construct a chain of if-and-only-if implications."
2. Prove P is equivalent to a second statement which is equivalent to a third statement and so forth until you reach Q.

This method sometimes requires more ingenuity than the first, but the result can be a short, elegant proof, as we see in the following example.

Theorem 2.2.3. *The standard deviation of a sequence of values $x_1, \ldots, x_n$ is zero iff all the values are equal to the mean.*

Definition. The *standard deviation* of a sequence of values $x_1, x_2, \ldots, x_n$ is defined to be:

$$\sqrt{\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \cdots + (x_n - \mu)^2}{n}} \tag{2.1}$$

where μ is the *mean* of the values:

$$\mu ::= \frac{x_1 + x_2 + \cdots + x_n}{n}$$

As an example, Theorem 2.2.3 says that the standard deviation of test scores is zero if and only if everyone scored exactly the class average. (We will talk a lot more about means and standard deviations in Part IV of the book.)

Proof. We construct a chain of "iff" implications, starting with the statement that the standard deviation (2.1) is zero:

$$\sqrt{\frac{(x_1-\mu)^2+(x_2-\mu)^2+\cdots+(x_n-\mu)^2}{n}}=0. \tag{2.2}$$

Since zero is the only number whose square root is zero, equation (2.2) holds iff

$$(x_1-\mu)^2+(x_2-\mu)^2+\cdots+(x_n-\mu)^2=0. \tag{2.3}$$

Squares of real numbers are always nonnegative, and so every term on the left hand side of equation (2.3) is nonnegative. This means that (2.3) holds iff

$$\text{Every term on the left hand side of (2.3) is zero.} \tag{2.4}$$

But a term $(x_i-\mu)^2$ is zero iff $x_i=\mu$, so (2.4) is true iff

$$\text{Every } x_i \text{ equals the mean.}$$

■

2.2.4 Proof by Contradiction

In a *proof by contradiction* or *indirect proof*, you show that if a proposition were false, then some false fact would be true. Since a false fact can't be true, the proposition had better not be false. That is, the proposition really must be true.

EDITING NOTE:

So proof by contradiction would be described by the inference rule

Rule.

$$\frac{\neg P \longrightarrow \mathbf{F}}{P}$$

■

Proof by contradiction is *always* a viable approach. However, as the name suggests, indirect proofs can be a little convoluted. So direct proofs are generally preferable as a matter of clarity.

Method: In order to prove a proposition P by contradiction:

1. Write, "We use proof by contradiction."

2. Write, "Suppose P is false."

3. Deduce something known to be false (a logical contradiction).

4. Write, "This is a contradiction. Therefore, P must be true."

As an example, we will use proof by contradiction to prove that $\sqrt{2}$ is irrational. Recall that a number is *rational* if it is equal to a ratio of integers. For example, $3.5 = 7/2$ and $0.1111\cdots = 1/9$ are rational numbers.

Theorem 2.2.4. *$\sqrt{2}$ is irrational.*

Proof. We use proof by contradiction. Suppose the claim is false; that is, $\sqrt{2}$ is rational. Then we can write $\sqrt{2}$ as a fraction n/d where n and d are positive integers. Furthermore, let's take n and d so that n/d is in *lowest terms*, namely, there is no number greater than 1 that divides both n and d).

Squaring both sides gives $2 = n^2/d^2$ and so $2d^2 = n^2$. This implies that n is a multiple of 2. Therefore n^2 must be a multiple of 4. But since $2d^2 = n^2$, we know $2d^2$ is a multiple of 4 and so d^2 is a multiple of 2. This implies that d is a multiple of 2.

So the numerator and denominator have 2 as a common factor, which contradicts the fact that n/d is in lowest terms. So $\sqrt{2}$ must be irrational. ■

EDITING NOTE:

Potential Pitfall

Often students use an indirect proof when a direct proof would be simpler. Such proofs aren't wrong; they just aren't excellent. Let's look at an example. A function f is *strictly increasing* if $f(x) > f(y)$ for all real x and y such that $x > y$.

Theorem 2.2.5. *If f and g are strictly increasing functions, then $f + g$ is a strictly increasing function.*

Let's first look at a simple, direct proof.

Proof. Let x and y be arbitrary real numbers such that $x > y$. Then:

$$f(x) > f(y) \qquad \text{(since } f \text{ is strictly increasing)}$$

$$g(x) > g(y) \qquad \text{(since } g \text{ is strictly increasing)}$$

Adding these inequalities gives:

$$f(x) + g(x) > f(y) + g(y)$$

Thus, $f + g$ is strictly increasing as well. ■

Now we *could* prove the same theorem by contradiction, but this makes the argument needlessly convoluted.

Proof. We use proof by contradiction. Suppose that $f + g$ is not strictly increasing. Then there must exist real numbers x and y such that $x > y$, but

$$f(x) + g(x) \leq f(y) + g(y)$$

This inequality can only hold if either $f(x) \leq f(y)$ or $g(x) \leq g(y)$. Either way, we have a contradiction because both f and g were defined to be strictly increasing. Therefore, $f + g$ must actually be strictly increasing. ■

■

A proof of a proposition P by contradiction is really the same as proving the implication **T** IMPLIES P by contrapositive. Indeed, the contrapositive of T IMPLIES P is NOT(P) IMPLIES **F**. As we saw in Section 2.2.2(**???**), such a proof would be begin by assuming NOT(P) in an effort to derive a falsehood, just as you do in a proof by contradiction.

Pitfall

No matter how you think about it, it is important to remember that when you start by assuming NOT(P), you will derive conclusions along the way that are not necessarily true. (Indeed, the whole point of the method is to derive a falsehood.) This means that you cannot rely on such intermediate results after the proof is completed, for example that n is even in the proof of Theorem 2.2.4). There was not much risk of that happening in the proof of Theorem 2.2.4, but when you are doing more complicated proofs that build up from several lemmas, some of which utilize a proof by contradiction, it will be important to keep track of which follow from a (false) assumption in a proof by contradiction.

2.3 *Good* Proofs in Practice

One purpose of a proof is to establish the truth of an assertion with absolute certainty. Mechanically checkable proofs of enormous length or complexity can accomplish this. But humanly intelligible proofs are the only ones that help someone understand the subject. Mathematicians generally agree that important mathematical results can't be fully understood until their proofs are understood. That is why proofs are an important part of the curriculum.

To be understandable and helpful, more is required of a proof than just logical correctness: a good proof must also be clear. Correctness and clarity usually go together; a well-written proof is more likely to be a correct proof, since mistakes are harder to hide.

In practice, the notion of proof is a moving target. Proofs in a professional research journal are generally unintelligible to all but a few experts who know all the terminology and prior results used in the proof. Conversely, proofs in the first weeks of an introductory course like *Mathematics for Computer Science* would be regarded as tediously long-winded by a professional mathematician. In fact, what we accept as a good proof later in the term will be different than what we consider to be a good proof in the first couple of weeks of this course. But even so, we can offer some general tips on writing good proofs:

State your game plan. A good proof begins by explaining the general line of reasoning. For example, "We use case analysis" or "We argue by contradiction."

Keep a linear flow. Sometimes proofs are written like mathematical mosaics, with juicy tidbits of independent reasoning sprinkled throughout. This is not good. The steps of an argument should follow one another in an intelligible order.

A proof is an essay, not a calculation. Many students initially write proofs the way they compute integrals. The result is a long sequence of expressions without explanation, making it very hard to follow. This is bad. A good proof usually looks like an essay with some equations thrown in. Use complete sentences.

Avoid excessive symbolism. Your reader is probably good at understanding words, but much less skilled at reading arcane mathematical symbols. So use words where you reasonably can.

Revise and simplify. Your readers will be grateful.

Introduce notation thoughtfully. Sometimes an argument can be greatly simplified by introducing a variable, devising a special notation, or defining a new term. But do this sparingly since you're requiring the reader to remember all that new stuff. And remember to actually *define* the meanings of new variables, terms, or notations; don't just start using them!

Structure long proofs. Long programs are usually broken into a hierarchy of smaller procedures. Long proofs are much the same. Facts needed in your proof that are easily stated, but not readily proved are best pulled out and proved in preliminary lemmas. Also, if you are repeating essentially the same argument over and over, try to capture that argument in a general lemma, which you can cite repeatedly instead.

Be wary of the "obvious". When familiar or truly obvious facts are needed in a proof, it's OK to label them as such and to not prove them. But remember that what's obvious to you, may not be—and typically is not—obvious to your reader.

Most especially, don't use phrases like "clearly" or "obviously" in an attempt to bully the reader into accepting something you're having trouble proving. Also, go on the alert whenever you see one of these phrases in someone else's proof.

Finish. At some point in a proof, you'll have established all the essential facts you need. Resist the temptation to quit and leave the reader to draw the "obvious" conclusion. Instead, tie everything together yourself and explain why the original claim follows.

The analogy between good proofs and good programs extends beyond structure. The same rigorous thinking needed for proofs is essential in the design of critical computer systems. When algorithms and protocols only "mostly work" due to reliance on hand-waving arguments, the results can range from problematic to catastrophic. An early example was the Therac 25, a machine that provided radiation therapy to cancer victims, but occasionally killed them with massive overdoses due to a software race condition. A more recent (August 2004) example involved a single faulty command to a computer system used by United and American Airlines that grounded the entire fleet of both companies—and all their passengers!

It is a certainty that we'll all one day be at the mercy of critical computer systems designed by you and your classmates. So we really hope that you'll develop the ability to formulate rock-solid logical arguments that a system actually does what you think it does!

2.3.1 Problems

Class Problems

Homework Problems

Part II

Mathematical Data Types

Part III

Counting

Part IV

Probability