MIT 6.854/18.415: Advanced Algorithms	Spring 2016

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1 Overview

Recall:

- Weak Duality
- Projection onto a convex set
- Farkas' Lemma

Today:

- Strong Duality
- Zero Sum Games
- Complementary Slackness + relation to strong and weak duality

2 Farkas' Lemma

Recall standard form of a linear program:

(primal) $\max c^T x$ s.t. $Ax = b$, $x \geq 0$

(dual) $\min y^T b$ s.t. $y^T A \geq c^T$

And the original form of Farkas' lemma:

Lemma 1 (Farkas'). *Exactly 1 of the following holds:*

(1) $\exists x$ *s.t.* $Ax = b$, $x \geq 0$

(2) $\exists y$ *s.t.* $y^T A \geq 0$, $y^T b < 0$

That is, either there is a feasible x, or there is a y that certifies no such x exists.

We now prove that an alternate form of Farkas' lemma holds.

Lemma 2 (Farkas', alternate form). *Exactly 1 of the following holds:*

(1′) $\exists x$ *s.t.* $Ax \leq b$

(2′) $\exists y$ *s.t.* $y^T A = 0$, $y \geq 0$, $y^T b < 0$

Proof. We map $(1')$ into the form (1). Consider the following linear system.

$$\begin{pmatrix} A & -A & I \end{pmatrix} \begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} = b, \begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} \geq 0 \tag{$1''$}$$

or expanded out

$$Ax^+ - Ax^- + s = b, x^+ \geq 0, x^- \geq 0, s \geq 0$$

s represents "slack" variables (the amount of room remaining for each constraint), and A is the same as in $(1')$.

Claim 3. *$1''$ and $1'$ are equivalent. We can convert any solution of $1'$ to a solution of $1''$ and vice versa.*

Suppose we have a solution to $1'$, x with $Ax \leq b$. Then we can construct a solution to $1''$ as follows.

$$\begin{cases} x^+ = max(x, 0) \\ x^- = -min(x, 0) \\ s = b - Ax \end{cases}$$

This is a solution to $1''$.

Conversely, suppose we are given a solution x^+, x^-, s to the $1''$ system. Then,

$$x = x^+ - x^-$$

is a valid solution.

Now looking back at the original Farkas' lemma, we find the corresponding $2''$ having the form

$$y^T \begin{pmatrix} A & -A & I \end{pmatrix} \geq 0, y^T b < 0$$

$1''$ has no solution if and only if $2''$ holds.

$2''$ is equivalent to $2'$ because $y^T A \geq 0$ and $-y^T A \geq 0$ implies $y^T A = 0, y \geq 0, y^T b < 0$. □

3 Strong Duality

Let $z^* \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ be the optimal value for (p). Note that $-\infty$ corresponds to (p) being infeasible and $+\infty$ corresponds to an unbounded objective value.

Let $w^* \in R \cup \{-\infty\} \cup \{+\infty\}$ be optimal value for (d). Here $-\infty$ corresponds to an unbounded objective value and $+\infty$ corresponds to (p) being infeasible.

Theorem 4. *If either (p) or (d) is feasible, $z^* = w^*$.*

Proof. Assume without loss of generality that (p) is feasible.

If (p) is feasible and (p) is unbounded, then $z^* = +\infty$. Then $w^* = +\infty$ by weak duality.

Otherwise, let x^* be an optimal solution to (p). Then $z^* = c^T x^*$.

We're looking for y with $b^T y \leq$ and $A^T y \geq c$ or equivalently,

$$\begin{pmatrix} -A^T \\ b^T \end{pmatrix} y \leq \begin{pmatrix} -c \\ z^* \end{pmatrix}$$

If there is no such y, then Farkas' lemma tells us that there exists an x, λ such that

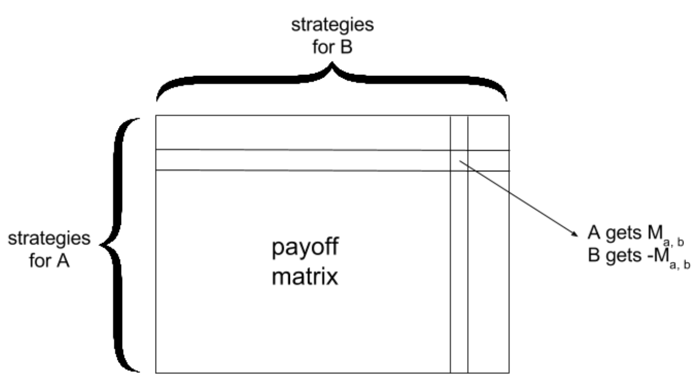
$$(*) \begin{pmatrix} x^T & -A \end{pmatrix} \begin{pmatrix} -A^T \\ B^T \end{pmatrix} = 0x \geq 0, \lambda \geq 0, -x^T c + \lambda z^* < 0$$

Case 1: If $\lambda > 0$, then rescale both x and λ by $\frac{1}{\lambda}$ to get $(x, \lambda) = (\frac{x}{\lambda}, 1)$. This is still feasible by (*). The new system satisfies $Ax = b$, $x \geq 0$ and $x^T x > z^*$. However, since z^* was optimal we have a contradiction.

Case 2: If $\lambda = 0$, then $Ax = 0$, $x \geq 0$ and $c^T x > 0$. Now, $x^* + x$ is feasible for (p) so $c^T(x^* + x) > c^T x^* = z^*$, but since z^* is optimal this is a contradiction. □

4 Zero Sum Games

A powerful application of strong duality is to zero sum games. A zero sum game associates with every strategy a for player A and b for player B, a known payoff of $M_{a,b}$ for A and $-M_{a,b}$ for B.



The Colonel Blotto Games are examples of zero-sum games. Imagine A has r armies and B has s armies. Both players divide their armies among 2 mountain passes. A gets -1 if he is outnumbered on either pass. Otherwise he gets 1.

Theorem 5. *(Von Neumann) There are randomized strategies* (x, y) *where* $x, y \geq 0$, $\sum x_i = 1$ *and* $\sum y_i = 1$, *which represent distributions among the strategies, and a value* $\mathcal{V}$ *such that*

$$x^T M \geq \mathcal{V}\mathbf{1} \tag{1}$$

$$My \leq \mathcal{V}\mathbf{1} \tag{2}$$

where $\mathbf{1}$ *is the vector of all 1s.*

Intuitively, (1) corresponds to the amount A can guarantee by playing x and (2) corresponds to the amount B can guarantee by playing y.

We present a sketch of the proof. We first set up (1) as an LP which maximizes $\mathcal{V}$. Then the dual of this linear program will minimize $\mathcal{V}$ and give (2). Use strong duality to finish. □

The $\mathcal{V}$ from above is called the game value. While the theorem above gives a powerful characterization of $\mathcal{V}$, $\mathcal{V}$ can also be computed.

5 Complementary Slackness

Lemma 6. *(Complementary Slackness) Let* x *and* y *be feasible for the primal and the dual respectively. Then both* x *and* y *are optimal if and only if* $x_i > 0$ *imples* $(y^T A)_i = c_i$

Proof. We follow the proof of weak duality. Because $y^T A \geq c^T$, $x \geq 0$ and $b = Ax$, we have

$$y^T b = y^T Ax \geq c^T x.$$

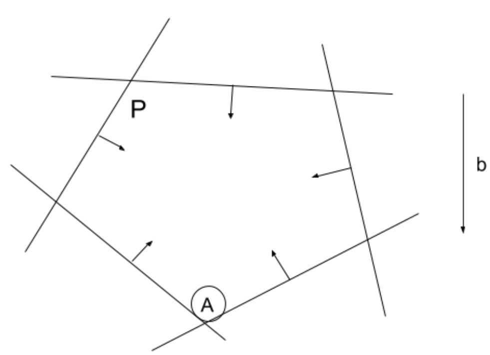
If for any i, we have $x_i > 0$ and $(y^T A)_i > c_i$, then this inequality becomes strict, i.e. $y^T Ax > c^T x$, and so x and y aren't optimal.
If for all i, $x_i > 0$ implies $(y^T A)_i = c_i$, then the inequality becomes equality, i.e. $y^T Ax = c^T x$, and so x and y are both optimal. □

6 Physics Interpretation

We will give a physical interpretation of duality through physics. Let

$$p = \{y | A^T y \geq c\}.$$

Pictorally the setup looks like:



with gravity in the $-b$ direction. We can create a dictionary between LP terms and physics terms as follows:

LPs	Physics
$-b$	gravity
rows of A^T	normals to walls
$\exists x \geq 0$, $x^T A^T = b$	forces balance at equilibrium
complementary slackness: $x_i > 0 \Rightarrow (A^T y)_i = c_i$	only walls touching, exert force

This can be turned into a proof of strong duality, but the details are subtle.