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Lecture 12 – From Separation to Optimization

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Last time we discussed zero-sum games, strong duality and complementary slackness. Strong duality tells us when either the primal or dual is feasible, then their optimal values are equal. In the context of zero-sum games, this is the best value both players can guarantee themselves (game value).

Up to this point we have only defined and stated properties about linear programs. Today, we will cover our first algorithm for solving linear programs. We will first find a way to determine if a set of constraints is feasible using the ellipsoid algorithm, and later, we will use feasibility to find a solution.

1 From Separation to Feasibility

1.1 Setup

Given a convex set P with

1. $P \subseteq B(0, R)$ (a Euclidean ball of radius R) and if $P \neq \varnothing$ then $P \supseteq B(a, r)$ for some center a.
2. a *separation oracle* that answers a query x by
   (a) asserting $x \in P$ or
   (b) asserting $x \notin P$ with a c such that $c^\top y < c^\top x$ for all $y \in P$

we would like to ouput any $x \in P$ or show $P = \varnothing$ (a *feasability problem*). We can think of $R \sim \text{poly}(n)$ and $r \sim \text{poly}(1/n)$.

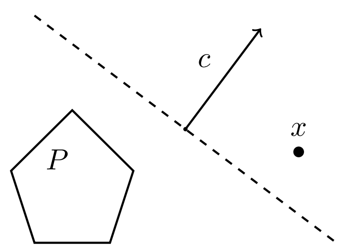

Figure 1: A geometric view of the separation oracle when $x \notin P$. The vector c describes a direction (its magnitude is irrelevant) and it specifies a hyperplane separating x and P.

How can we implement a separation oracle for LPs? For $P = \{x : Ax \leq b\}$, we can easily check $x \in P$. If $x \notin P$, then we can return any row of A that voilates our constraints.

1.2 Ellipsoid Algorithm

1.2.1 Intuition

Leveraging a separation oracle for P, the ellipsoid algorithm can determine feasibility. We will maintain an ellipsoid E_k containing P and at each step reduce its volume until its center is in P or the ellipsoid becomes small enough to convince us that P has no feasible solutions.

Definition 1. *An* **ellipsoid** *is* $E(a, A) = \{x : (x-a)^\top A^{-1}(x-a) \leq 1\}$ *where* A *is symmetric* $(A = A^\top)$ *and positive definite* $(x^\top Ax > 0 \ \ \forall x \neq 0)$.

For example, $B(0, R) = E(0, R^2 I)$.

1.2.2 The Algorithm

- Initialize $E_0 = B(0, R)$
- For $k = 0$ to m
  1. Let $E_k = E(a_k, A_k)$ be the current ellipsoid.
  2. Query the separation oracle at a_k.
     - If $a_k \in P$ output a_k.
     - Otherwise with answer c_k find E_{k+1} such that

$$E_{k+1} \supseteq E_k \cap \{x : c_k^\top x \leq c_k^\top a_k\} \tag{1}$$

- Output $P = \varnothing$.

Note that we have not yet defined m. We will first show how to find E_{k+1} and prove the correctness of the algorithm. In the proof, we will determine the appropriate value of m.

1.2.3 Finding E_{k+1}

Does E_{k+1} even exist and how do we find it?

We start by considering the special case where $E = E(0, I)$. Then without loss of generality we can assume $c_k = -\hat{e}_1$. So the intersection in 1 can be written as

$$E' \supseteq E \cap \{x : x_1 \geq 0\}$$

Let n be the dimension. We now claim the following E' satisfies the desired constraints.

$$E' = \left\{ x : \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i^2 \leq 1 \right\} \tag{2}$$

Claim 2. *If* $x \in E$ *and* $x_1 \geq 0$ *then* $x \in E'$.

Proof. Starting with the first term in 2,

$$\left(\frac{n+1}{n}\right)^2\left(x_1-\frac{1}{n+1}\right)^2=\left(\frac{n^2+2n+1}{n^2}\right)x_1^2-\frac{2(n+1)}{n^2}x_1+\frac{1}{n^2}$$

Note that

$$\left(\frac{n^2+2n+1}{n^2}\right)=\left(\frac{n^2-1}{n^2}+\frac{2n+2}{n^2}\right)$$

Therefore, we can rewrite the left-hand side of the constraint in 2 as

$$\begin{aligned}\left(\frac{n^2-1}{n^2}\right)\left(\sum_{i=1}^{n}x_i^2\right)+\frac{(2n+2)(x_1^2-x_1)}{n^2}+\frac{1}{n^2}&\le\frac{n^2-1}{n^2}\|x\|_2^2+0+\frac{1}{n^2}\\&\le\frac{n^2-1}{n^2}+\frac{1}{n^2}\le 1\end{aligned}$$

□

Claim 3.

$$\frac{\mathrm{vol}(E')}{\mathrm{vol}(E)}\le e^{-\frac{1}{2(n+1)}}$$

Proof. The volume of an ellipsoid is proportional to the product of its sidelengths.

$$\begin{aligned}\frac{\mathrm{vol}(E')}{\mathrm{vol}(E)}&=\left(\frac{n}{n+1}\right)\left[\left(\frac{n^2}{n^2-1}\right)^{1/2}\right]^{n-1}\\&=\left(1-\frac{1}{n+1}\right)\left(1+\frac{1}{n^2-1}\right)^{\frac{n-1}{2}}\\&\le e^{-\frac{1}{n+1}}e^{\frac{n-1}{2(n^2-1)}}=e^{-\frac{1}{2(n+1)}}\end{aligned}$$

□

Now, let us consider the general case $E_k=E(a,A)$. We use the following facts to reduce to the special case.

Fact 4. *A is positive definite if and only if $A=B^\top B$ for some invertible B.*

Fact 5. *Invertible linear transformations preserve volume ratios (can show with the Jacobian).*

Define the linear transform $y=T(x)=(B^{-1})^\top(x-a)$. Observe $A^{-1}=B^{-1}(B^{-1})^\top$. So

$$y^\top y\le 1\Leftrightarrow (x-a)^\top A^{-1}(x-a)\le 1$$

Thus, $y\in E(0,1)\Leftrightarrow x\in E(a,A)$. Thus, we can apply the special case and let

$$E_{k+1}=\{T^{-1}(y):y\in E'\}$$

Because we use an invertible linear transform it will satisfy 1, which completes the general case.

$$\begin{tikzcd} E_k = E(a, A) \arrow[r, "T"] & E(0, I) \arrow[d, "\text{special case}"] \\ E_{k+1} & E' \arrow[l, "T^{-1}"'] \end{tikzcd}$$

Figure 2: Reduction for the general case.

1.2.4 Setting the Number of Iterations

Since if $P \neq \varnothing$, $P \supseteq B(a, r)$, we know that the ratio of the volume of the final elipsoid to the volume of the initial elipsoid is bounded below by:

$$\frac{\text{vol}(E_f)}{\text{vol}(E_0)} \geq \left(\frac{r}{R}\right)^n$$

Therefore, we can bound the number of iterations m needed of the ellipsoid algorithm from:

$$\frac{\text{vol}(E_m)}{\text{vol}(E_0)} \leq e^{-\frac{m}{2(n+1)}} \leq \left(\frac{r}{R}\right)^{2n}$$

Setting $m = O(n^2 \ln \frac{R}{r})$ satisfies this condition.

2 From Feasibility to Optimization

To optimize the set of constraints and find a solution, we:

- Check feasbility for the primal (P). If this is infeasible, there is no solution we are done.
- Check feasbility for the dual (D). If this is infeasible, there is no solution we are done. This stems from strong duality, which tells us that P has unboundedly good solutions.
- Lastly, we set up a joint feasibility LP with the following constraints: $Ax = b$, $x \geq 0$, $y^\top A \geq c^\top$ and $c^\top x = y^\top b$, and check feasibility. If the problem is infeasible, there is no solution. Otherwise, the point from the feasibility problem satisfies all constraints and is an optimum.