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1 Semidefinite Programming

Earlier we saw a framework for approximating NP-hard problems by relaxing integer linear programming (ILP) to general linear programming (LP). Here we see a problem for which this method does not work and introduce a more power technique, semidefinite programming, to solve it.

2 MAXCUT

Given $G = (V, E)$ we want to choose a subset $U \subseteq V$ so as to maximize $|E(U, V \backslash U)|$. I.e. we want to maximiaze the number of edges connecting a node in U with a node outside of U.

Observe that the maximum cut equals $|E|$ iff G is bipartite.

Let's give a first attempt at solving this problem via integer linear programming:

Integer Linear Program:

$$\begin{aligned} \max \sum_{(u,v)\in E} & Z_{(u,v)} \\ \text{subject to } Z_{(u,v)} &\leq X_u + X_v \\ Z_{(u,v)} &\leq (1 - X_u) + (1 - X_v) \\ Z_{(u,v)}, X_v &\in \{0, 1\} \end{aligned}$$

$Z_{(u,v)}$ can only be set to 1 when $X_u \neq X_v$. Setting $X_u = 1$ corresponds to placing u in U and we can increase our objective function by 1 for every v where $X_v = 0$ (i.e. where v was not also placed in U).

ILP relaxation: If we relax the integer constraint $Z_{u,v}, X_v \in \{0, 1\}$ to $Z_{u,v}, X_v \in [0, 1]$ we can satisfy the LP constraints by setting every $Z_{(u,v)} = 1$ and every $X_v = \frac{1}{2}$. This achieves a value of $|E|$. However, for the complete graph, MAXCUT $\approx \frac{|E|}{2}$. Accordingly, we can't expect any rounding strategy to achieve better than a 1/2-approximation.

And in fact, we can trivially obtain a 1/2 approximation in expectation: assign each node independently with probability $\frac{1}{2}$ to U.

3 Positive Semidefinite Matrices

Let $X \in \mathbb{R}^{n \times n}$ be symmetric. We say X is positive semidefinite (PSD or $X \succeq 0$) if the following equivalent statements are true:

1. $\forall a \in \mathbb{R}^n \ a^T X a \geq 0$
2. $X = B^T B$ for some B
3. All of X's eigenvalues are non-negative

4 Semidefinite Programs (SDP)

The standard form of a semidefinite program is analogous to the standard form of a linear program. In the following equations, let C, X, and A_i be $n \times n$ matrices.

$$\begin{aligned} \min_X \langle C, X\rangle &= \sum_{i,j} c_{i,j} x_{i,j} \text{ (the Frobenius product of } C \text{ and } X) \\ \text{s.t. } \langle A_i, X\rangle &= b_i \quad \forall i \in (1, ..., m) \\ X &\succeq 0 \end{aligned}$$

Note that this program corresponds exactly to linear programming when all matrices are diagonal. The feasible region for the SDP is $\{X \mid \langle A_i, X\rangle = b_i \ \ \forall i, a^T X a \geq 0 \ \ \forall a\}$. So, the requirement that X be positive semidefinite effectively creates an infinite number of linear constraints on X.

We can solve semidefinite programs using either the ellipsoid method or interior-point methods. However, unlike linear programs we can only obtain solutions to within arbitrary accuracy, not exact solutions. This is because the bit-complexity of solutions to linear programs are bounded by a function of the size of the original problem, meaning that if we converge "close enough" for a linear program we can obtain an exact answer. This property does not hold for semidefinite programs.

4.1 Duality

The dual of the program above can be written as

$$\begin{aligned} \max_y \ & b^T y \\ \text{s.t. } & \sum_{i=1}^m y_i A_i + S = C \\ & S \succeq 0 \end{aligned}$$

where y is a length-m vector and S is an $n \times n$ matrix.

4.2 Basic Facts

Fact 1 $\langle A, X \rangle = \text{Tr}(A^T X)$

This fact follows from the definition of matrix multiplication.

Fact 2 $\text{Tr}(AB) = \text{Tr}(BA)$

More generally, *cyclic permutations* of the order in which matrices are multiplied do not affect the trace. E.g. $\text{Tr}(ABC) = \text{Tr}(CAB) = \text{Tr}(BCA)$.

4.3 Weak Duality

Lemma 1. *Let X be symmetric, then $X \succeq 0 \Leftrightarrow \langle A, X \rangle \geq 0 \quad \forall A \succeq 0$*

Proof Suppose $X \not\succeq 0$. Then $a^T X a < 0$ for some vector a.
Let $A = aa^T$. Clearly, $A \succeq 0$.
Then $\langle A, X \rangle = \text{Tr}(A^T X) = \text{Tr}(aa^T X) = \text{Tr}(a^T X a) < 0$

Now suppose $X \succeq 0$
Let $A \succeq 0$. Then $A = BB^T = \sum_i b_i b_i^T$ for some matrix B with columns $b_1, b_2, ...b_n$.
Then $\langle A, X \rangle = \sum_i b_i^T X b_i \geq 0$ by the same logic as above.

Lemma 2 (Weak Duality)**.** *If x/y are feasible for (P)/(D) then $b^T y \leq \langle C, X \rangle$.*

Proof By the feasibility of y, we have

$$\langle C, X \rangle = \left\langle \sum y_i A_i, X \right\rangle + \langle S, X \rangle .$$

By the feasibility of x, we have

$$\left\langle \sum y_i A_i, X \right\rangle = b^T y.$$

And by Lemma 1, we have

$$\langle S, X \rangle \geq 0.$$

Thus

$$b^T y = \langle C, X \rangle - \langle S, X \rangle \leq \langle C, X \rangle .$$

So any solution for the dual lower bounds the minimum of the primal.

4.4 Strong Duality

Warning: We won't cover details, but strong duality "usually holds" for semidefinite programs. Specifically, it holds under the following condition:

Proposition 3 (Slater's condition)**.** *Strong duality holds if the feasible region has an interior point.*

5 Goemans-Williamson

Semidefinite programming provides a generalization of linear programming that is often much more powerful for solving hard approximation problems. Here we will see a famous relax and round procedure for MAXCUT based on SDPs. Specifically, consider the following program:

$$\begin{aligned} \max \sum_{(u,v)\in E} & \frac{1}{2} - \frac{1}{2} X_{uv} \\ \text{s.t. } & X_{uu} = 1, \forall u \\ & X \succeq 0 \end{aligned}$$

Why is this a relaxation to MAXCUT?

We can construct a feasible solution to this SDP from a solution to MAXCUT

Let

$$x_u = \begin{cases} 1, & \text{if } u \in U, \\ -1, & \text{otherwise.} \end{cases}$$

Set $X = xx^T$. The object value is the number of edges across the cut because X_{uv} is -1 if u and v are on opposite sides of the cut and X_{uv} equals 1 otherwise.

6 Hyperplane Rounding

Goemans and Williamson show how to round this semidefinite program to obtain the following approximation guarantee:

Theorem 4 (Goemans-Williamson [1])**.** *Let*

$$\alpha_{gw} = \min_{0\leq\theta\leq\pi} \frac{2}{\pi} \frac{\theta}{1-\cos\theta} \approx 0.87856\ldots$$

There is an algorithm with obtains an α_{gw}-approximation for MAXCUT in expectation.

We take X which is the optimal solution to the SDP. Since X is positive semidefinite and has diagonal entries equal to 1, it can be written as $X = YY^T$ for some Y. Accordingly, there are vectors $\{y_u\}$ so that $X_{uv} = \langle y_u, y_v \rangle$.

Choose a vector a uniformly on the sphere.

Set $x_u = \text{sgn}(\langle a, y_u \rangle)$.

We want to analyze the expected contribution of each edge (u, v) to our rounded solution.

The contribution to the SDP is $\frac{1}{2} - \frac{1}{2} \langle y_u, y_v \rangle = \frac{1}{2} \cos\theta$ where θ is the angle between vectors y_u and y_v.

For a contribution to the cut, we have that a cuts edge (u, v) if and only if its orthogonal hyperplane lies between the vectors y_u and y_v. If we assume, without loss of generality, that $0 \leq \theta \leq \pi$, then this occurs with probability $\frac{\theta}{\pi}$. Accordingly, the expected contribution of (u, v) to our cut value is $\frac{\theta}{\pi}$.

Thus, the worst case contribution for the edge as a fraction of its contribution to the SDP is:

$$\min_{0 \le \theta \le \pi} \frac{\frac{\theta}{\pi}}{\frac{1-\cos\theta}{2}} = \min_{0 \le \theta \le \pi} \frac{2\theta}{\pi(1-\cos\theta)}$$

as desired.

The Goemans-Williamson rounding scheme gives the best known approximation to MAXCUT and it *may be* the best approximation possible via any efficient algorithm.

Theorem 5 (Knot, Kindler, Mossel, O'Donnell [2])(Mossel, O'Donnell, Oleszkiewicz [3])**.** *Assuming the "Unique Games Conjecture" it is NP-hard to approximate MAXCUT better than* α_{gw}

The UGC, or Unique Games Conjecture, is a controversial, far-reaching conjecture in complexity theory. It states that there exist constant limits for the best approximation algorithms for certain NP-hard problems, and makes some statements about what those bounds are. Many believe it to be true in some form, many believe it to be false in some form.

References

[1] Goemans, Michel X., and David P. Williamson."Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming." Journal of the ACM (JACM) 42, no. 6 (1995): 1115-1145.

[2] Khot, Subhash, Guy Kindler, Elchanan Mossel, and Ryan O'Donnell. "Optimal inapproximability results for MAX-CUT and other 2-variable CSPs?." SIAM Journal on Computing 37, no. 1 (2007): 319-357.

[3] Mossel, Elchanan, Ryan O'Donnell, and Krzysztof Oleszkiewicz. "Noise stability of functions with low influences: invariance and optimality." In Foundations of Computer Science, 2005. FOCS 2005. 46th Annual IEEE Symposium on, pp. 21-30. IEEE, 2005.