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Prof. Ankur Moitra *Scribe: Bryan Cai, Prafulla Dhariwal, Logan Mercer, Isaac Grosof*

1 Compressed Sensing

In compressed sensing, we want to solve

$$\min \|x\|_0 \text{ s.t. } Ax = b. \tag{P_0}$$

$\|x\|_0$ refers to the number of nonzero entries in x.

Here A is $m \times n$ and $m \ll n$, so there are many solutions. Among the solutions, we want to find the sparsest. This problem has a huge number of applications: MRI, single pixel camera, etc.

It is NP-hard, but we'll give conditions under which we can solve it anyway. Let's start by considering a relaxed version of the problem that we can solve efficiently:

$$\min \|x\|_1 \text{ s.t. } Ax = b \tag{P_1}$$

This can be rewritten as

$$\min \sum y_i \text{ s.t. } Ax = b, x \leq y, -x \leq y$$

If A has certain properties, the optimal solution to (P_1) will also be an optimal solution to (P_0), allowing us to solve (P_0) easily.

2 Restricted Isometry Property

We say that A has restricted isometry property (RIP) (k, δ_k) if for all x with $\|x\|_0 \leq k$ we have

$$(1 - \delta_k)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_k)\|x\|_2^2$$

Roughly, this means that on sparse vectors, A behaves similarly to an orthogonal matrix. Recall that if Q is an orthogonal matrix, $\|x\|_2^2 = \|Qx\|_2^2 \; \forall x$.

3 Sample Compressed Sensing Theorem

Theorem 1 (Candes-Tao [1]). *If $Ax = b$ and $\|x\|_0 \leq k$, and A has the RIP for $(2k, \delta_{2k})$ and for $(3k, \delta_{3k})$, and $\delta_{2k} + \delta_{3k} < 1$, the uniquely optimal solution to (P_1) is x.*

Fact: Random $m \times n$ matrix A (independent, Gaussian entries), when scaled appropriately, will satisfy the above RIP with $m = \Theta(k \log \frac{n}{k})$.

4 Almost Euclidean Subspace

In what follows, we care about the subspace which is the kernel of A.

We say that a subspace $\Gamma \subseteq \mathbb{R}^n$ is C-Almost Euclidean (C-AE) if for all $v \in \Gamma$ we have

$$\frac{1}{\sqrt{n}}\|v\|_1 \leq \|v\|_2 \leq \frac{C}{\sqrt{n}}\|v\|_1$$

Informally, we want $\Gamma \cap \{x \mid \|x\|_1 \leq 1\}$ to be approximately a sphere

Claim 2. *$\frac{1}{\sqrt{n}}\|v\|_1 \leq \|v\|_2$ for all v, regardless of Γ.*

Proof. If $u_i = sign(v_i)$, then $\|v\|_1 = \langle v, u\rangle \leq \|v\|_2 \cdot \|u\|_2 \leq \|v\|_2\sqrt{|supp(v)|} \leq \|v\|_2\sqrt{n}$ □

Consider $v \in \Gamma$, $v \neq 0$, $S = \frac{n}{C^2}$.

Lemma 3. *If $v \in \Gamma$, $v \neq 0$, then $|supp(v)| \geq S$*

Proof. From above,

$$\|v\|_1 \leq \|v\|_2\sqrt{|supp(v)|} \leq \frac{C}{\sqrt{n}}\|v\|_1\sqrt{|supp(v)|} \tag{1}$$

so it must be that $\sqrt{|supp(v)|} \geq \sqrt{n}/C$. □

Analogy: linear error correcting codes

$$e = \{Ax \big| x \in \{0,1\}^k\}$$

over $GF(2)$ need all nonzero Ax's to have many 1's.

Lemma 4. *Let $v \in \Gamma$, $v \neq 0$, $T \subset [n]$, $|T| \leq \frac{S}{16}$. Then,*

$$\|v_T\|_1 \leq \frac{\|v\|_1}{4} \tag{2}$$

where v_T is v restricted to T.

In words, this lemma says that the ℓ_1 norm of v cannot be concentrated in too small a set of vertices.

Proof. We have

$$\|v_T\|_1 \leq \sqrt{|T|}\|v_T\|_2 \leq \sqrt{|T|}\|v\|_2 \leq \frac{C\sqrt{|T|}}{\sqrt{n}}\|v\|_1 \leq \frac{C\sqrt{\frac{n}{16C^2}}}{\sqrt{n}}\|v\|_1 = \frac{\|v\|_1}{4} \tag{3}$$

□

Theorem 5. *If $Ax = b$ and $\|x\|_0 \leq \frac{S}{16} = \frac{n}{16C^2}$ and $\Gamma = \ker(A)$ is C-AE then x is the uniquely optimal solution to (P_1).*

Proof. Let w be any other potential solution to (P_1). We can write $w = x + v$, $v \in \Gamma$ since $\Gamma = \ker(A)$. Let $T = supp(x)$. Then, we have

$$\begin{aligned}\|w\|_1 = \|w_T\|_1 + \|w_{\overline{T}}\|_1 &\geq \|x_T\|_1 - \|v_T\|_1 + \|v_{\overline{T}}\|_1 \\ &= \|x_1\| - 2\|v_T\|_1 + \|v\|_1 \\ &\geq \|x\|_1 + \frac{\|v\|_1}{2} \\ &\geq \|x\|_1\end{aligned}$$

The second to last last inequality follows because $v \in \Gamma$ and so $\|v_T\|_1 \leq \frac{\|v\|_1}{4}$. ☐

Theorem 6 (Kashin [3], Garnaev-Gluskin [2])**.** *A random subspace $\Gamma \subset \mathbb{R}^n$ of $\dim(\Gamma) = n - m$ is C-AE (whp) with*

$$C \leq \sqrt{\frac{n}{m} \log \frac{n}{m}} \tag{4}$$

Proof. By Theorem 5 we can obtain sparse recovery up to sparsity:

$$\|x\|_0 = \frac{S}{16} = \frac{n}{16C^2} = \Omega(\frac{m}{\log \frac{n}{m}})$$

☐

What happens if x is not exactly k-sparse? Let $\sigma_k(x) = \min_{\|w\|_0 \leq k} \|x - w\|_1$

This measures how far x is, in the l_1 norm, from being k-sparse.

Theorem 7. *Let $Ax = b$, with $\Gamma = \ker(A)$ is C-AE. Let $S = \frac{n}{C^2}$. If w is an optimal solution to (P_1), then*

$$\|x - w\|_1 \leq 4\sigma_{\frac{S}{16}}(x) \tag{5}$$

So, even when x is not exactly k-sparse we can recover a vector that well approximates x in the sense that does nearly as well as the best k-sparse approximation to x.

Proof. Let T be the $\frac{S}{16}$ largest magnitude coordinates of x. Then

$$\|x - w\|_1 = \|(x - w)_T\|_1 + \|(x - w)_{\overline{T}}\|_1 \leq \|(x - w)_T\|_1 + \|x_{\overline{T}}\|_1 + \|w_{\overline{T}}\|_1 \tag{6}$$

Because w is optimal for (P_1),

$$\|w_{\overline{T}}\|_1 = \|w\|_1 - \|w_T\|_1 \leq \|x\|_1 - \|w_T\|_1 \tag{7}$$

So, we get

$$\|x - w\|_1 \leq \|(x - w)_T\|_1 + \|x_{\overline{T}}\|_1 + \|x\|_1 - \|w_T\|_1 \tag{8}$$

Note that

$$\|x_{\overline{T}}\|_1 + \|x\|_1 - \|w_T\|_1 = 2\|x_{\overline{T}}\|_1 + \|x_T\|_1 - \|w_T\|_1 \leq 2\|x_{\overline{T}}\|_1 + \|(x - w)_T\|_1 \tag{9}$$

Combining all the above gives

$$\|x-w\|_1 \leq 2\|(x-w)_T\|_1 + 2\|x_{\overline{T}}\|_1 \leq \frac{\|x-w\|_1}{2} + 2\sigma_{\frac{S}{16}}(x) \tag{10}$$

The last inequality uses Lemma 4. Finally, we conclude that

$$\frac{\|x-w\|_1}{2} \leq 2\sigma_{\frac{S}{16}}(x) \tag{11}$$

which gives the result. □

References

[1] Emmanuel Candes and Terence Tao. Decoding by linear programming. IEEE Trans. on Information Theory, 51(12):4204–4215, 2005.

[2] Andrey Garnaev and Efim Gluskin. The widths of a Euclidean ball. Sovieth Math. Dokl., pages 200–204, 1984.

[3] Boris S. Kashin. Diameters of certain finite-dimensional sets in classes of smooth functions. Izv. Akad. Nauk SSSR, Ser. Mat., 41 (1977), pp. 334–351.