Lecture #12

Last Time: Strong Duality, Zero Sum Games and Complementary Slackness

If either (P) or (D) are feasible, their optimal values are equal (what is this called in zero sum games?)
game value

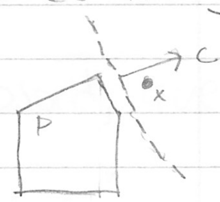
Today: From Separation to Optimization and Back (i.e. Ellipsoid Algorithm)

Setup: Given a convex set P with

"not too big, not too small"

(1) $P \subseteq B(0,R) = \{x \mid \|x\|_2 \leq R\}$ "ball of radius R"
and promise that if $P \neq \emptyset$, $P \supseteq B(a,r)$ for some a

(2) separation oracle that answers queries x by



- asserting $x \in P$ OR
- asserting $x \notin P$ and giving c s.t.

$$\forall y \in P, \quad c^T y < c^T x$$

Goal: Output any $x \in P$ or show $P = \emptyset$ "feasibility problem"

How can we implement a separation oracle for LPs?

e.g. $P = \{x \mid Ax \leq b\}$

Given x, check whether $Ax \leq b$ and if not output any violated row of A

Idea: Maintain ellipsoid $E_k \supseteq P$, reduce its volume in each step

An ellipsoid is

$$E(a, A) = \{x \mid (x-a)^T A^{-1} (x-a) \le 1\}$$

where A is symmetric (i.e. $A = A^T$) and positive definite (i.e. $x^T A x > 0 \ \forall x \neq 0$)

e.g. $B(0, R) = E(a, R^2 \cdot I)$

Ellipsoid Algorithm

Initialize $E_0 = B(0, R)$

For $k = 0$ to m

Let $E_k = (a_k, A_k)$ be current ellipsoid, query separation oracle at a_k

If $a_k \in P$, output a_k

Else given answer c_k

Find $E_{k+1} \supseteq P \cap \{x \mid c_k^T x < c_k^T a_k\}$ of smaller volume

Output $P = \emptyset$

Main question: Is there such an E_k, and how do we find it?

Let's consider a special case

$$E = E(0, I) \text{ then wlog let } c_k = -e_1$$

Let's set

$$E' = \left\{ x \;\middle|\; \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2} x_i^2 \leq 1 \right\}$$

Claim #1 If $x \in E$ and $x_1 \geq 0$ then $x \in E'$

Proof:

$$\left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 = \underbrace{\left(\frac{n^2+2n+1}{n^2}\right)}_{\frac{n^2-1}{n^2} + \frac{2n+2}{n^2}} x_1^2 - \frac{2(n+1)x_1}{n^2} + \frac{1}{n^2}$$

Now if we add $\frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i^2$ we get:

$$\left(\frac{n^2-1}{n^2}\right) \underbrace{\left(\sum_{i=1}^{n} x_i^2\right)}_{\leq 1} + \underbrace{\frac{(2n+2)(x_1^2 - x_1)}{n^2}}_{\leq 0} + \frac{1}{n^2}$$

$$\leq \frac{n^2-1}{n^2} + \frac{1}{n^2} = 1 \quad \boxtimes$$

Claim #2 $\frac{\mathrm{vol}(E')}{\mathrm{vol}(E)} \leq e^{-\frac{1}{2(n+1)}}$

Proof: The volume of an Ellipsoid is proportional to product of its side lengths, thus

$$\frac{vol(E')}{vol(E)} = \frac{\underbrace{\left(\frac{n}{n+1}\right)}_{\left(1-\frac{1}{n+1}\right)}\left[\underbrace{\left(\frac{n^2}{n^2-1}\right)}_{\left(1+\frac{1}{n^2-1}\right)}^{\frac{1}{2}}\right]^{n-1}}{1}$$

$$\leq e^{-\frac{1}{n+1}} e^{\frac{n-1}{2(n^2-1)}}$$

$$= e^{-\frac{1}{n+1}} e^{\frac{1}{2(n+1)}} = e^{-\frac{1}{2(n+1)}} \qquad \blacksquare$$

How can we handle general case $E(a, A)$?

Let's reduce to the special case

Fact #1: A is positive definite iff $A = B^T B$ for some invertible B

Fact #2: invertible linear transformations preserve volume ratios (follows from Jacobian)

Let $y = T(x) = (B^{-1})^T (x - a)$, then

$$y^T y \leq 1 \iff (x-a)^T \underbrace{B^{-1} (B^{-1})^T}_{A^{-1}} (x-a) \leq 1$$

Thus $y \in E(0, I) \iff x \in E(a, A)$

Putting it all together we have:

$$\begin{array}{ccc} E_k = E(a, A) & \xrightarrow{T} & E(0, I) \\ & & \downarrow \\ E_{k+1} & \xleftarrow{T^{-1}} & E' \end{array}$$

Here we assumed separating hyperplane is $-e_1$, but easy to compose T with a rotation to make this so.

Now we have $\mathrm{vol}(E_m) \leq \mathrm{vol}(E_0)\, e^{-\frac{m}{2(n+1)}}$

If we set $m = O(n^2 (\ln \frac{R}{r}))$ then

$$\frac{\mathrm{vol}(E_m)}{\mathrm{vol}(E_0)} \leq e^{-2n(\ln \frac{R}{r})} = \left(\frac{r}{R}\right)^{2n}$$

Recall, if $P \neq \emptyset$ then $P \supseteq B(a, r)$ hence

$$\frac{\mathrm{vol}(E_m)}{\mathrm{vol}(E_0)} \geq \left(\frac{r}{R}\right)^{n}$$

but this is a contradiction so P must be empty

<u>Theorem</u>: If $m = O(n^2 \ln \frac{R}{r})$ then the ellipsoid method either returns $x \in P$ or correctly asserts $P = \emptyset$

From Feasibility to Optimization

<u>alternative</u>: binary search, gives exact when solution is rational with poly bit complexity

- Check feasibility for (P) and (D), if either unfeasible we are done
- Otherwise set up joint feasibility LP:

e.g. $\quad Ax = b,\ x \geq 0,\ y^T A \geq c^T,\ c^T x = y^T b$

what we've shown so far

separation $\Rightarrow$ Optimization

In fact the other direction is true

Optimization $\Rightarrow$ separation

The <u>polar dual</u> of P is defined as:

$$P^\circ = \{ y \mid y^T x \leq 1 \ \forall x \in P \}$$

<u>Fact</u>: If P is convex, $(P^\circ)^\circ = P$ $\left(\begin{array}{l} \text{actually in general} \\ (P^\circ)^\circ = \text{conv}(P) \end{array} \right)$

Now if we want to implement a separation oracle for P, then

- given x, solve $\max_{c \in P^\circ} c^T x$

- If optima $c^T x > 1$ then $x \notin P$ because

$$c^T x' \leq 1 \ \forall x' \in P \text{ but } \underbrace{c^T x > 1}_{\text{separating hyperplane}}$$

- If optima $c^T x \leq 1$ then $x \in P$ because

$$\forall c \in P^\circ,\ c^T x \leq 1 \Rightarrow x \in (P^\circ)^\circ \overset{\text{by fact}}{=} P$$

second

Thus we have

$$
\begin{tikzcd}
\text{Separation on } P \arrow[d, Rightarrow, "\text{Ellipsoid}"] & \text{optimization on } P^{\circ} \arrow[l, Rightarrow, "\text{Polarity}"'] \\
\text{Optimization on } P \arrow[r, Rightarrow, "\text{Polarity}"] & \text{Separation on } P^{\circ} \arrow[u, Rightarrow, "\text{Ellipsoid}"']
\end{tikzcd}
$$

first

ask how to fill in rest

<u>Test question</u>: Is P° just the LP dual? No, they live in different dimensional spaces (in general)

But if P is defined by a polynomial # of inequalities then it is easy to define P° by a polynomial # of inequalities too