# Disentangling Gaussians

Ankur Moitra, MIT

November 6th, 2014 — Dean's Breakfast

# The Gaussian Distribution

The Gaussian distribution is defined as ($\mu$ = mean, $\sigma^2$ = variance):

$$\mathcal{N}(\mu, \sigma^2, x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

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**Central Limit Theorem:** The sum of independent random variables $X_1, X_2, ...., X_s$ converges in distribution to a Gaussian:

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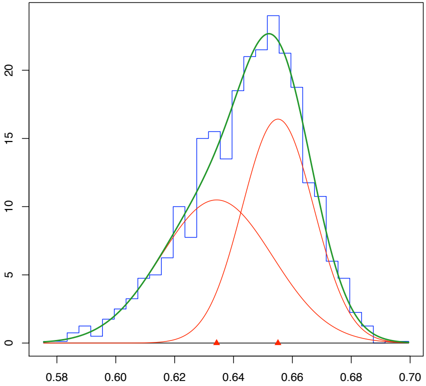
This distribution is ubiquitous — e.g. used to model height, velocities in an ideal gas, annual rainfall, ...

# Karl Pearson (1894) and the Naples Crabs

(figure due to Peter Macdonald)



# Gaussian Mixture Models

$$F(x) = w_1 F_1(x) + (1 - w_1) F_2(x), \text{ where } F_i(x) = \mathcal{N}(\mu_i, \sigma_i^2, x)$$

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Pearson invented the **method of moments**, to attack this problem...

# Pearson's Sixth Moment Test

**Claim**

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# Provable Guarantees?

In Contributions to the Mathematical Theory of Evolution (attributed to George Darwin):

> *"Given the probable error of every ordinate of a frequency curve, what are the probable errors of the elements of the two normal curves into which it may be dissected?"*

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- Are the parameters of a mixture of two Gaussians uniquely determined by its moments?
- Are these polynomial equations robust to errors?

# A View from Theoretical Computer Science

Suppose our goal is to **provably** learn the parameters of each component within an additive $\epsilon$:

Goal

*Output a mixture* $\widehat{F} = \widehat{w}_1 \widehat{F}_1 + \widehat{w}_2 \widehat{F}_2$ *so that there is a permutation* $\pi : \{1, 2\} \rightarrow \{1, 2\}$ *and for* $i = \{1, 2\}$

$$|w_i - \widehat{w}_{\pi(i)}|, |\mu_i - \widehat{\mu}_{\pi(i)}|, |\sigma_i^2 - \widehat{\sigma}^2_{\pi(i)}| \leq \epsilon$$

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Is there an algorithm that takes poly$(1/\epsilon)$ samples and runs in time poly$(1/\epsilon)$?

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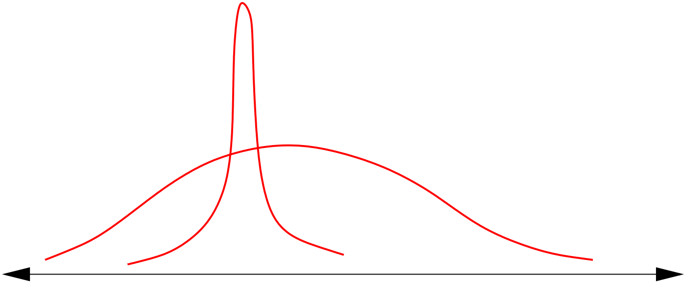
# Identifiability through the Tails



**Approach:** Find the parameters of the component with largest variance (it dominates the behavior of $F(x)$ at infinity)

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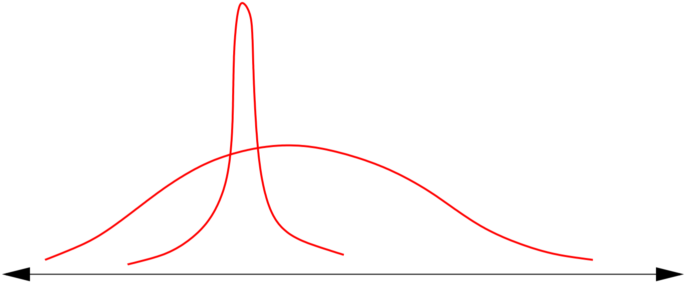

**Approach:** Find the parameters of the component with largest variance (it dominates the behavior of $F(x)$ at infinity); subtract it off and continue

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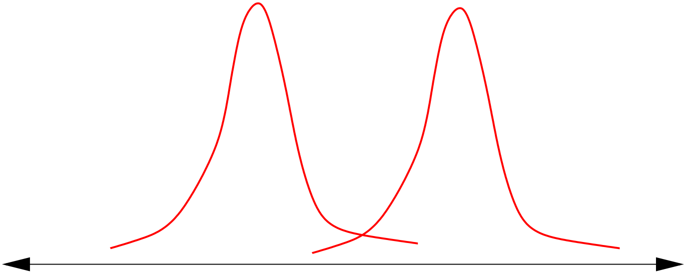

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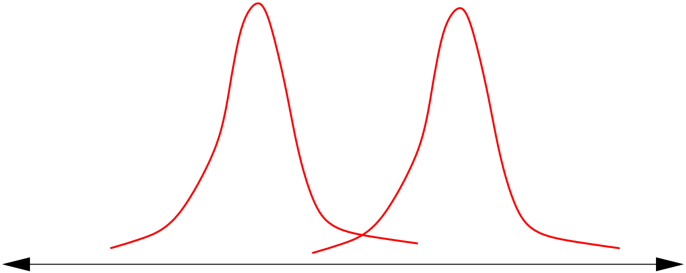

**Approach:** Cluster samples based on which component generated them; output the empirical mean and variance of each cluster

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  assumes almost **non-overlapping** components

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**[Kalai, Moitra, Valiant]** (studies $n$-dimensional version too):

- Reduce to the one-dimensional case
- Analyze Pearson's sixth moment test (with noisy estimates)

# Our Results

Suppose $w_1 \in [\epsilon^{10}, 1 - \epsilon^{10}]$ and $\int |F_1(x) - F_2(x)| dx \geq \epsilon^{10}$

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See also **[Moitra, Valiant]** and **[Belkin, Sinha]** for mixtures of $k$ Gaussians

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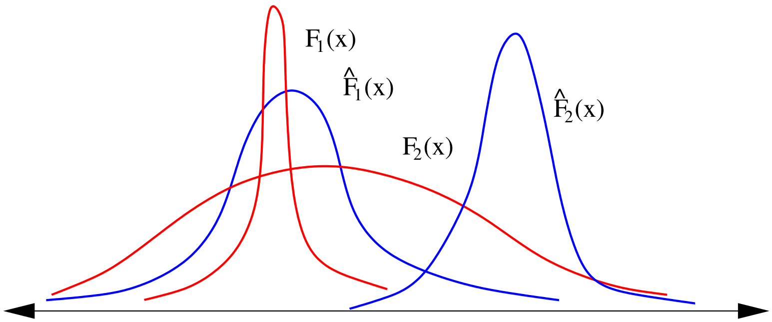
*Do any two different mixtures $F$ and $\widehat{F}$ differ on at least one of the first six moments?*

# Method of Moments



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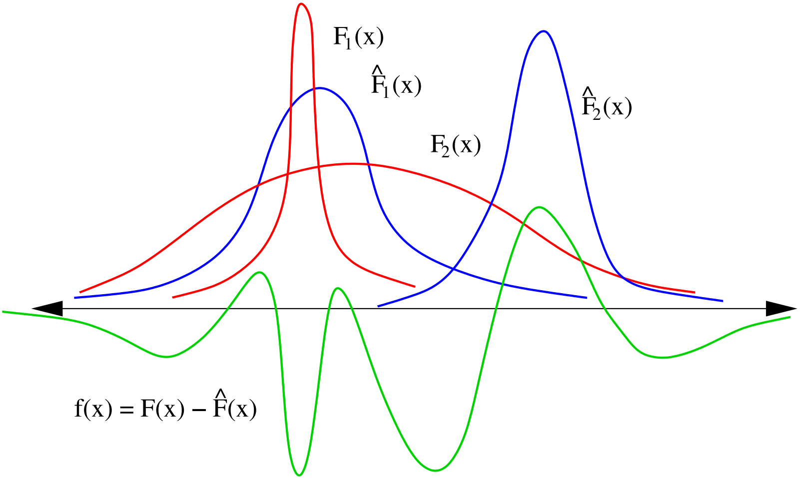

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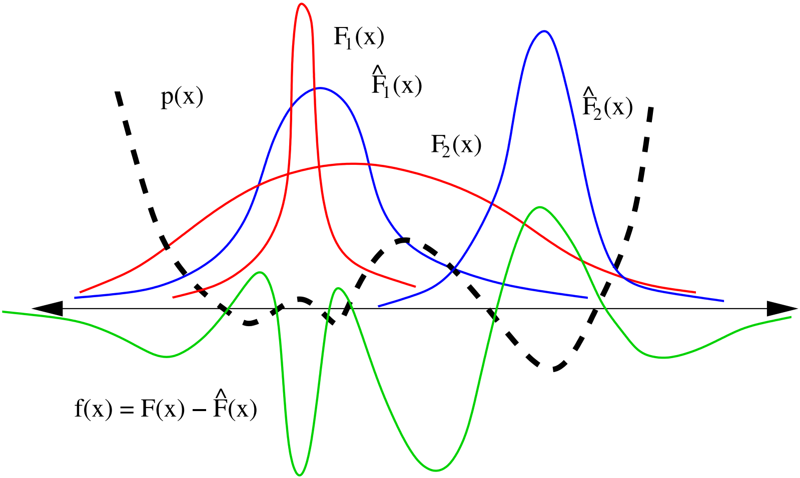

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So $\exists_{r \in \{1,2,\ldots,6\}}$ such that $|M_r(F) - M_r(\widehat{F})| > 0$

Our goal is to prove the following:

### Proposition

*If* $f(x) = \sum_{i=1}^{k} \alpha_i \mathcal{N}(\mu_i, \sigma_i^2, x)$ *is not identically zero,* $f(x)$ *has at most* $2k - 2$ *zero crossings (*$\alpha_i$*'s can be negative).*

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We will do it through properties of the **heat equation**

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# The Heat Equation

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*If the initial heat distribution on a one-dimensional infinite rod* $(\kappa)$ *is* $f(x) = f(x, 0)$ *what is the heat distribution* $f(x, t)$ *at time* $t$*?*

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Alternatively, this is called a **convolution**:

$$f(x, t) = \int_{z=-\infty}^{\infty} f(x + z)\mathcal{N}(0, \sigma^2, z)dz = f(x) * \mathcal{N}(0, \sigma^2)$$

# The Key Facts

**Theorem (Hummel, Gidas)**

*Suppose $f(x) : \mathbb{R} \to \mathbb{R}$ is analytic and has $N$ zeros. Then*

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**Fact**

$$\mathcal{N}(\mu_1, \sigma_1^2, x) * \mathcal{N}(\mu_2, \sigma_2^2, x) = \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2, x)$$

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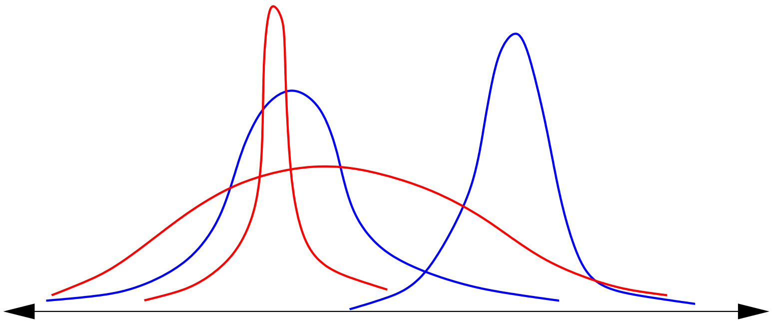
Start with $k = 3$ (at most 4 zero crossings),

Let's prove it for $k = 4$ (at most 6 zero crossings)

......

# Bounding the Number of Zero Crossings



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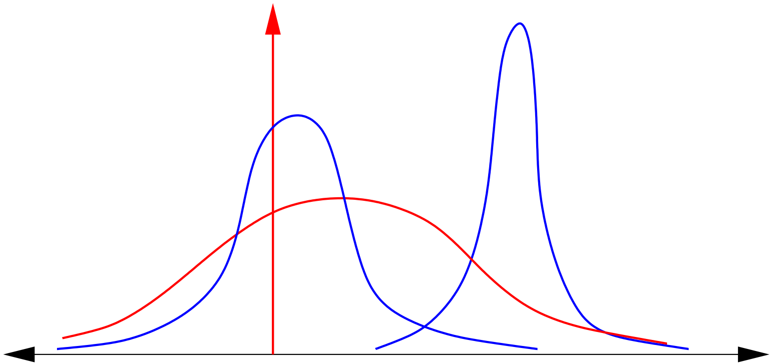

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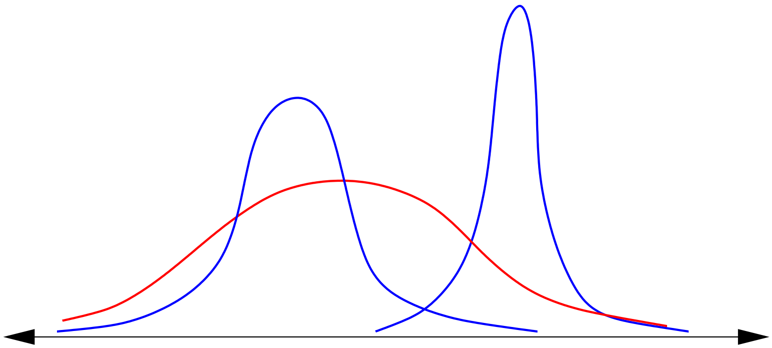

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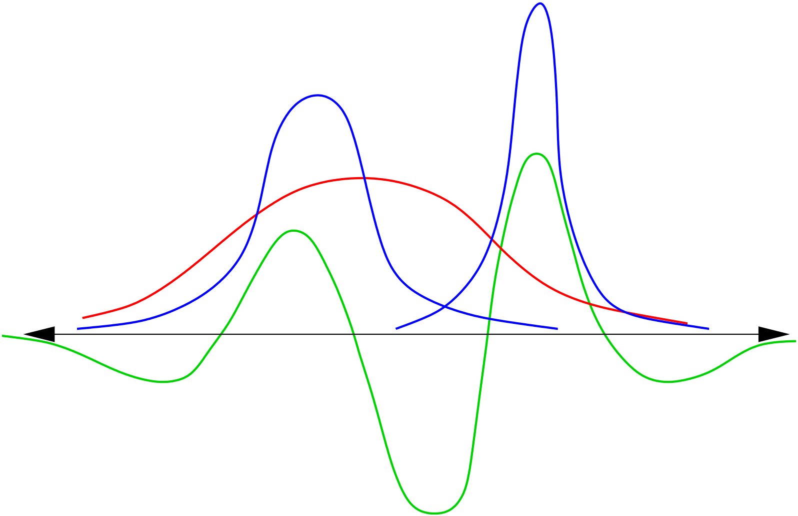

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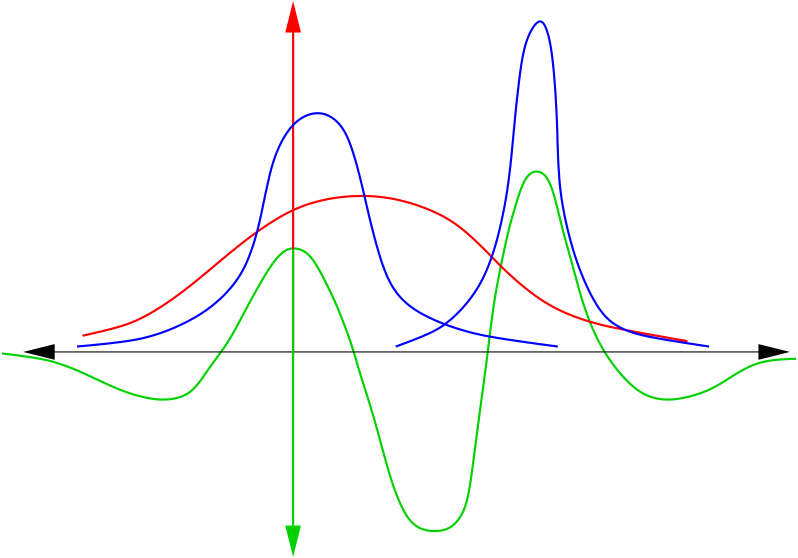

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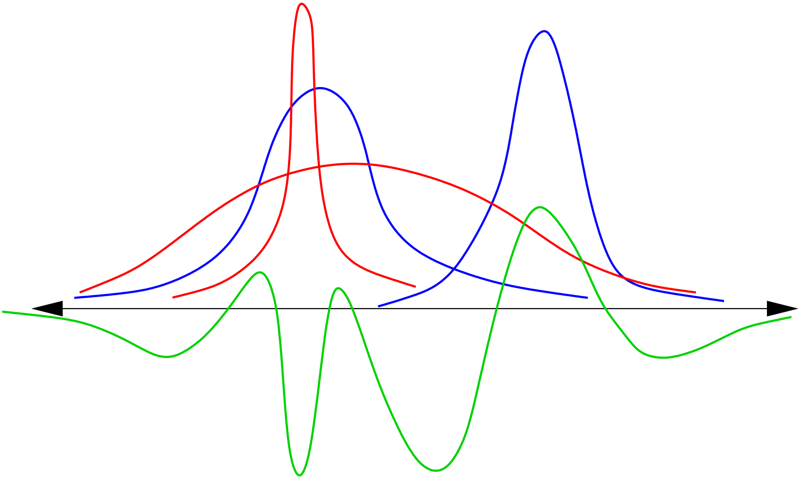

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**Claim**

*Let $\theta$ be the true parameters; then the only solutions to*

$$\left\{\widehat{\theta} \in \Theta | M_r(\widehat{\theta}) = M_r(\theta) \text{ for } r = 1, 2, ...6\right\}$$

*are $(w_1, \mu_1, \sigma_1, \mu_2, \sigma_2)$ and the relabeling $(1 - w_1, \mu_2, \sigma_2, \mu_1, \sigma_1)$*

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Are these equations stable, when we are given **noisy** estimates?

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$$\sum_{i=1}^{2} |w_i - \widehat{w}_{\pi(i)}| + |\mu_i - \widehat{\mu}_{\pi(i)}| + |\sigma_i^2 - \widehat{\sigma}^2_{\pi(i)}| \leq \epsilon$$

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Hence, close enough estimates for the first six moments guarantee that the parameters are close too!

# A Univariate Learning Algorithm

Our algorithm:

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And $\widehat{\theta}$ must be close to $\theta$, because solutions to this system of polynomial equations are **stable**

# Summary and Discussion

- Here we gave the first efficient algorithms for learning mixtures of Gaussians with provably minimal assumptions

**Key words:** method of moments, polynomials, heat equation

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- Here we gave the first efficient algorithms for learning mixtures of Gaussians with provably minimal assumptions

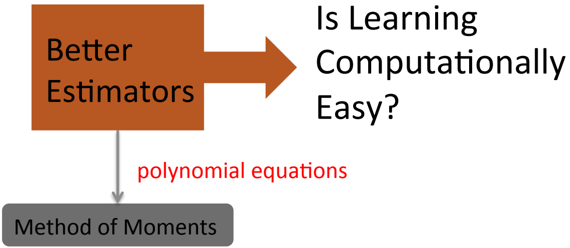
**Key words:** method of moments, polynomials, heat equation

- **Computational intractability** is everywhere in machine learning/statistics
- Currently, most approaches are **heuristic** and have no provable guarantees
- Can we design new algorithms for some of the fundamental problems in these fields?

# My Work

Is Learning Computationally Easy?

# My Work




e.g. mixtures of Gaussians

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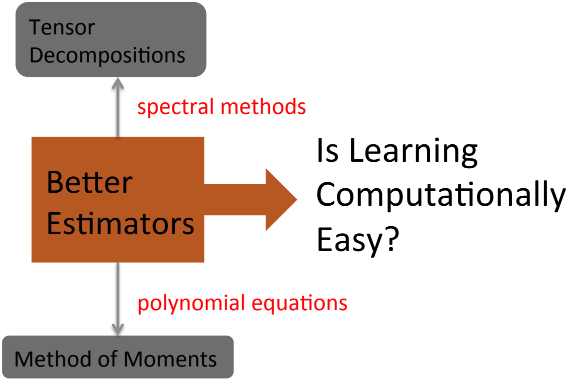


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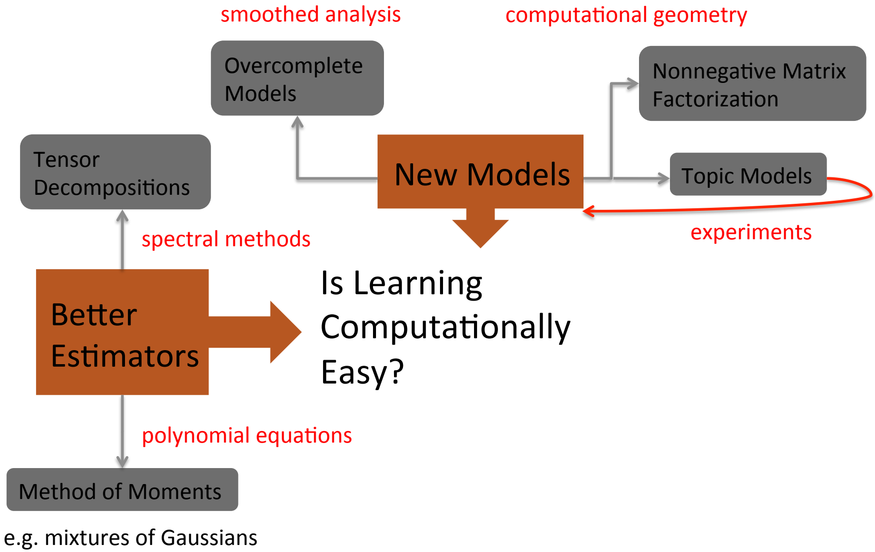
smoothed analysis
computational geometry
Overcomplete Models
Nonnegative Matrix Factorization
Tensor Decompositions
New Models
Topic Models
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spectral methods
Is Learning Computationally Easy?
Better Estimators
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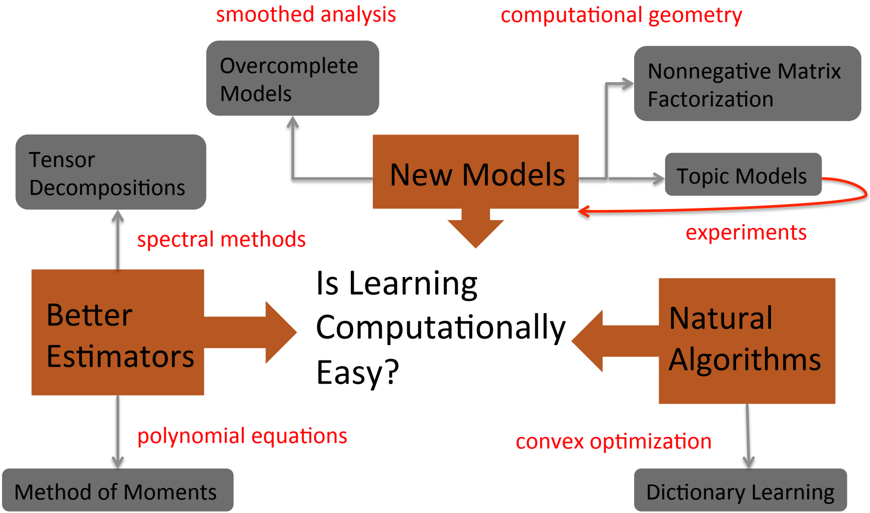


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Thanks!