Honeycombs and Sums of Hermitian Matrices, Revisited

Ankur Moitra (MIT)

SPUR Lecture Series, July 29th

based on joint work with Alex Postnikov (MIT) and
Dora Woodruff (MIT)

A FUNDAMENTAL QUESTION

Hermann Weyl

Weyl (1912): Given the eigenvalues of Hermitian matrices A and B, what can we say about the eigenvalues of their sum A + B?

A FUNDAMENTAL QUESTION

In particular, suppose I tell you the eigenvalues of A and B are

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \text{ and } \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$$

respectively...

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respectively. What can we say about the eigenvalues

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$$

of A + B?

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More exotic inequalities too

A MASTER ANSWER?

Alfred Horn

Horn (1962): Recursive algorithm to generate tons of valid inequalities…

A MASTER ANSWER?

Alfred Horn

Horn (1962): Recursive algorithm to generate tons of valid inequalities, but are they sufficient?

A MASTER ANSWER?

Horn’s inequalities all have the form

$$\sum_{i \in U} \nu_i \leq \sum_{j \in S} \lambda_j + \sum_{k \in T} \mu_k$$

for **admissible** triples (S, T, U)

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Horn's Conjecture: These inequalities are sufficient too

i.e. if all these exponentially many inequalities hold, there are Hermitian matrices A, B and A+B whose eigenvalues are λ, μ and ν respectively

OUTLINE

Part I: Horn's Problem

- A Celebrated Theorem of Knutson-Tao
- Proof Ingredients

Part II: Honeycombs

- As Graph Embeddings
- As Zero Tension Diagrams

Part III: Defining Axioms

- Proofs of the Axioms
- Building Matrices from Honeycombs, and Vice-Versa

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Theorem [Knutson, Tao '99]: Horn's conjecture is true and moreover the sums of eigenvalues of Hermitian matrices are characterized by **honeycombs**

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(3) **[Speyer]** via Vinnikov curves

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Honeycombs also lead to a linear program for deciding if λ, μ and ν are a **Horn triple**, i.e. A, B and A+B exist

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Honeycombs also lead to a linear program for deciding if λ, μ and ν are a **Horn triple**, i.e. A, B and A+B exist

This is not at all obvious, because Horn just gave a list of exponentially many inequalities

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THE PROOF?

Many excellent surveys and expositions, e.g.

Honeycombs and Sums of Hermitian Matrices

Allen Knutson and Terence Tao

2005 Conant Prize

In 1912 Hermann Weyl [W] posed the following problem: given the eigenvalues of two $n \times n$ Hermitian matrices A and B, how does one determine all the possible sets of eigenvalues of the sum $A + B$? When $n = 1$, the eigenvalue of $A + B$ is of course just the sum of the eigenvalue of A and the eigenvalue of B, but the answer is more complicated in higher dimensions. Weyl's partial answers to this problem have since had many direct applications to perturbation theory, quantum measurement theory, and the spectral theory of self-adjoint operators. The purpose of this article is to describe the complete resolution to this problem, based on recent breakthroughs [Kl], [HR], [KT], [KTW].

To standardize the notation, we shall always write the eigenvalues of an $n \times n$ Hermitian matrix as a weakly decreasing n-tuple $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$ of real numbers. Thus, for instance, the eigenvalues of diag$(3, 2, 5, 3)$ are $(5, 3, 3, 2)$.

To illustrate Weyl's problem, suppose that $n = 2$ and that A, B have eigenvalues $(3, 0)$ and $(5, 0)$ respectively. Then one can easily verify that $A + B$ can have eigenvalues $(8, 0)$ or $(5, 3)$ or, more generally, $(8 - a, a)$ for any $0 \leq a \leq 3$. This turns out to be the complete set of possibilities; $A + B$ cannot have eigenvalues $(9, -1)$ or $(7, 0)$ or $(4, 4)$, etc.

Allen Knutson is professor of mathematics at the University of California Berkeley. His e-mail address is `allenk@math.berkeley.edu`. *His research was partially conducted for the Clay Mathematics Institute.*

Terence Tao is professor of mathematics at the University of California Los Angeles. His e-mail address is `tao@math.ucla.edu`. *He is supported by the Clay Mathematics Institute and by grants from the Sloan and Packard Foundations.*

Let us denote the eigenvalues of A, B, and $A + B$ as λ, μ, and ν respectively; thus λ_2 is the second largest eigenvalue of A, etc. It is fairly easy to obtain necessary conditions on the triple λ, μ, ν. For instance, from the simple observation that the trace of $A + B$ must equal the sum of the traces of A and B, we obtain the condition

$$\nu_1 + \ldots + \nu_n = \lambda_1 + \ldots + \lambda_n + \mu_1 + \ldots + \mu_n. \tag{1}$$

Another immediate constraint is that

$$\nu_1 \leq \lambda_1 + \mu_1, \tag{2}$$

since the largest eigenvalue of $A + B$ is at most the sum of A's and B's individual largest eigenvalues. (Exercise for the reader: Show equality occurs exactly when the same vector is a principal eigenvector for both matrices.) Weyl found a number of similar necessary conditions, such as the statement $\nu_{i+j+1} \leq \lambda_{i+1} + \mu_{j+1}$ whenever $0 \leq i, j, i + j < n$. When $n = 1, 2$ these conditions are both necessary and sufficient; for higher dimensions many other necessary conditions were found by later authors. All of these conditions took the form of homogeneous linear inequalities (e.g., $\nu_1 + \nu_2 \leq \lambda_1 + \lambda_2 + \mu_1 + \mu_2$). These inequalities were generally proven by "minimax" methods, but there did not appear to be a general scheme that would produce a systematic and complete list of these inequalities.

This problem was studied extensively by Alfred Horn [Ho]. Among other things, he showed that a complete set of necessary conditions could be given by (1), together with a list of linear inequalities of the form

$$\nu_{k_1} + \ldots + \nu_{k_r} \leq \lambda_{i_1} + \ldots + \lambda_{i_r} + \mu_{j_1} + \ldots + \mu_{j_r} \tag{3}$$

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February 2001 Notices of the AMS 175

Is there a simple answer to where honeycombs come from?

THE PROOF?

Many deep tools

Sums of Hermitian matrices

THE PROOF?

Many deep tools

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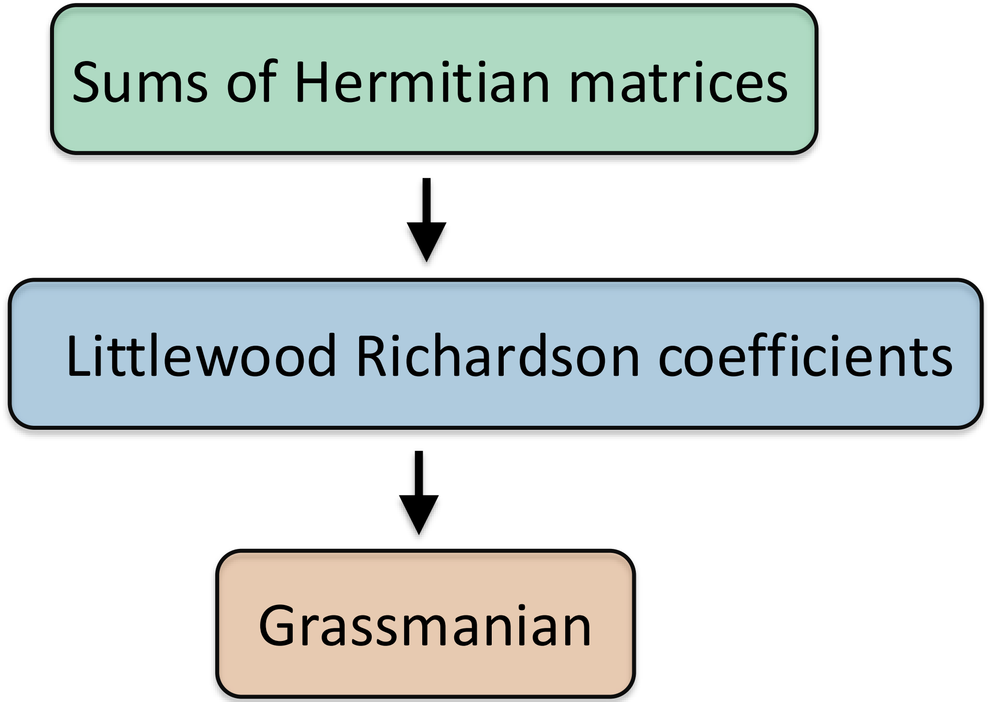
↓

Littlewood Richardson coefficients

A triple (S, T, U) is admissible if and only if the corresponding Littlewood-Richardson coefficient is nonzero

THE PROOF?

Many deep tools



The Littlewood Richardson coefficients are the structure constants for the cohomology ring of a Grassmannian in its Schubert basis

THE PROOF?

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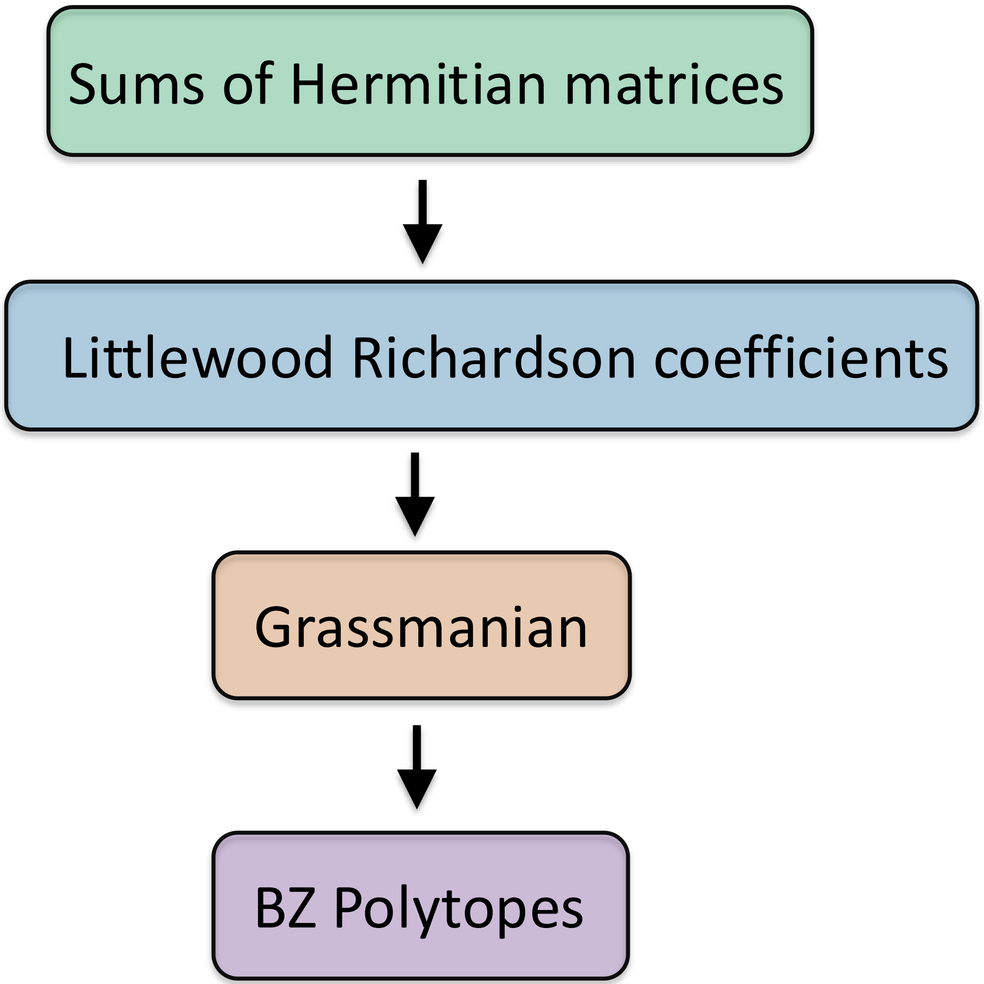

Berenstein-Zelevinsky polytopes are polyhedral models for tensor product multiplicities

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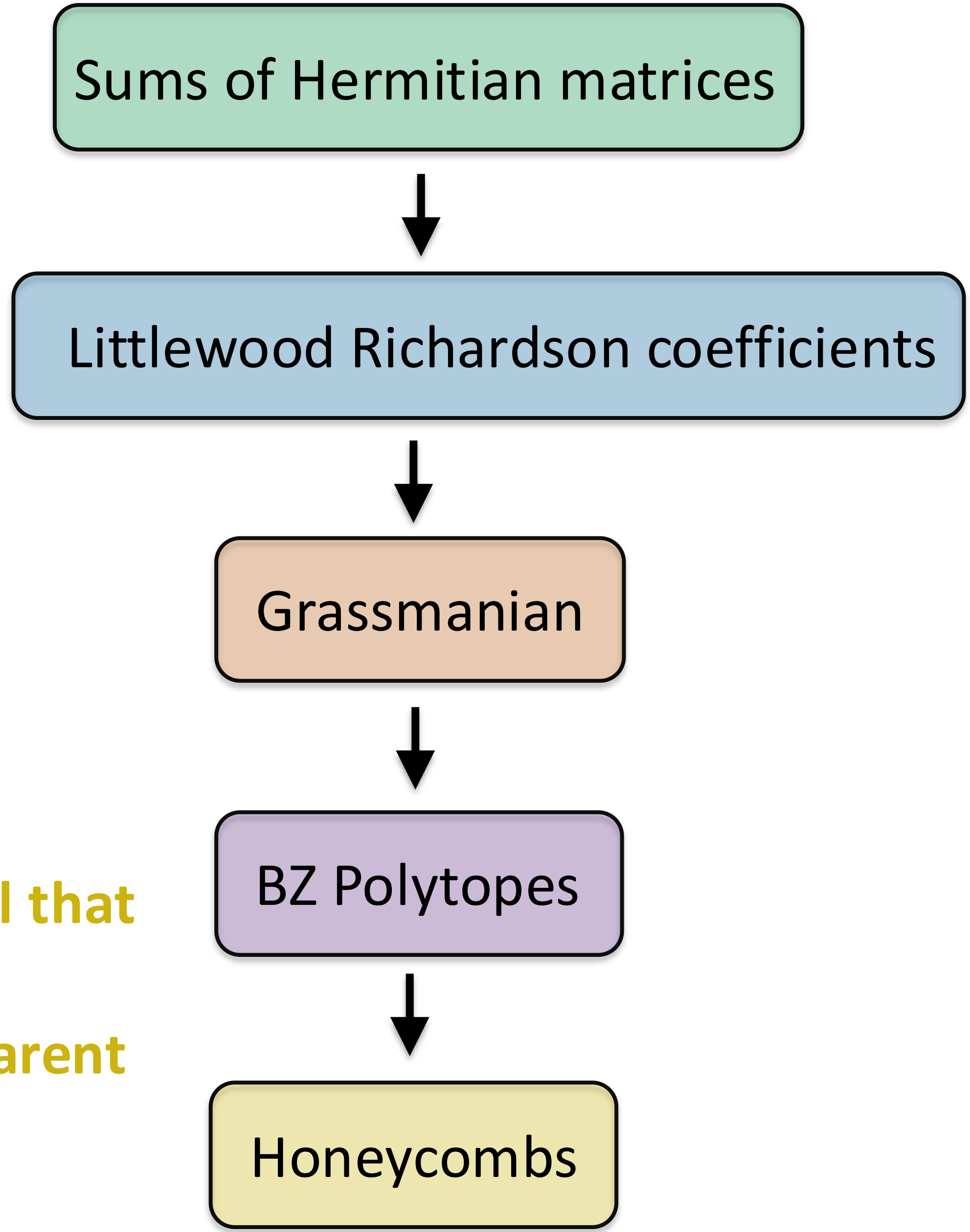

An equivalent model that makes the planar geometry more apparent

THE PROOF?

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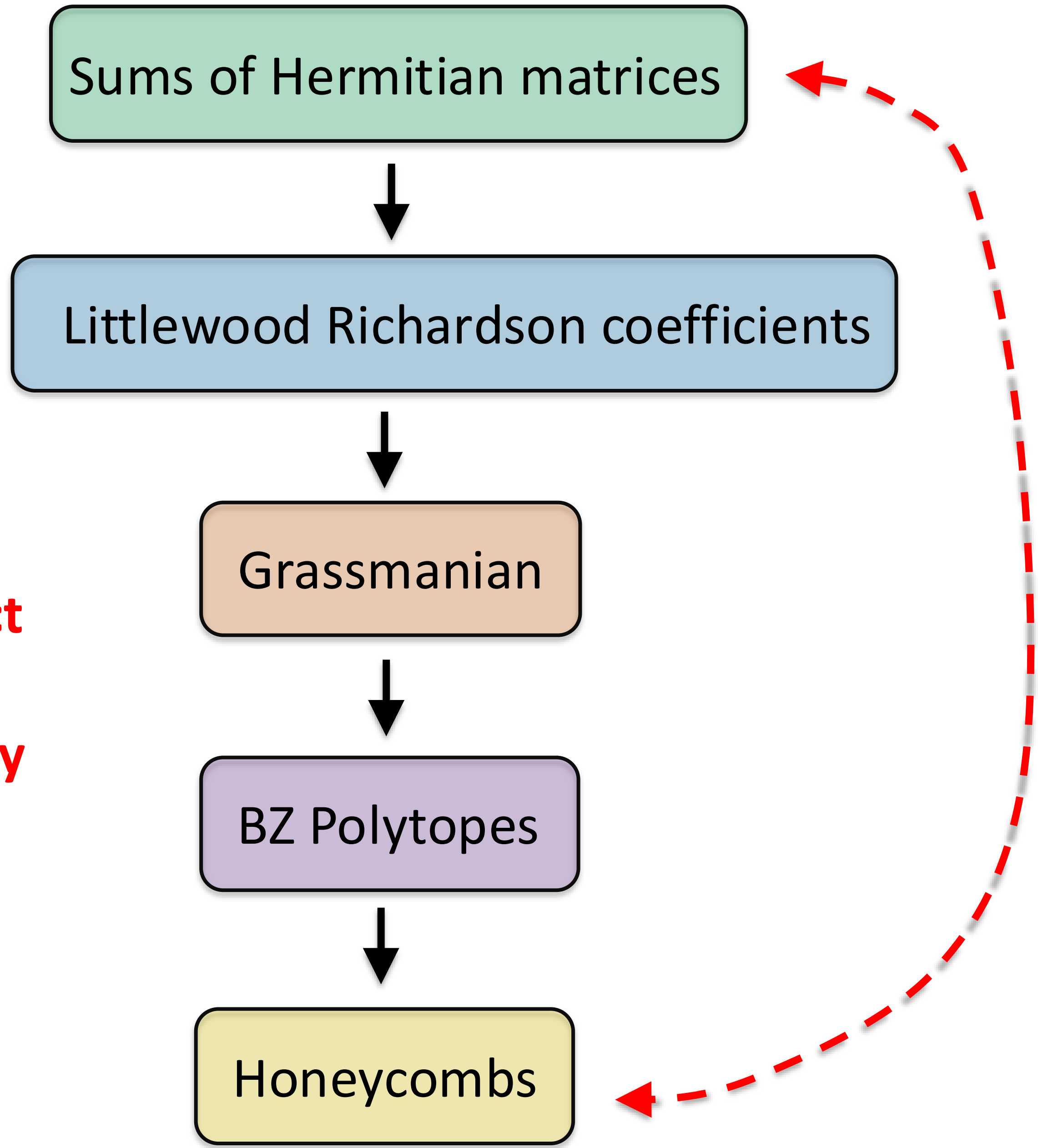

We give a new, direct proof just based on induction + convexity

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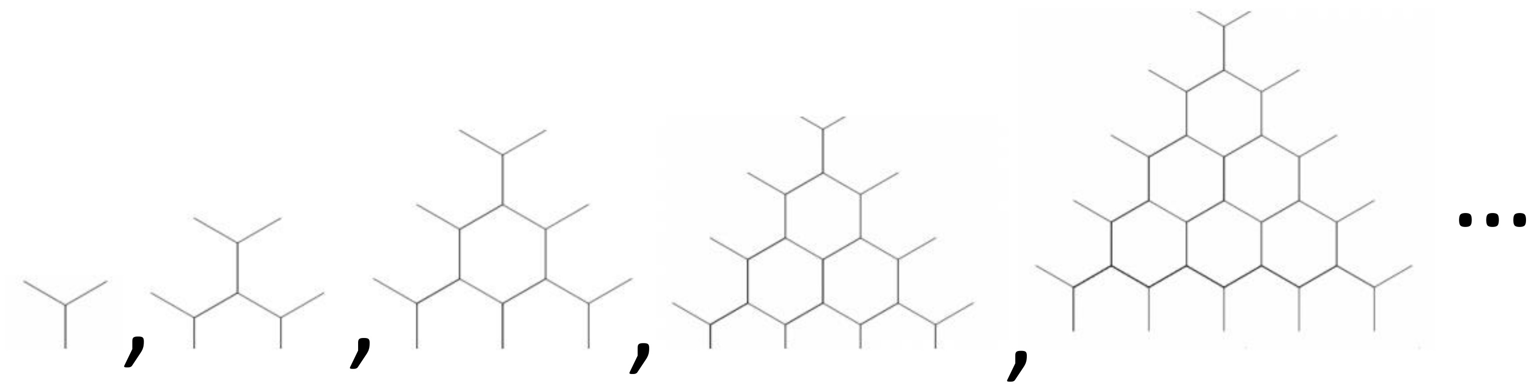

H_1 H_2 H_3 H_4 H_5

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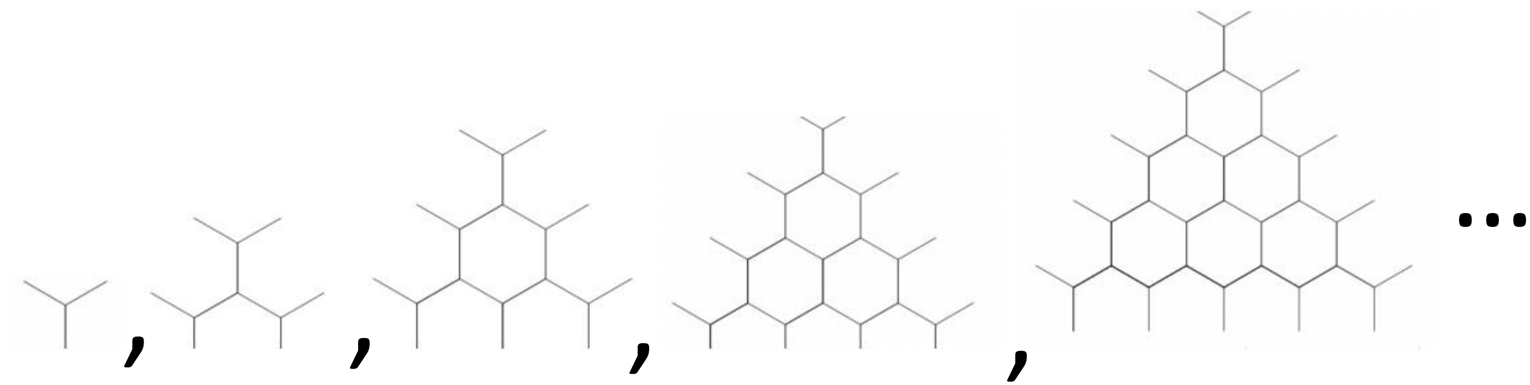

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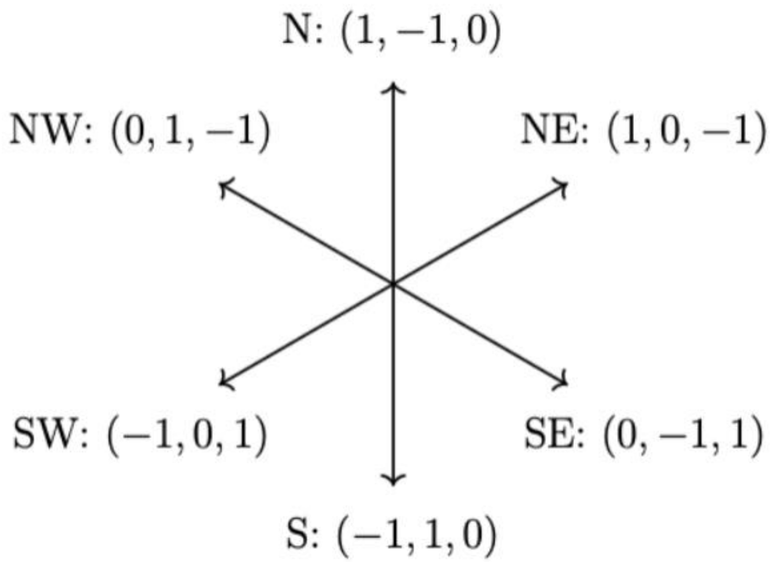
into $\mathbb{R}^3_{\Sigma=0}$ where x + y + z = 0, along **cardinal directions**

HONEYCOMBS

Where the **cardinal directions** are



HONEYCOMBS

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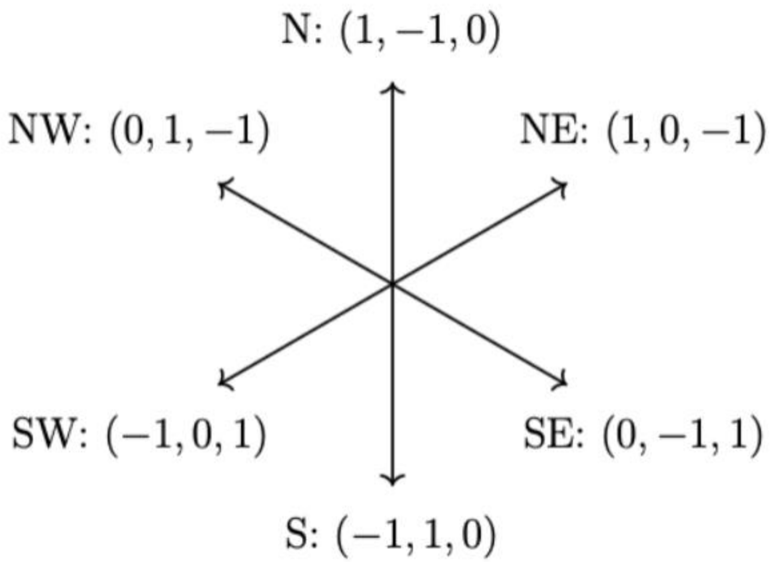

Notice each segment has a coordinate that remains constant

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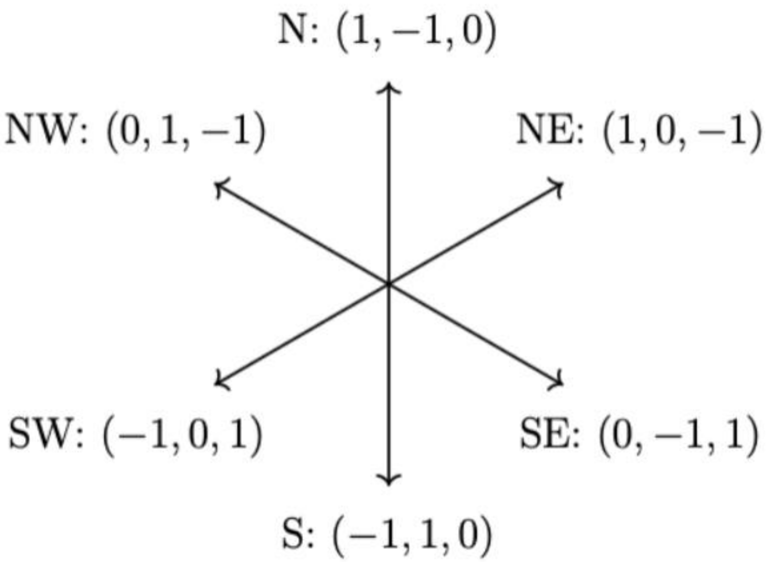

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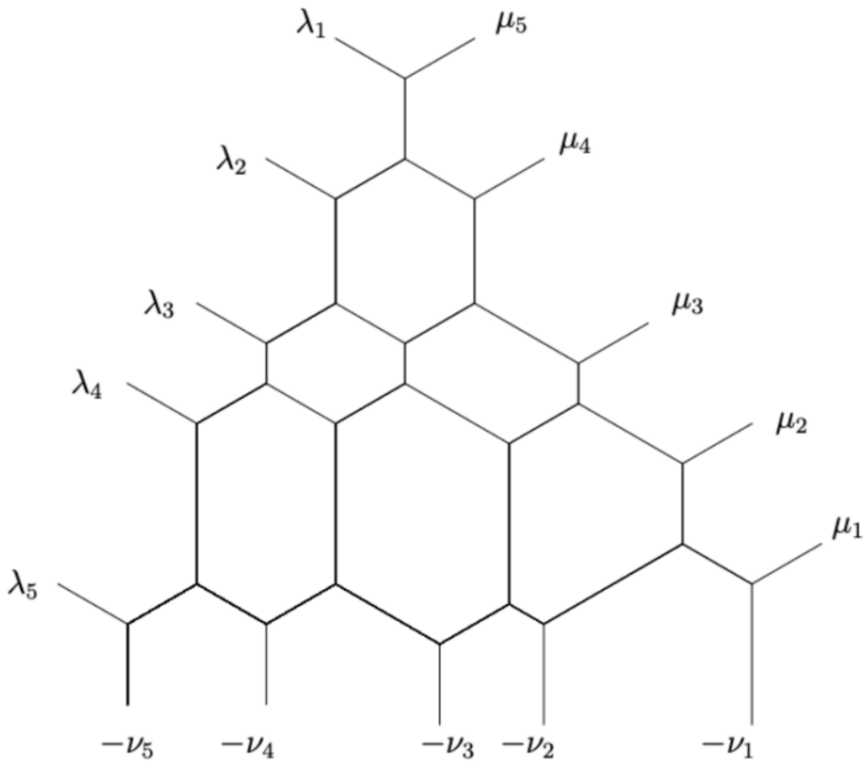
Finally, the 3n boundary rays have coordinates λ, μ and -ν and **each segment has nonnegative length**

HONEYCOMBS

For example



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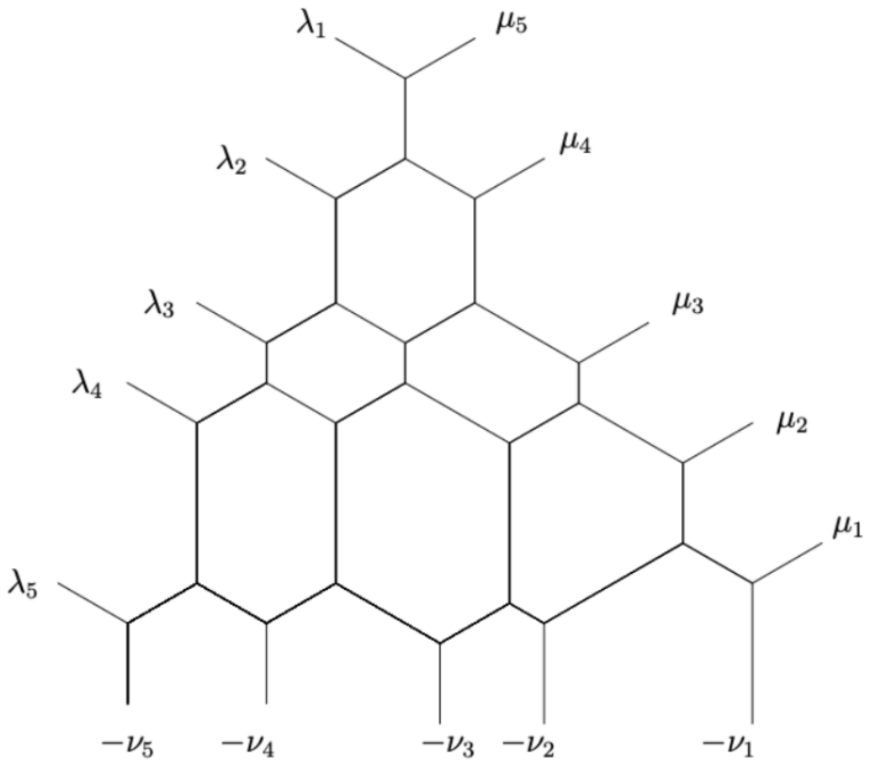

⚠ **Caution:** Edges can have zero length too

HONEYCOMBS

As an aside, this is what leads to a compact linear program:

Nonnegativity constraints are linear inequalties

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There are other ways to define honeycombs

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Definition 2: Diagram with segments and boundary rays and multiplicities so that in every neighborhood around a vertex it the finite union of segments along cardinal directions with **zero tension***…

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***i.e. the sum of directions times multiplicity is zero**

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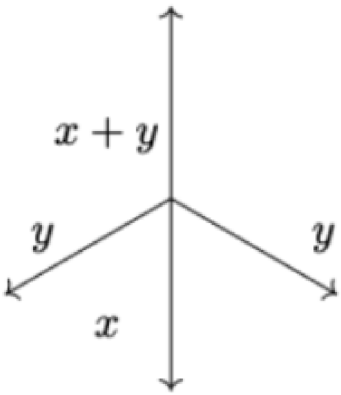

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These two definitions are cryptomorphic

Proposition [Knutson, Tao]: These two definitions of honeycombs are equivalent

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It’s easy to see that vertices in Definition 1 are zero tension

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It's easy to see that vertices in Definition 1 are zero tension

The more interesting direction is, going the other direction, how to resolve vertices with three or more outgoing segments

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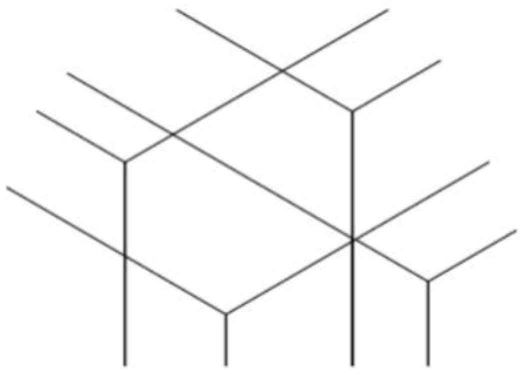

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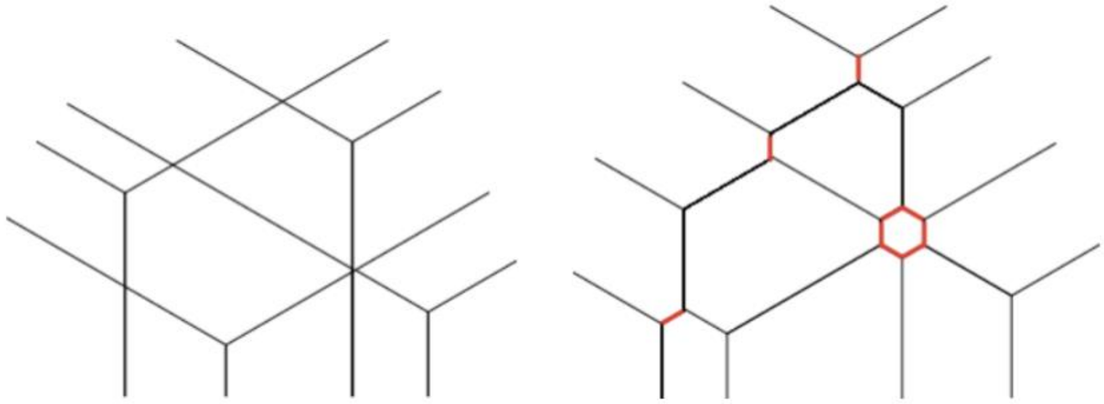

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We distill four key axioms, and show that both honeycombs and Horn triples satisfy them

And from these axioms, we can build one from the other

AN AXIOMATIC CHARACTERIZATION

The four axioms are

(1) **Base Case:** For n = 1 can get any $(\lambda, \mu, \lambda + \mu)$

AN AXIOMATIC CHARACTERIZATION

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(1) **Base Case**

(2) **Direct Sum:** For n and m can get

$$(\lambda \oplus \lambda', \mu \oplus \mu', \nu \oplus \nu')$$

provided that (λ, μ, ν) and (λ', μ', ν') are both possible

AN AXIOMATIC CHARACTERIZATION

The four axioms are

(1) **Base Case**

(2) **Direct Sum**

(3) **Convexity:** For any λ, μ the set of all possible ν is a **convex polytope**

AN AXIOMATIC CHARACTERIZATION

The four axioms are

(1) **Base Case**

(2) **Direct Sum**

(3) **Convexity**

(4) **Vertex Splitting:** If ν is a vertex of the polytope, can find nontrivial $|I| = |J| = |K|$ s.t.

$(\lambda_I, \mu_J, \nu_K)$ and $\left(\lambda_{\bar{I}}, \mu_{\bar{J}}, \nu_{\bar{K}}\right)$ are possible

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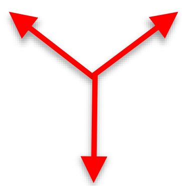	**obvious**

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honeycombs	Horn triples
	$\begin{bmatrix} A & 0 \\ 0 & A' \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B' \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & C' \end{bmatrix}$

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(2) **Direct Sum:** For n and m can get

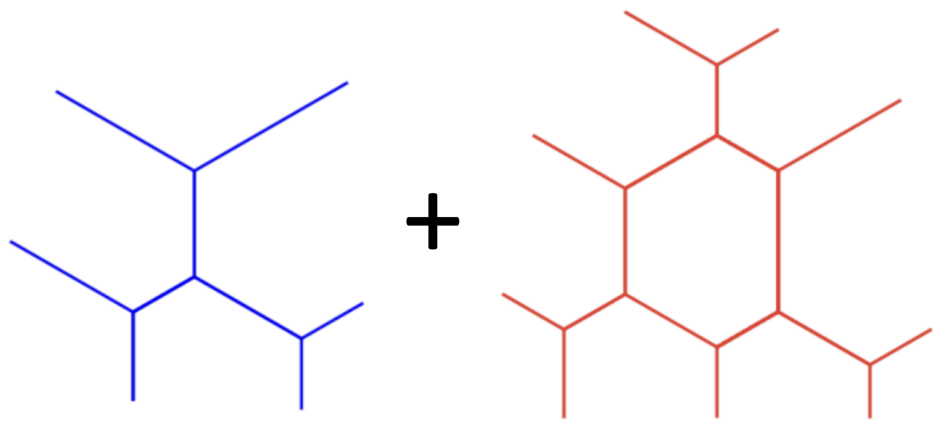
$$(\lambda \oplus \lambda', \mu \oplus \mu', \nu \oplus \nu')$$

provided that (λ, μ, ν) and (λ', μ', ν') are both possible

honeycombs	Horn triples
Definition 2, overlay diagrams	$\begin{bmatrix} A & 0 \\ 0 & A' \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B' \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & C' \end{bmatrix}$

PROOFS

For example



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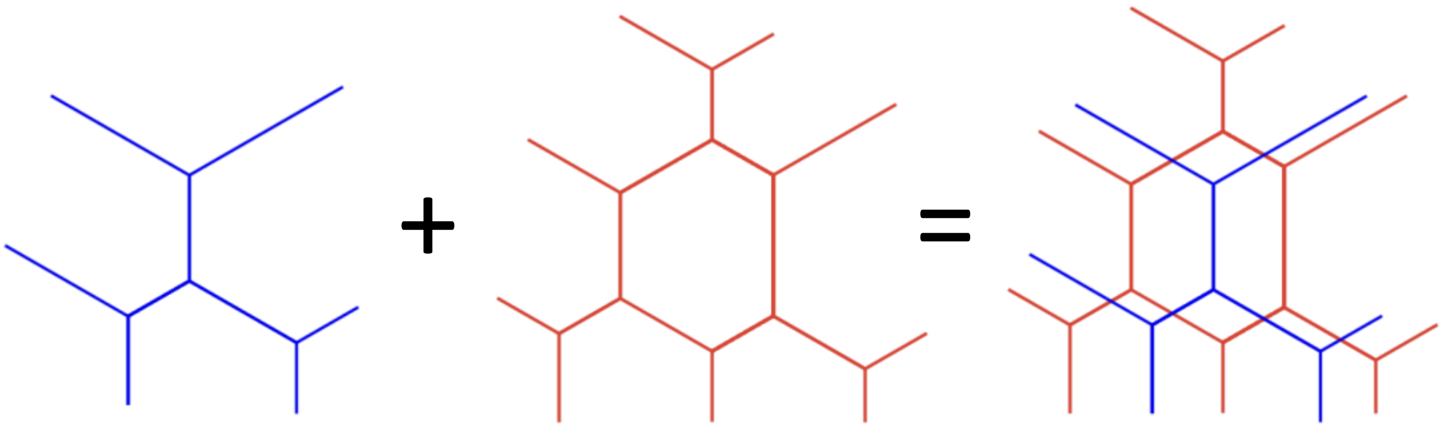

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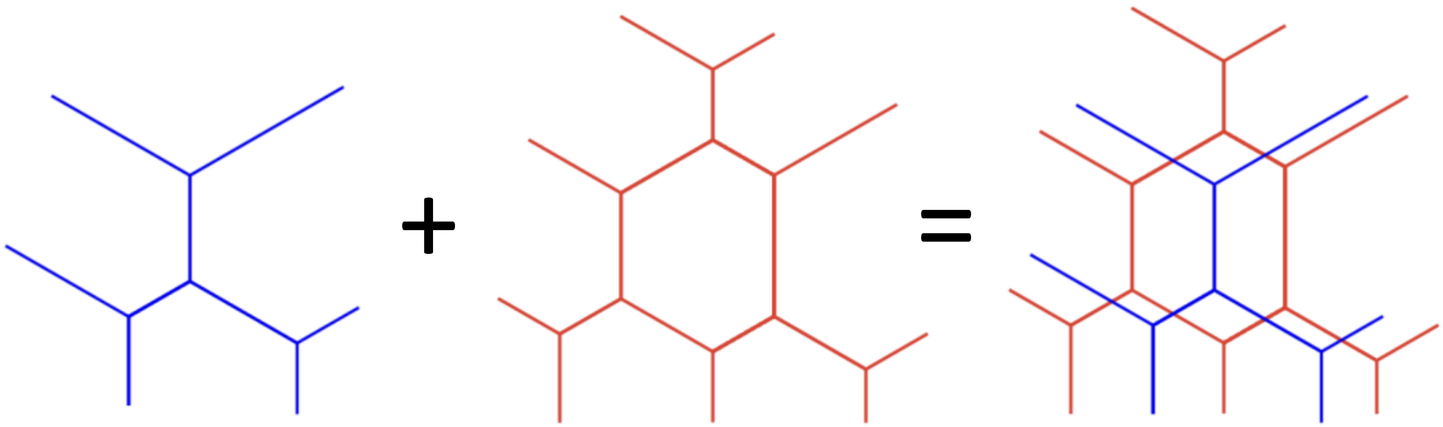

The sum of zero tension vertices is still zero tension

PROOFS

Now let's prove the axioms

(3) **Convexity:** For any λ, μ the set of all possible ν is a **convex polytope**

honeycombs	Horn triples

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honeycombs	Horn triples
Definition 1, explicit inequalities	**Convexity of the moment map Kirwan, Atiyah-Guillemin-Sternberg**

PROOFS

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(4) **Vertex Splitting:** If ν is a vertex of the polytope, can find nontrivial $|I| = |J| = |K|$ s.t.

$$(\lambda_I, \mu_J, \nu_K) \text{ and } \left(\lambda_{\bar{I}}, \mu_{\bar{J}}, \nu_{\bar{K}}\right) \text{ are possible}$$

honeycombs	Horn triples

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honeycombs	Horn triples
New object: Frameworks	**Horn proved for symmetric matrices, can generalize argument**

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Essentially, variations on the definitions where all edges have **positive length**

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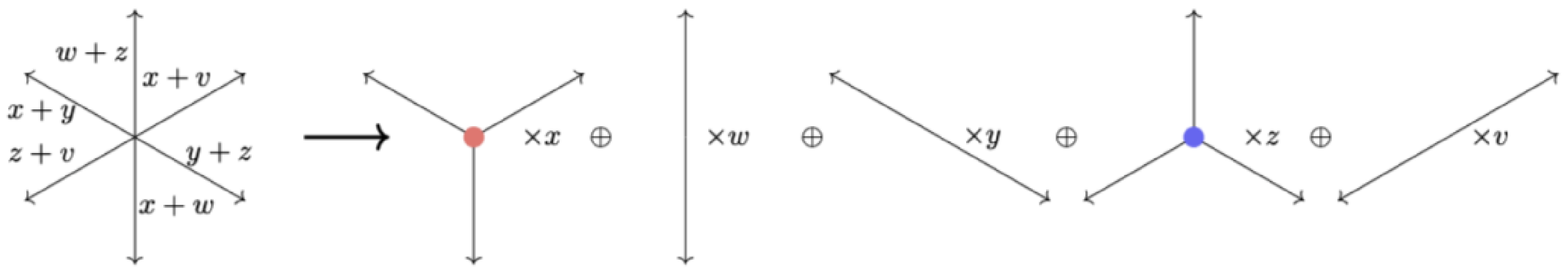

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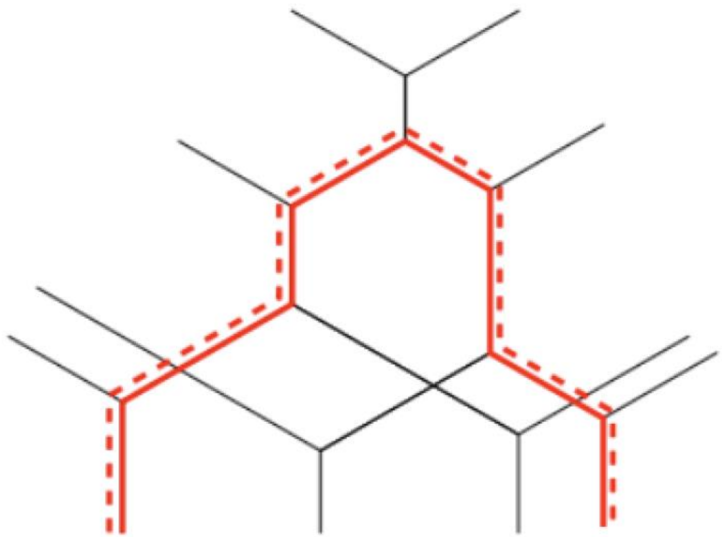

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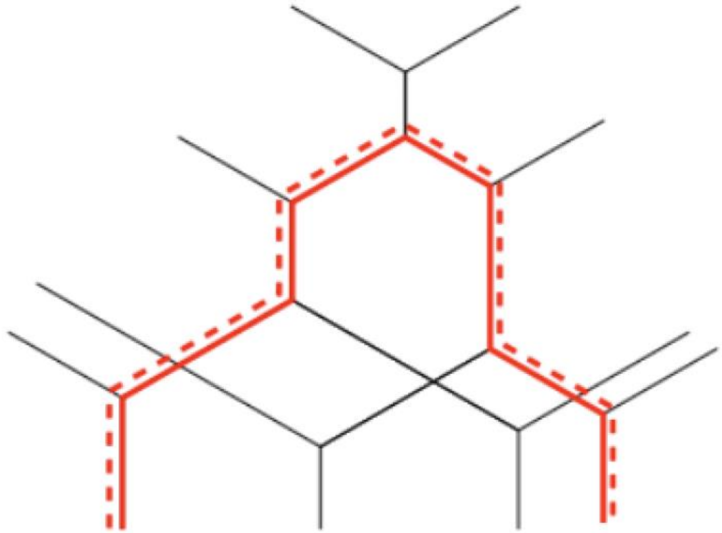

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FRAMEWORKS

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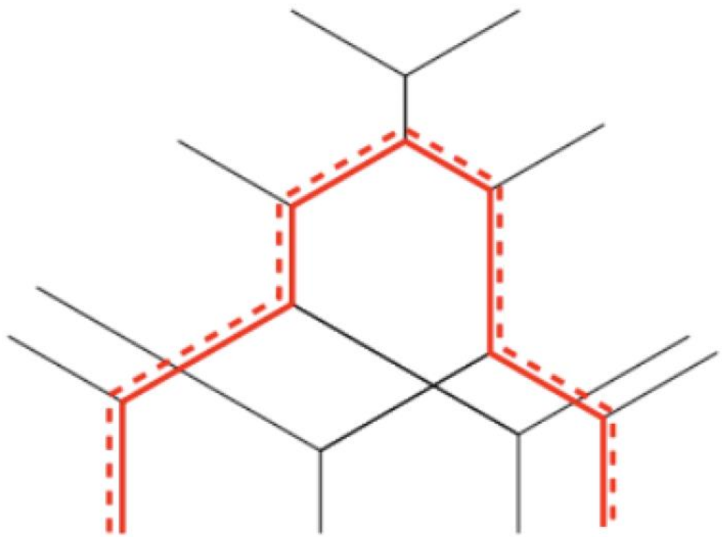

This path can **breathe**, hence honeycomb can’t be a vertex

Now if there are no paths that can breathe, can peel apart the framework to get two frameworks

HORN'S PROOF

Can prove vertex splitting for Horn triples by considering

$$D_\lambda + e^{-T} U^\dagger D_\mu U e^T$$

where T is skew-Hermitian and U is unitary

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Main Idea: Vary the eigenvectors of B and study how the eigenvalues of A + B change

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$$\Downarrow$$

Some stochastic matrix is decomposable, gives block diagonalization

OUTLINE

Part I: Horn's Problem

- A Celebrated Theorem of Knutson-Tao
- Proof Ingredients

Part II: Honeycombs

- As Graph Embeddings
- As Zero Tension Diagrams

Part III: Defining Axioms

- Proofs of the Axioms
- Building Matrices from Honeycombs, and Vice-Versa

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AN INDUCTIVE PROOF

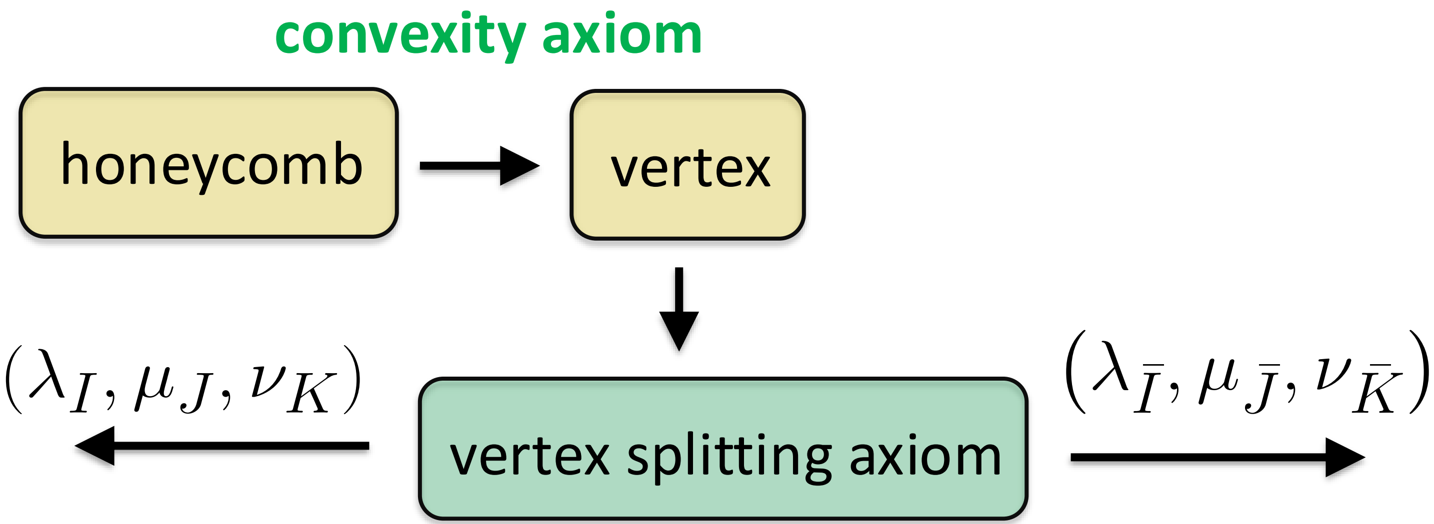
honeycomb

From any honeycomb, how do we build A, B and A+B?

AN INDUCTIVE PROOF

$$\boxed{\text{honeycomb}} \xrightarrow{\text{convexity axiom}} \boxed{\text{vertex}}$$

AN INDUCTIVE PROOF



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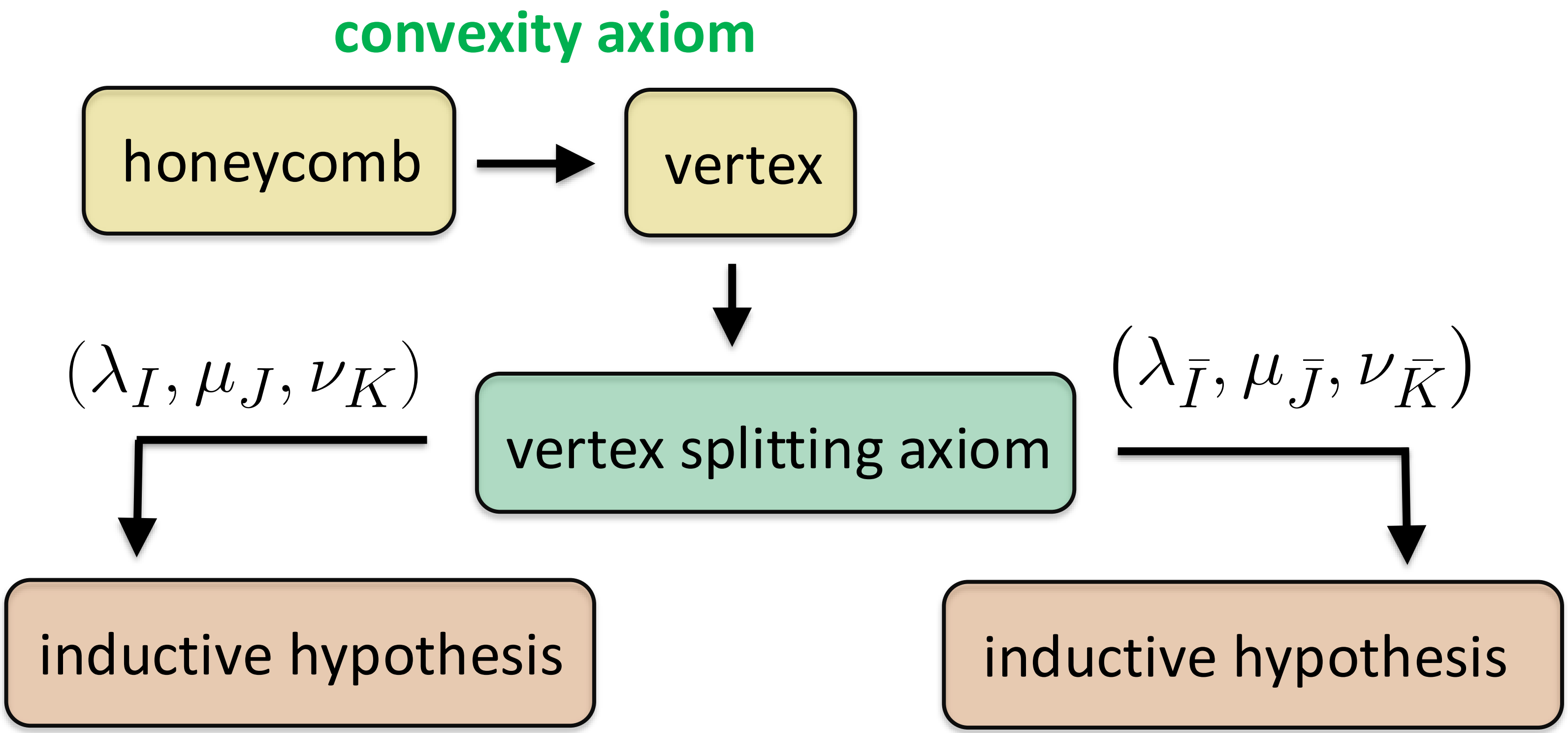

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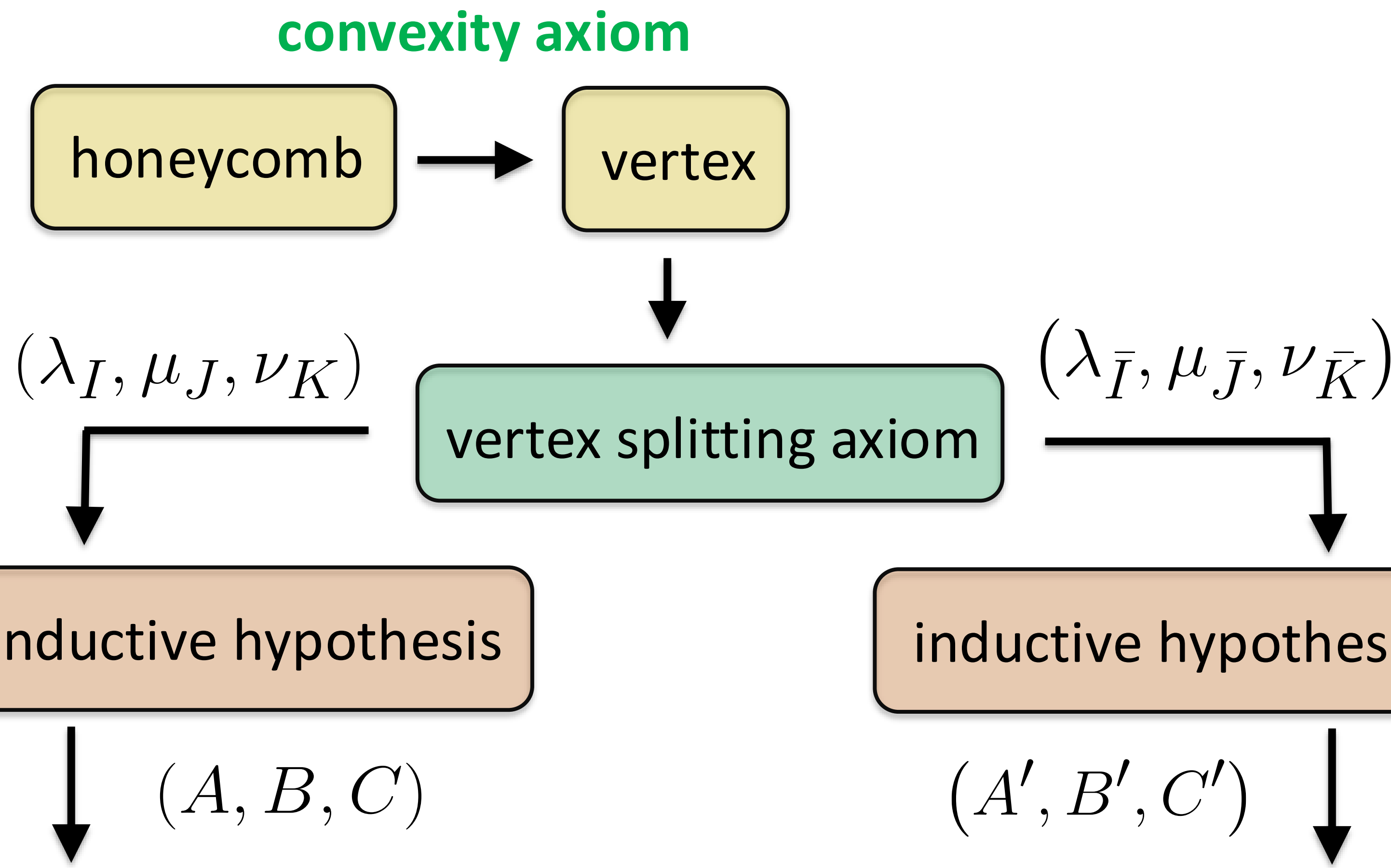

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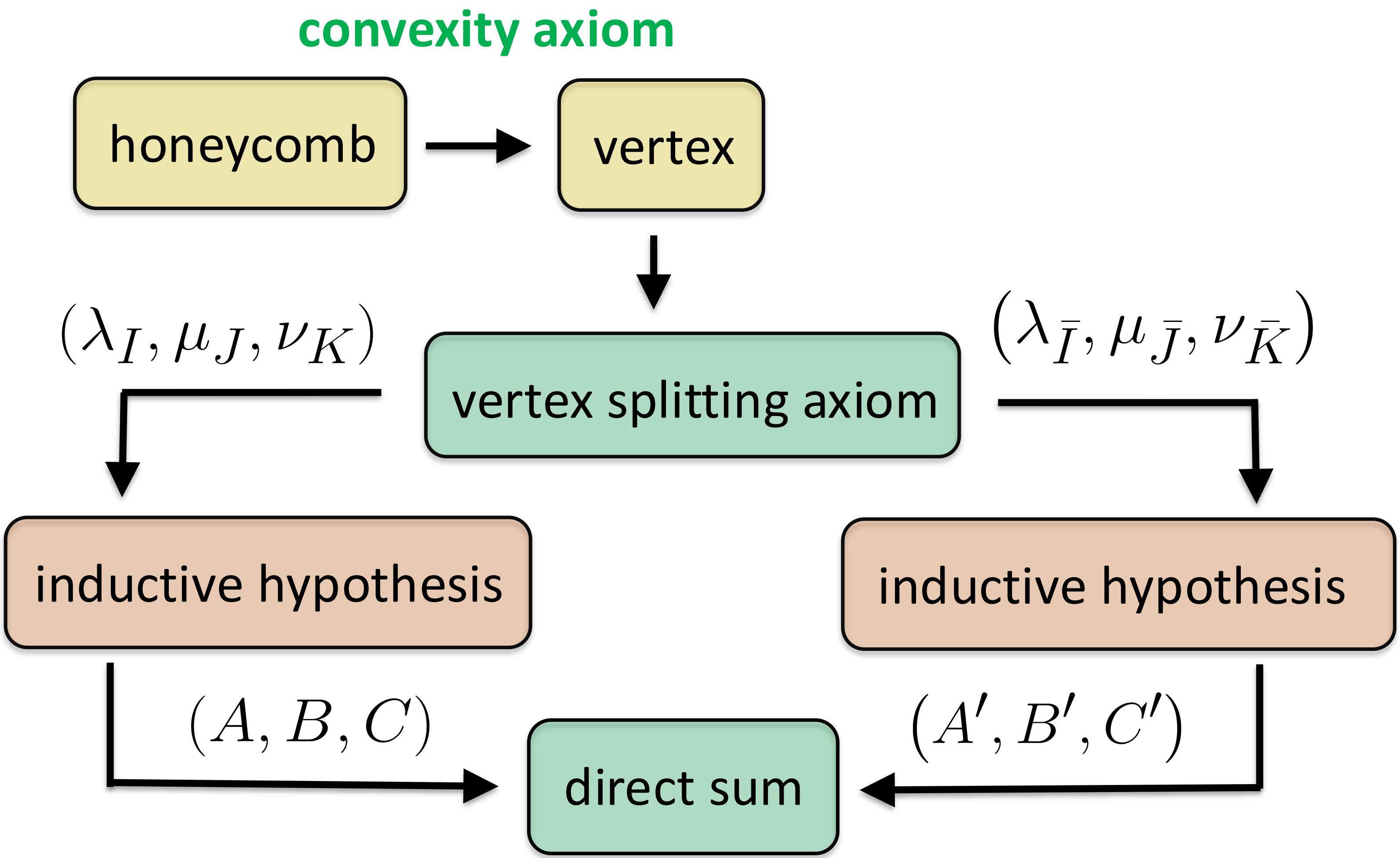

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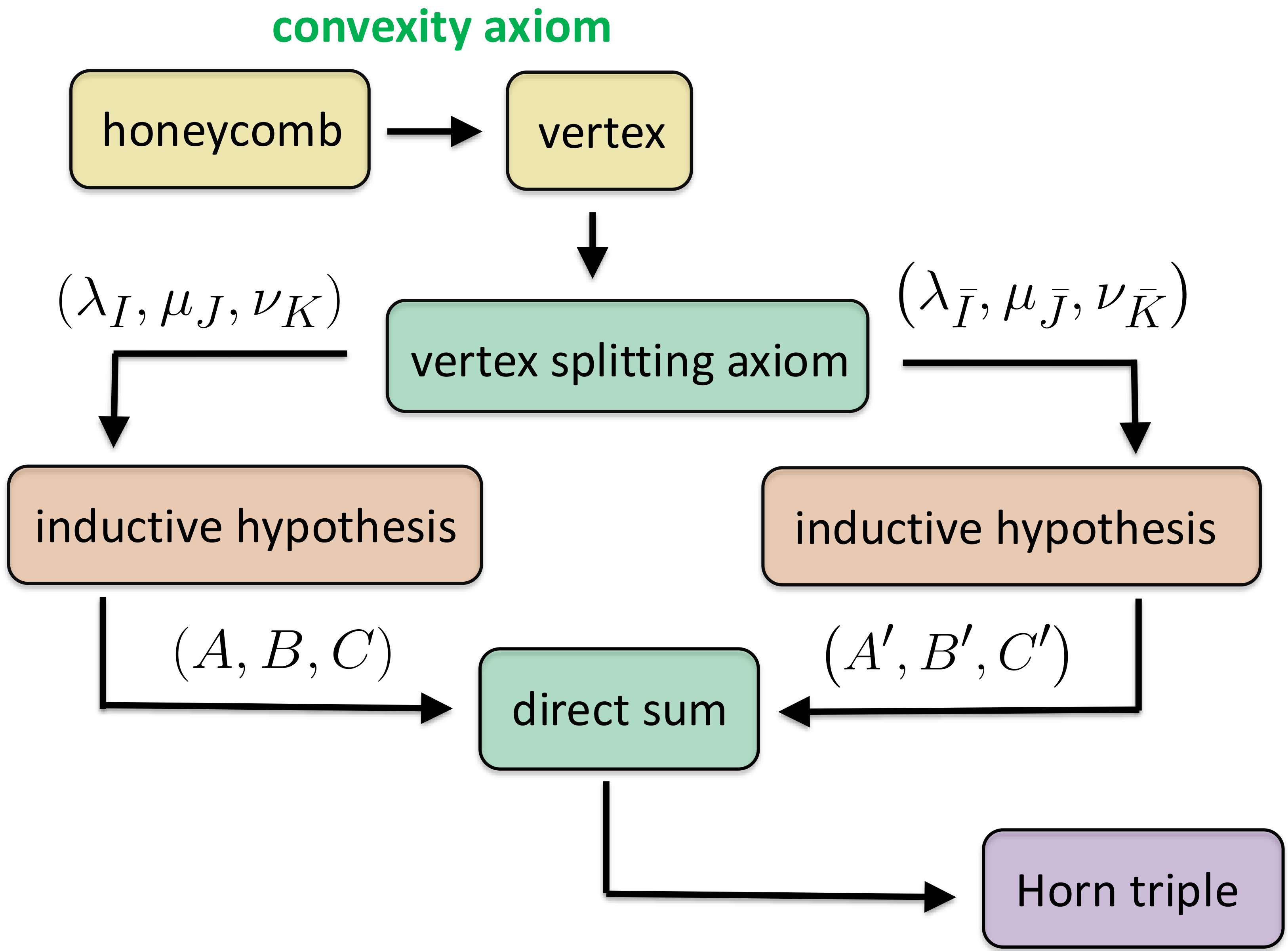
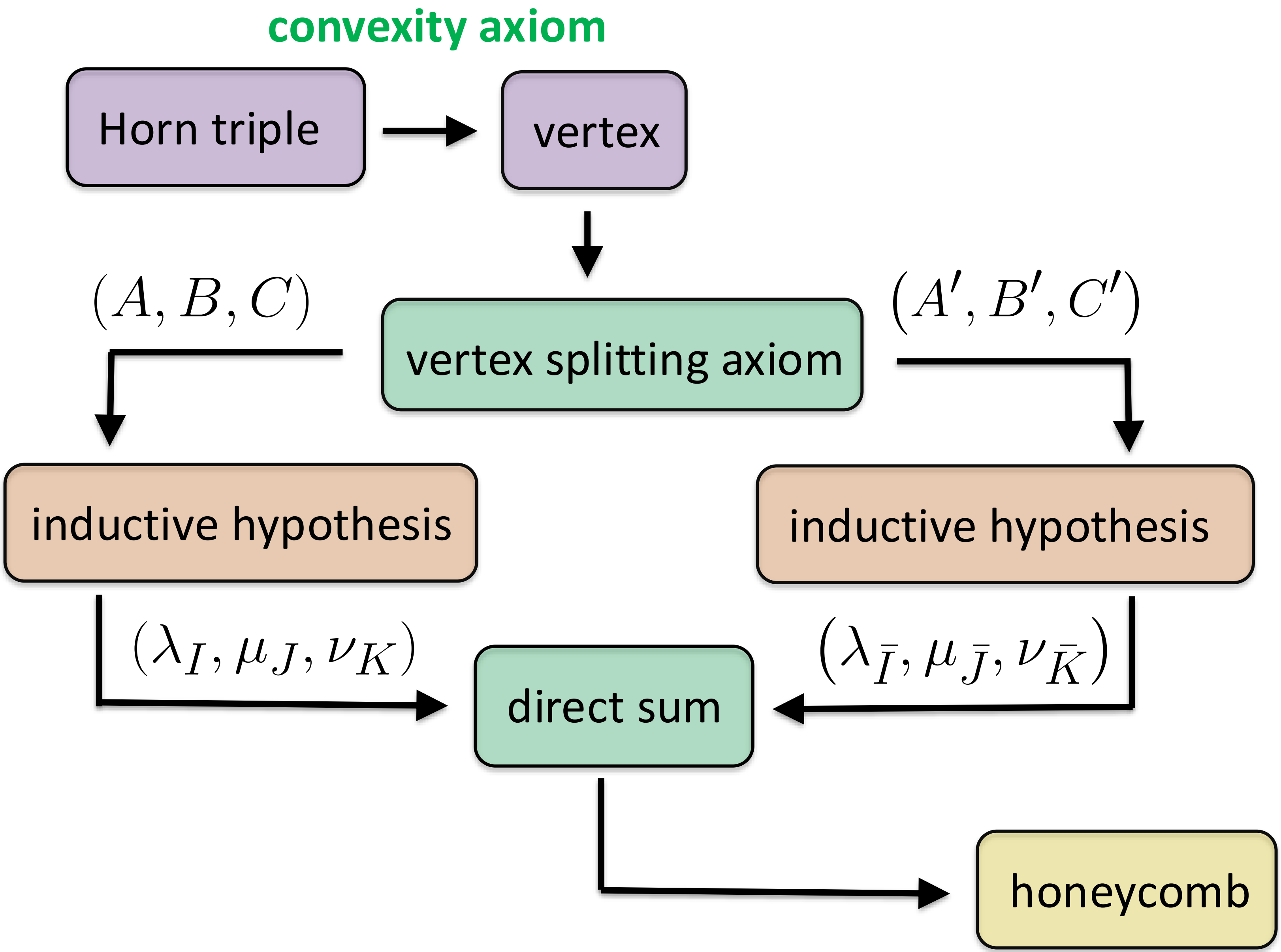


AN INDUCTIVE PROOF

The same proof works in reverse

AN INDUCTIVE PROOF



COMMENTS

This gives a more direct view on where honeycombs come from

“Families of objects satisfying the same defining properties, but in more obvious ways”

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"Families of objects satisfying the same defining properties, but in more obvious ways"

In this way, they are a **model organism**

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In particular, easier to reason at vertices

Summary:

- **Horn's problem**: Given eigenvalues of Hermitian matrices A and B, what can we say about the eigenvalues of A+B?
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- New insight: Honeycombs as **model organisms** that more obviously satisfy same defining axioms

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Thanks! Any Questions?