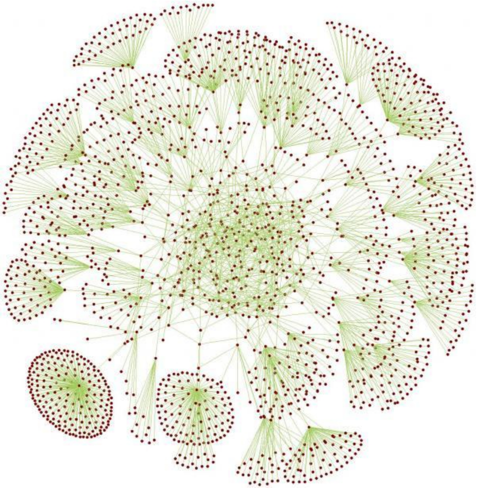
Vertex Sparsification

Ankur Moitra, IAS

February 15th, 2012



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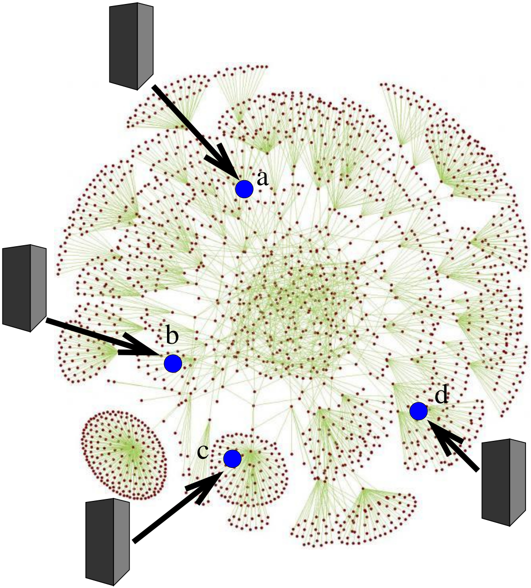

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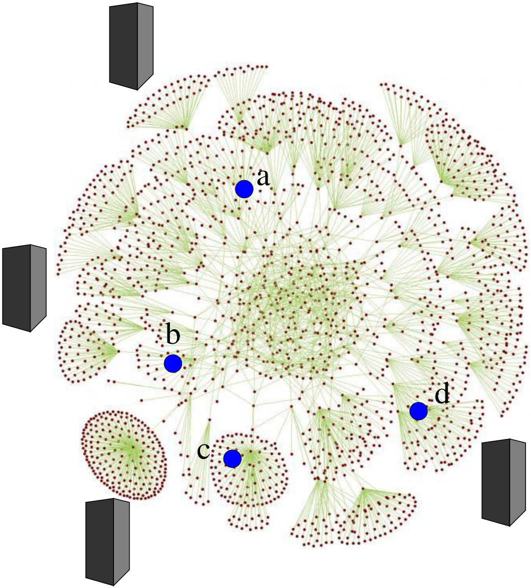

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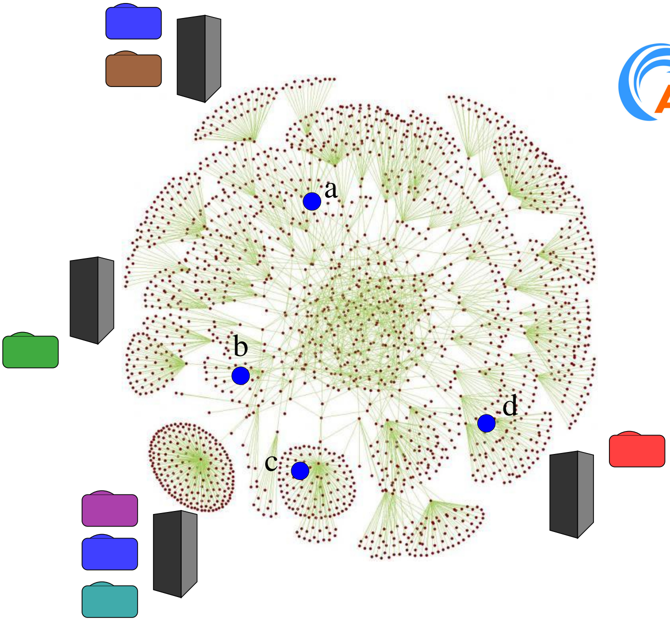

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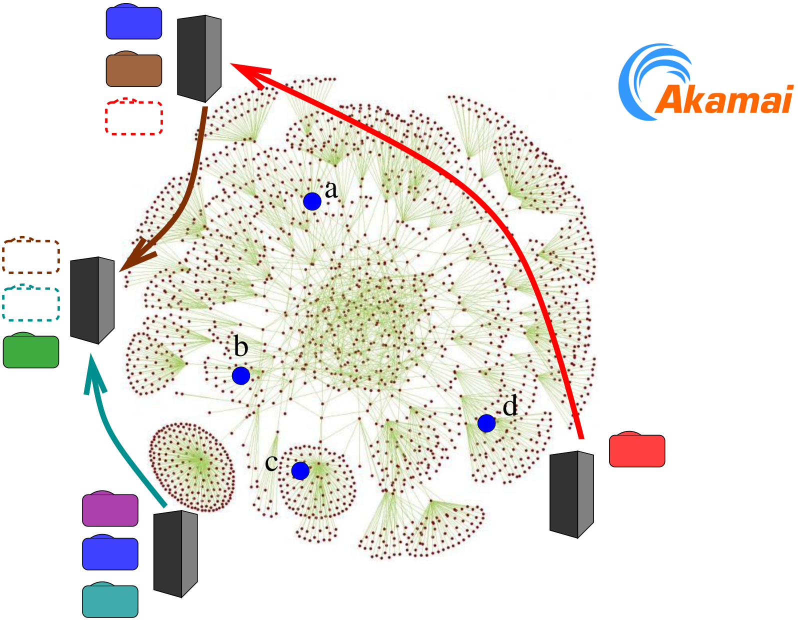

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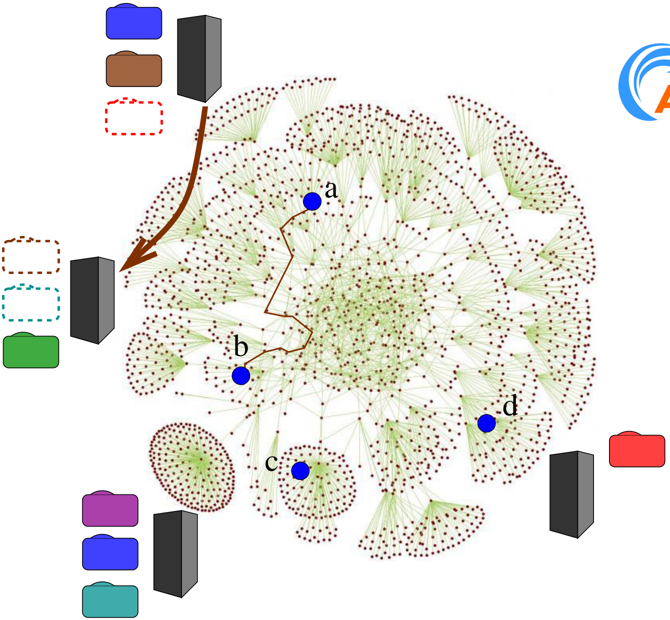
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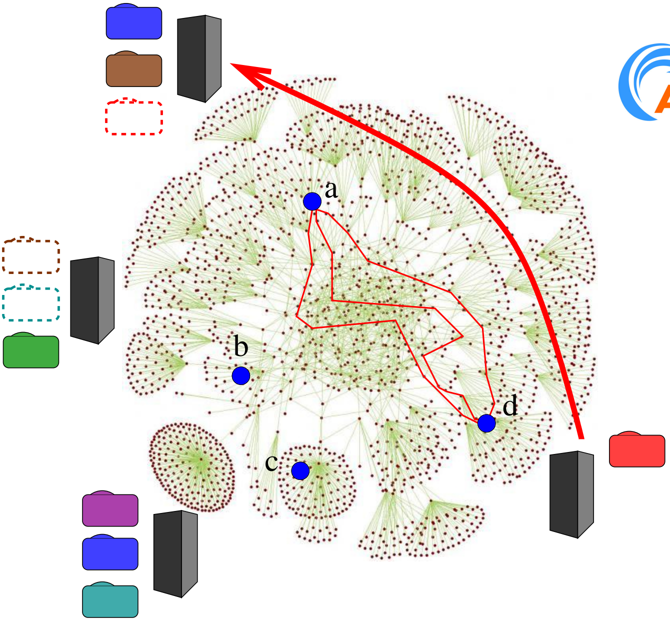

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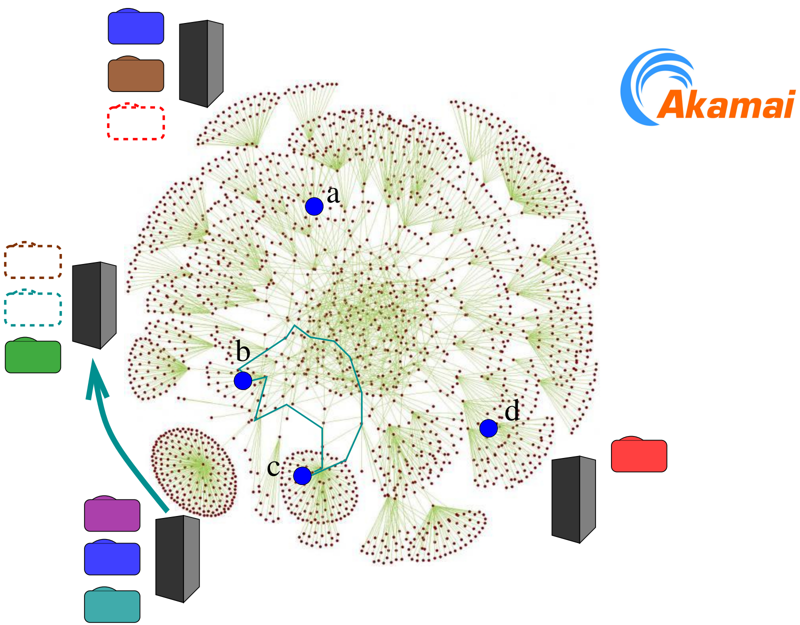
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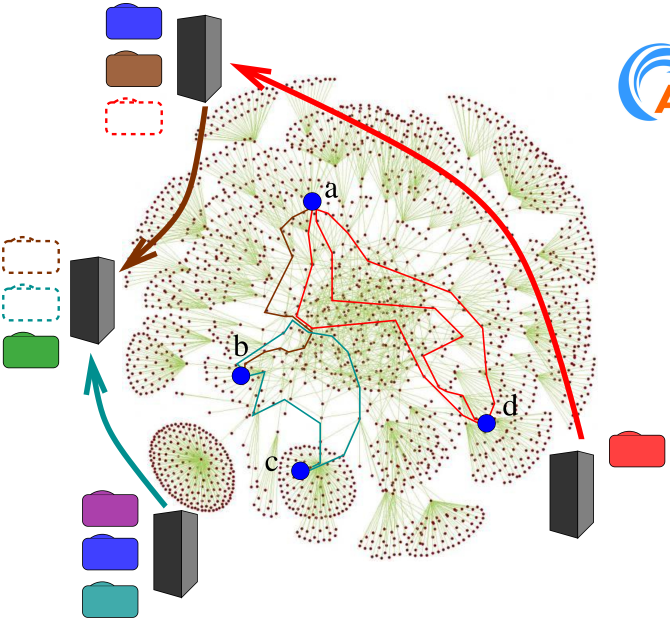
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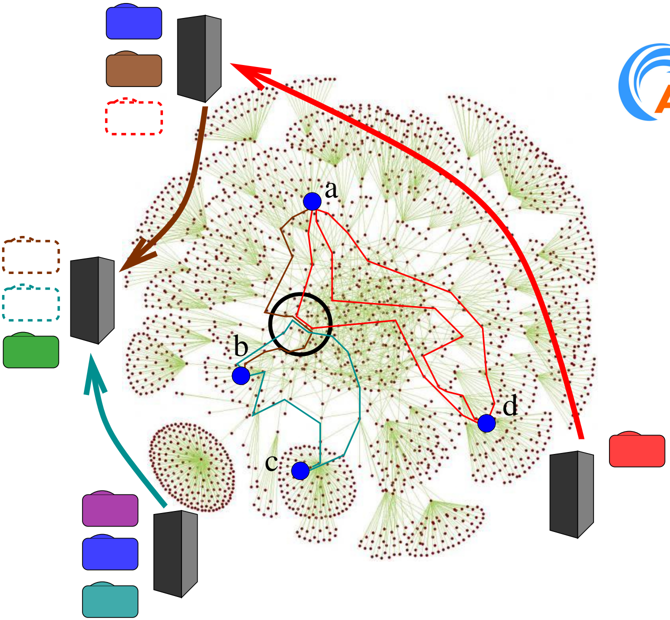
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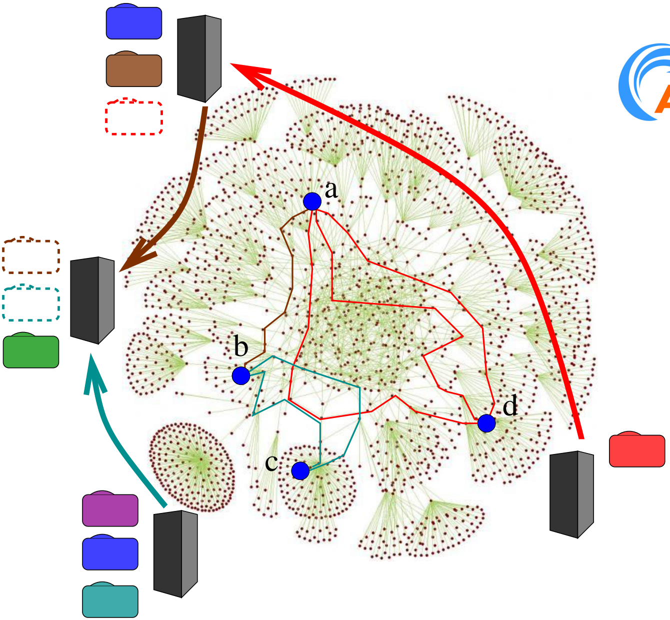

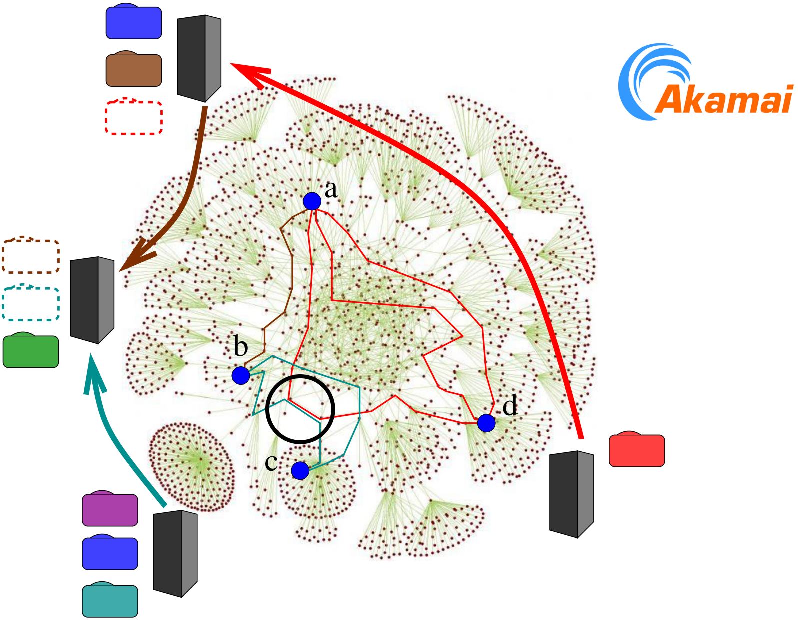
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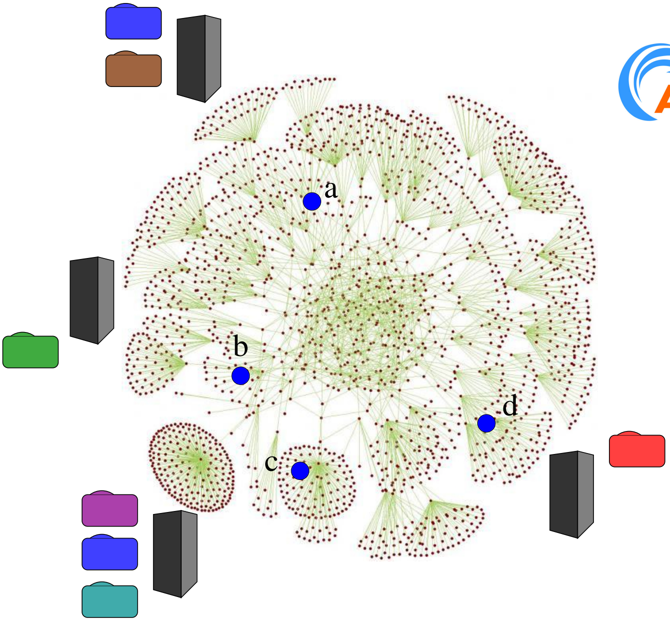

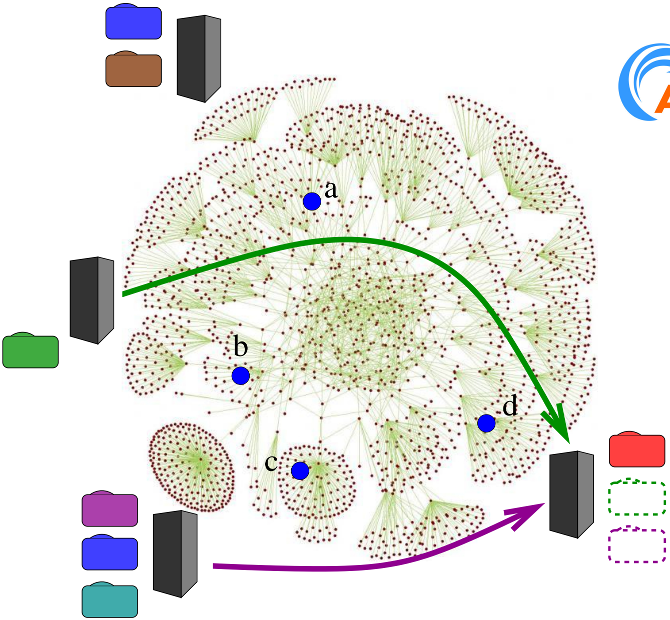

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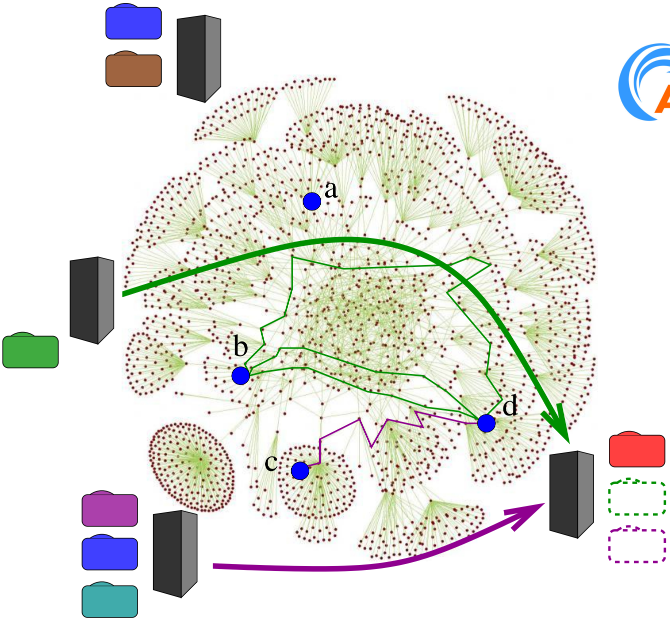
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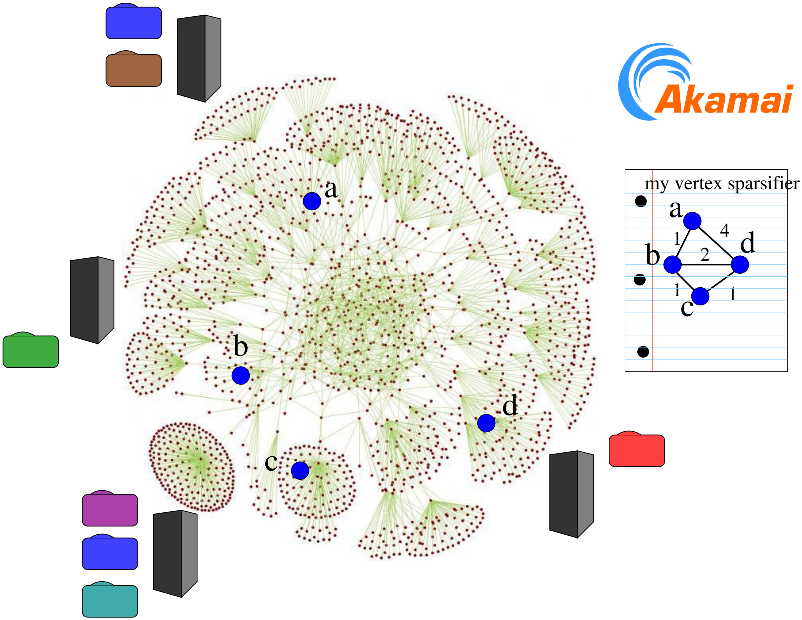
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my vertex sparsifier
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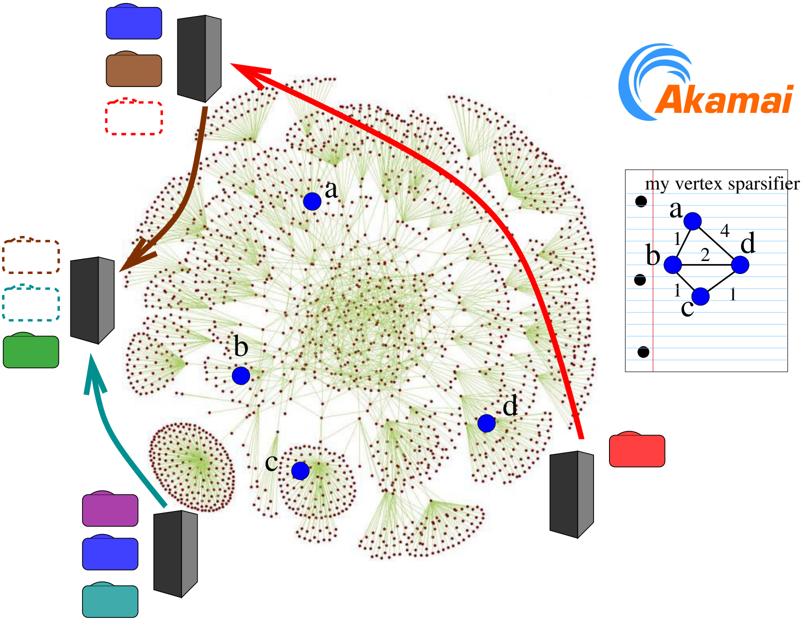
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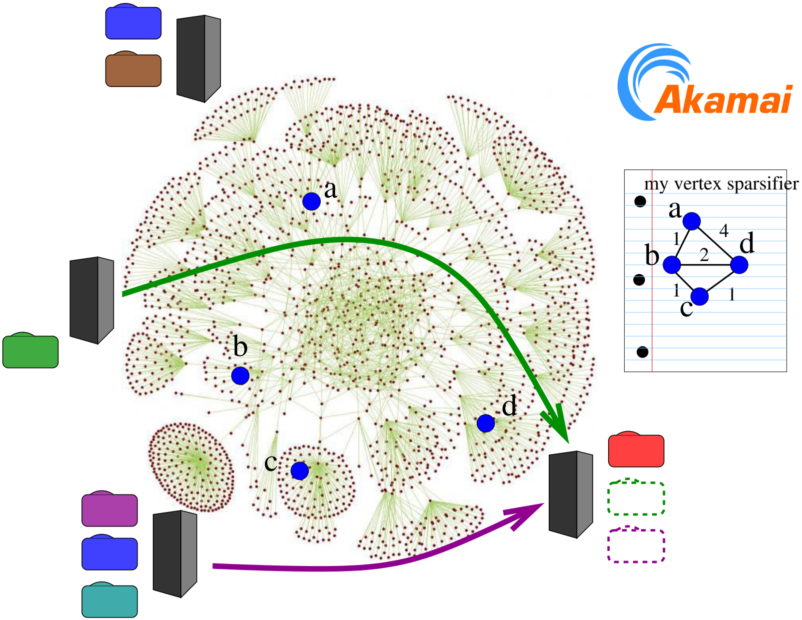

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- Save **SPACE** (store a much smaller network)
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- **UNIFIES** many rounding algorithms for graph partitioning

Outline

- Introduction
  - Minimum Congestion Routing
  - Application: Routing
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  - Definitions
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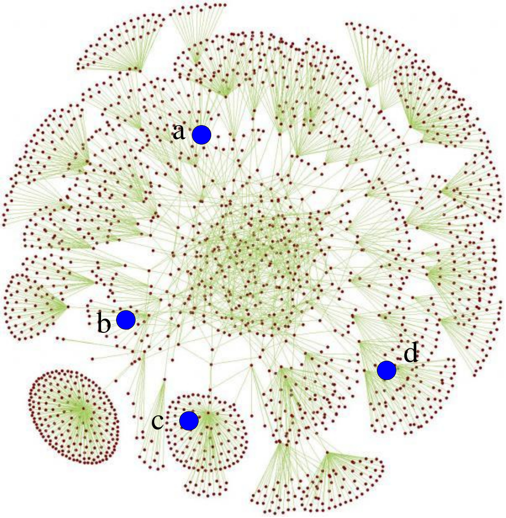

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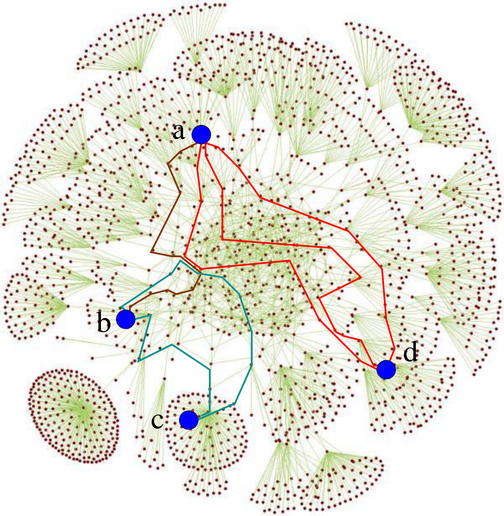

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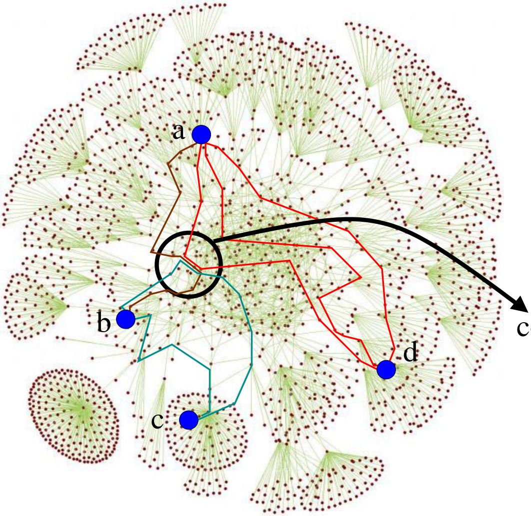

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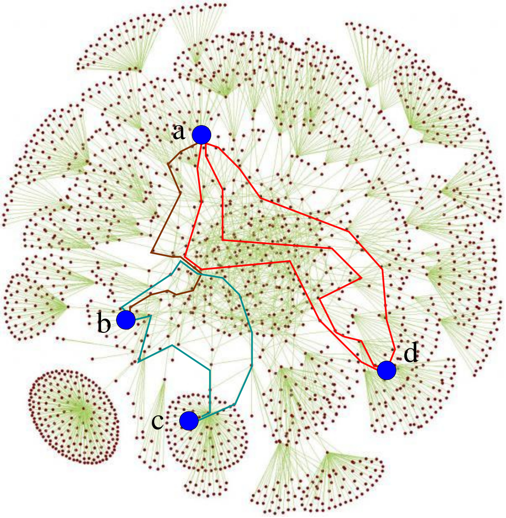

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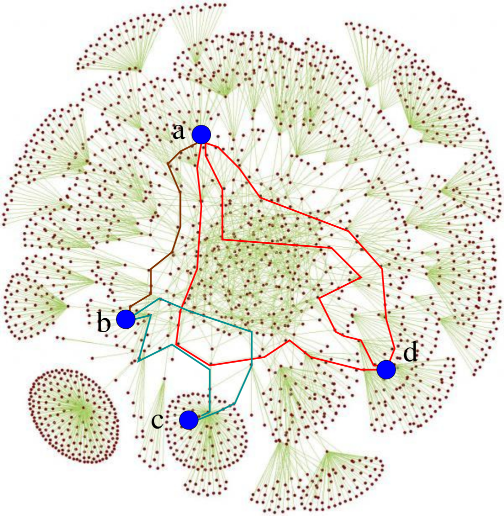

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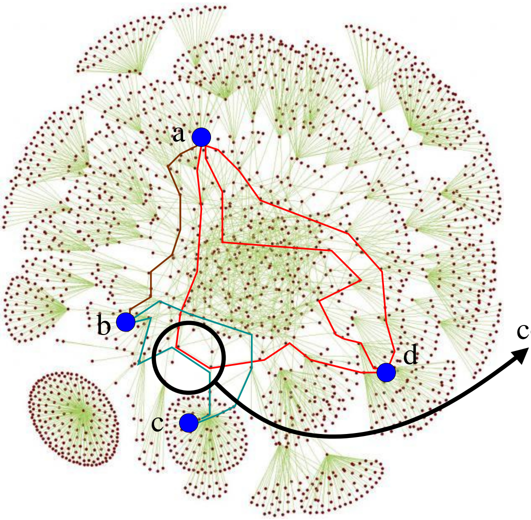

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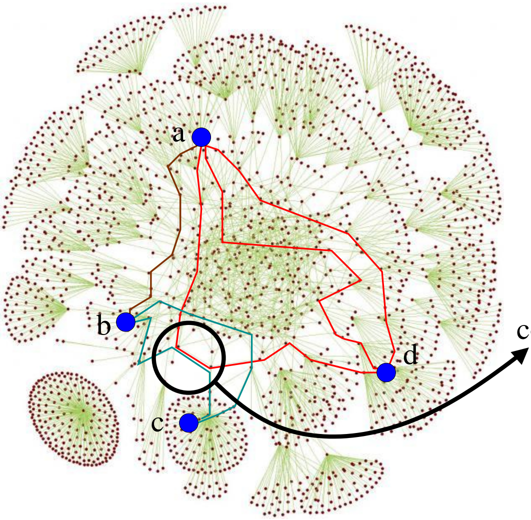
min congestion $\longleftrightarrow$ max throughput

(min max)



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Should good quality vertex sparsifiers exist?

Results (Good Vertex Sparsifiers Exist!)

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(Makarychev, Makarychev): $\tilde{\Omega}(\sqrt{\log |K|})$ *"quality" is necessary*

1. Moitra, "Approximation algorithms with guarantees independent of the graph size", FOCS 2009
2. Leighton, Moitra, "Extensions and limits to vertex sparsification", STOC 2010
3. Charikar, Leighton, Li, Moitra, "Vertex sparsifiers and abstract rounding algorithms", FOCS 2010
4. Makarychev, Makarychev, "Metric extension operators, vertex sparsifiers and Lipschitz extendability", FOCS 2010
5. Englert, Gupta, Krauthgamer, Räcke, Talgam-Cohen, Talwar, "Vertex sparsifiers: new results from old techniques", APPROX 2010
6. Chuzhoy. "On vertex sparsifiers with Steiner nodes", STOC 2012

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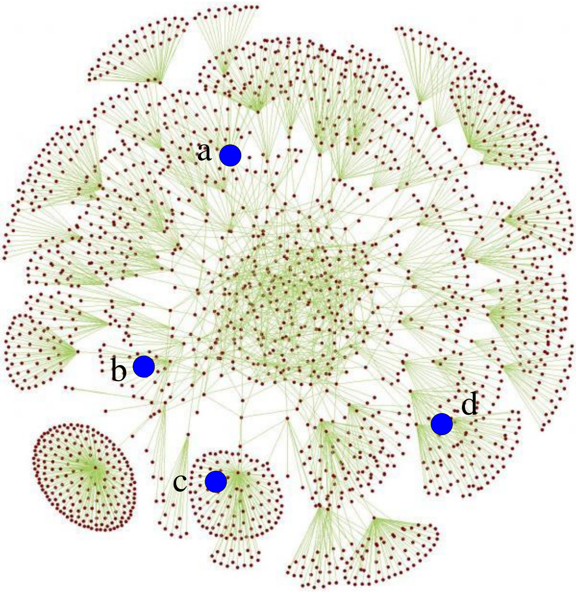
Claim

A good vertex sparsifier can be **SIMULATED** *with low overhead, in the original network*



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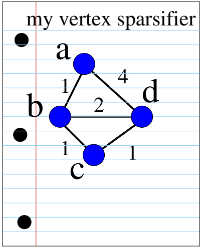

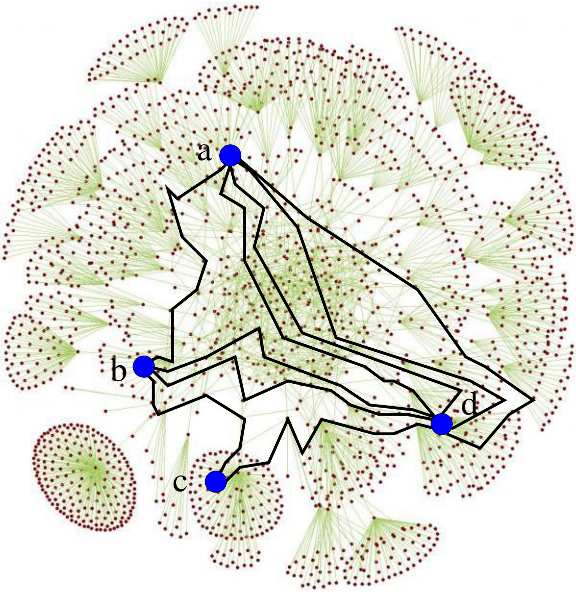

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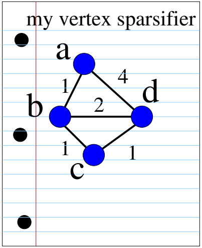

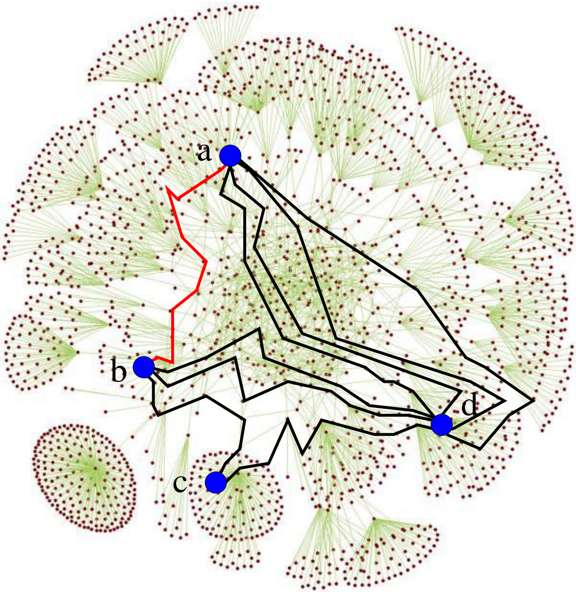

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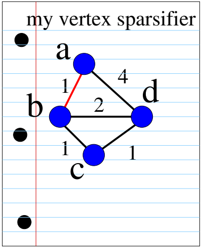

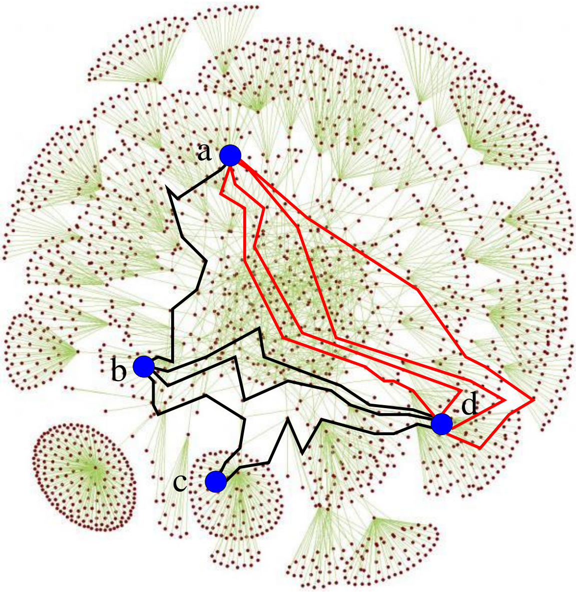

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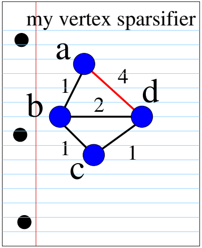

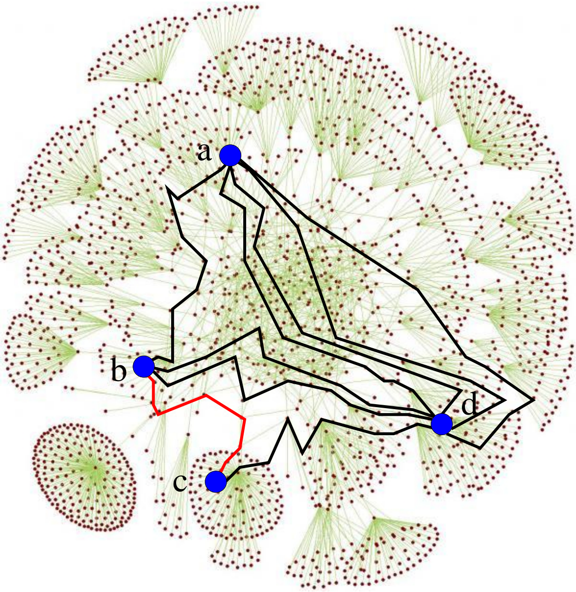

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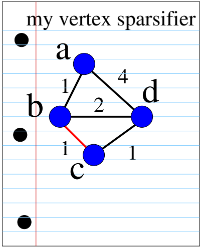

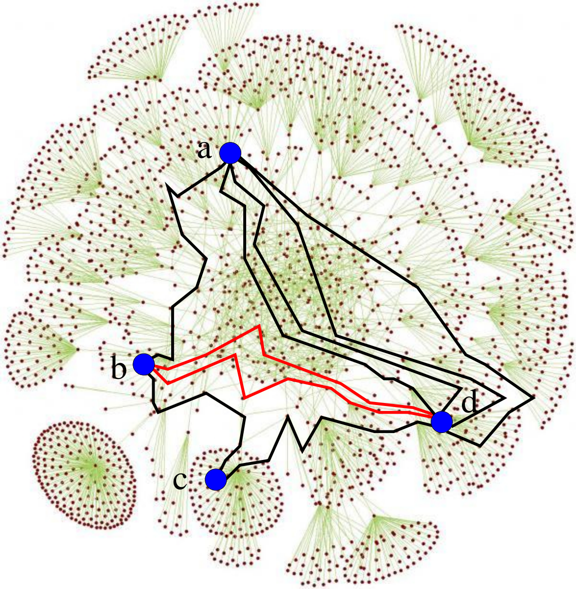

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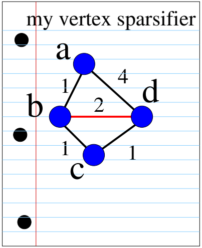

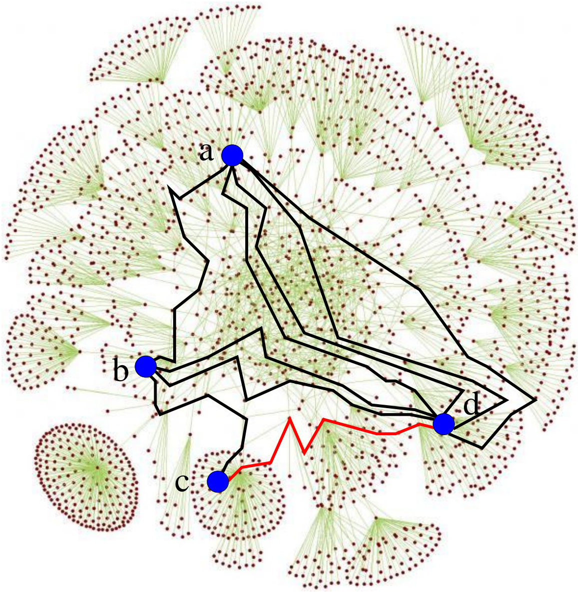

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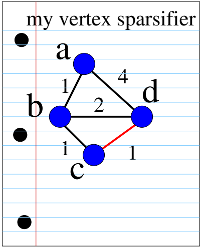

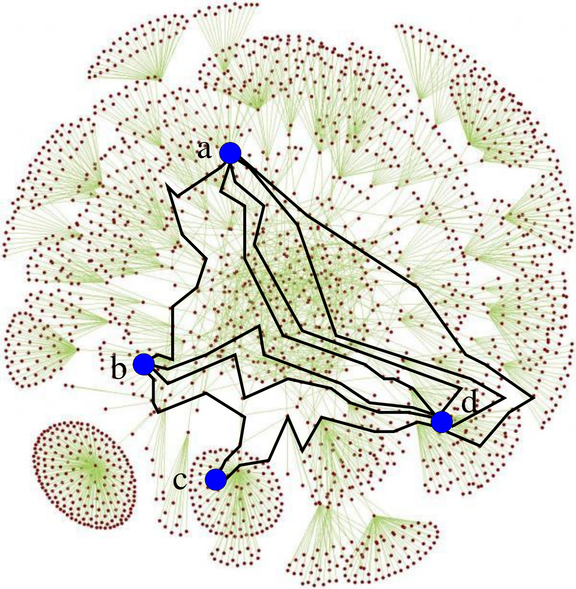

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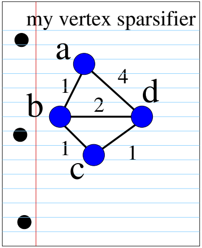

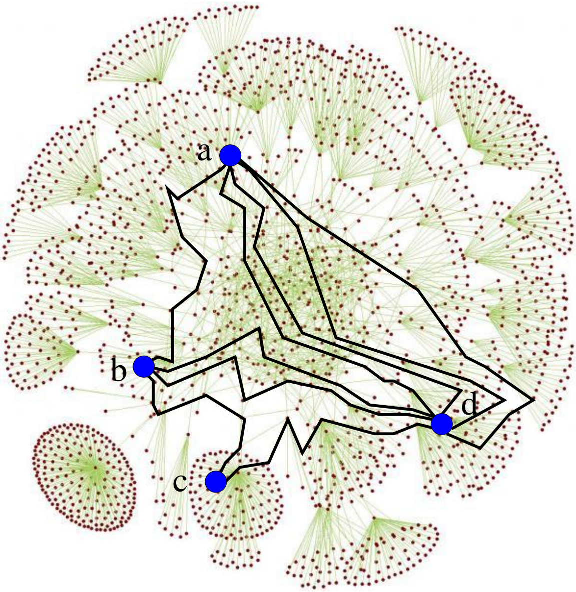

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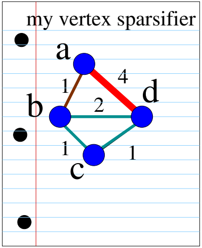

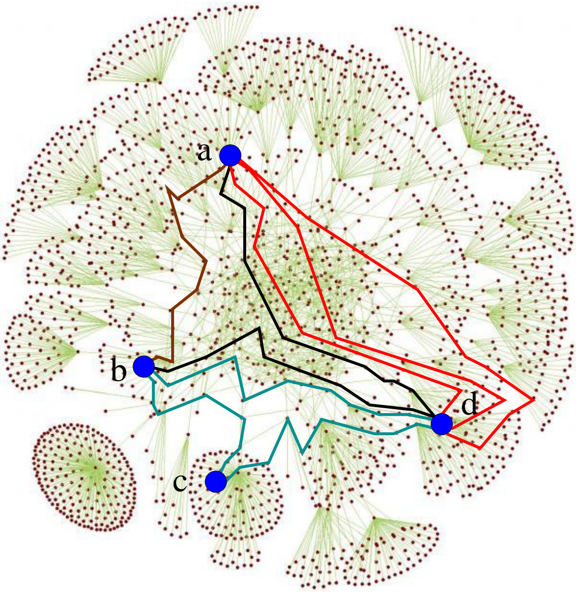

$$\vec{f} = \begin{array}{c} \left[\begin{array}{c} \color{brown}{x} \\ 0 \\ \color{red}{y} \\ \color{teal}{z} \\ 0 \\ 0 \end{array}\right] \end{array} \begin{array}{l} a \longleftrightarrow b \\ a \longleftrightarrow c \\ a \longleftrightarrow d \\ b \longleftrightarrow c \\ b \longleftrightarrow d \\ c \longleftrightarrow d \end{array}$$

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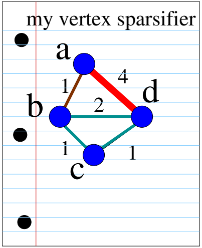

Solving a Sequence of Routing Problems

Observation

What we really want are good routing schemes in the original network!

There is a **CANONICAL** mapping of flows in H to flows in G:

Claim

A good vertex sparsifier can be **SIMULATED** *with low overhead, in the original network*

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For each routing request, run off-the-shelf algorithm on a 4 node network (instead of on a gigantic one)

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COMPARE: Räcke's oblivious routing scheme is $\Theta(\log |V|)$-competitive; ours is $O(\log |K|)$

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Graph Partitioning Made Easy

Graph Partitioning

- **Goal**: *cut few edges, disconnect terminals according to some constraints*

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Preserve flows $\Rightarrow$ *Preserve cuts*

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(Charikar, Leighton, Li, Moitra): Vertex sparsifiers yield many known approximation guarantees as a special case, and give new ones too!

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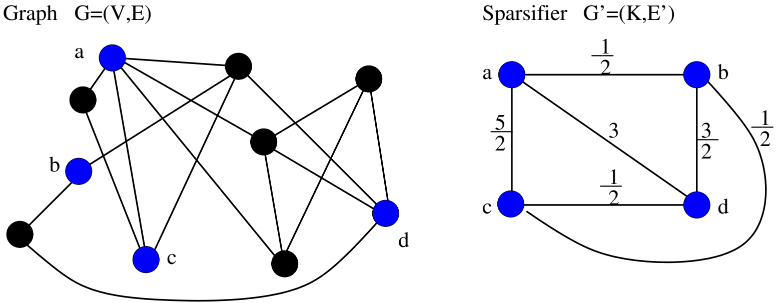
General Approach: Cut Sparsifiers



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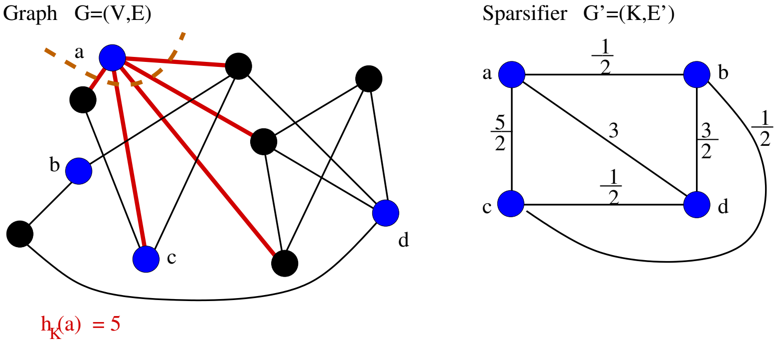
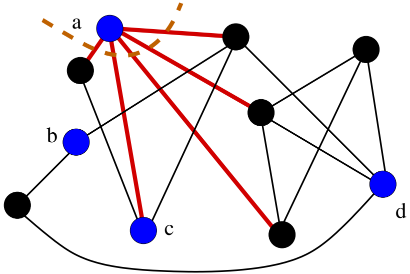


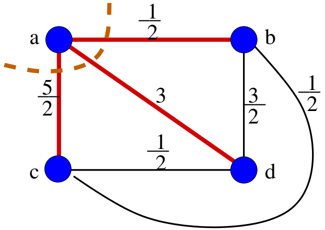
General Approach: Cut Sparsifiers

Graph $G=(V,E)$



$h_K(a) = 5$

Sparsifier $G'=(K,E')$



$h'(a) = 6$

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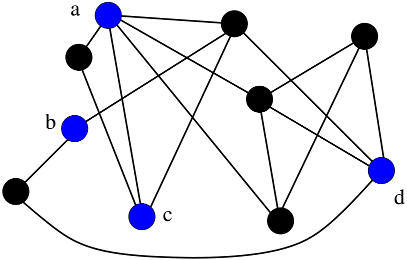

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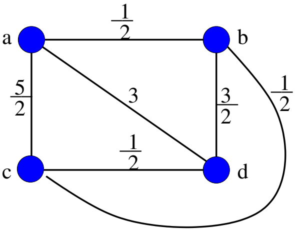

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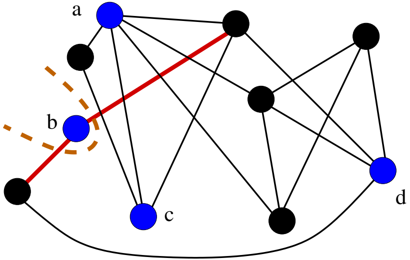

$h_K(a) = 5$
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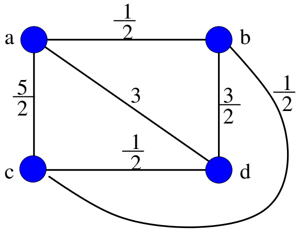

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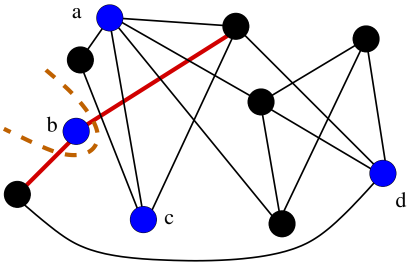

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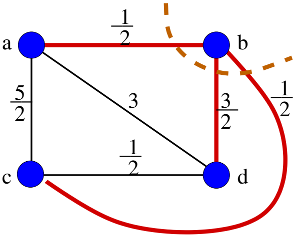

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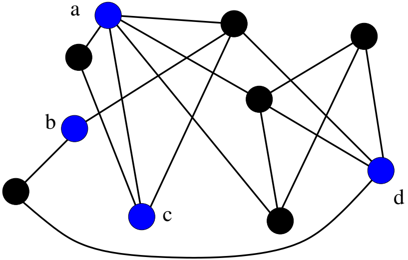

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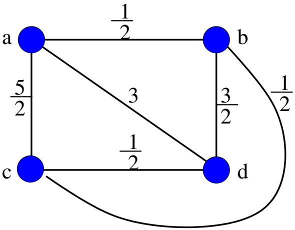

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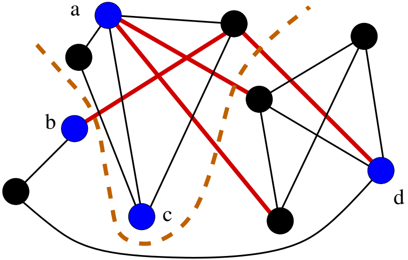

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$h_K(ac) = 4$

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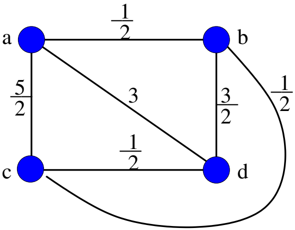

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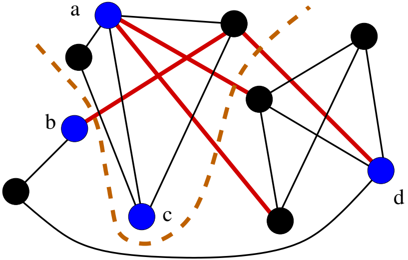

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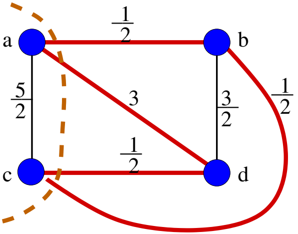

h'(a) = 6

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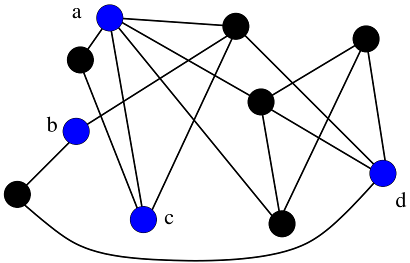

$h_K(a) = 5$ $\quad h_K(d) = 4$ $\quad h_K(ad) = 5$
$h_K(b) = 2$ $\quad h_K(ab) = 7$
$h_K(c) = 3$ $\quad h_K(ac) = 4$

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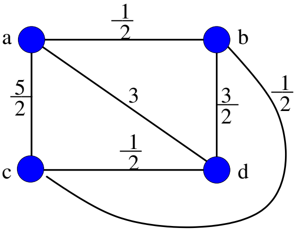

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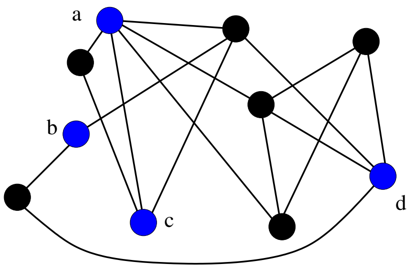

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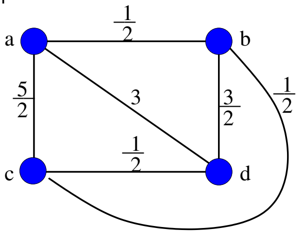

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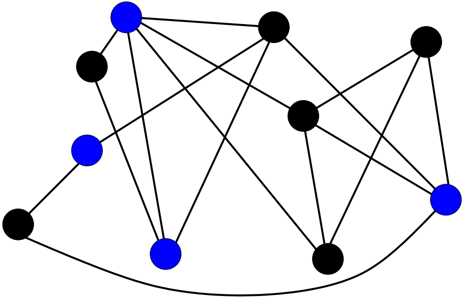
A Useful Primitive

Definition

G=(V,E)



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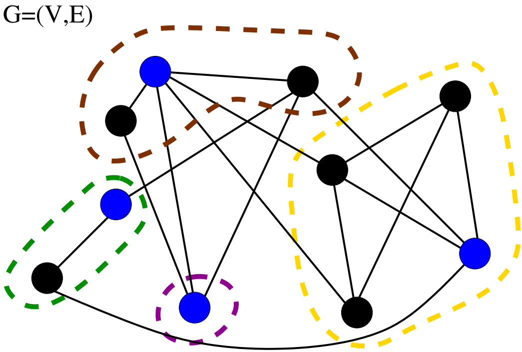

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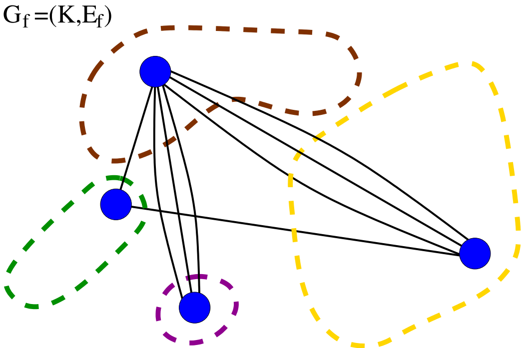
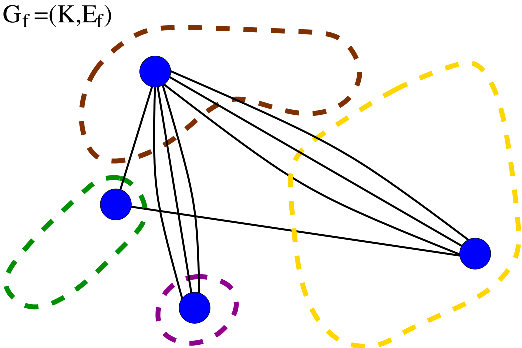


A Useful Primitive

Definition

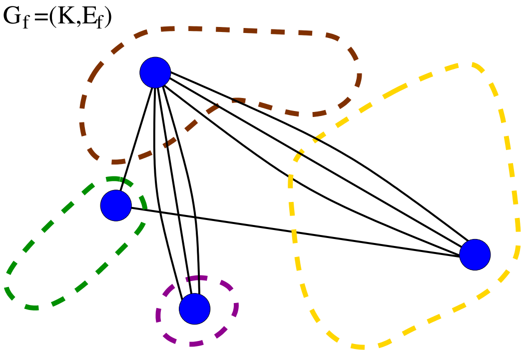
Let $f : V \to K$, is a 0-extension if for all $a \in K$, $f(a) = a$.



A Useful Primitive

Lemma

G_f is a Cut Sparsifier



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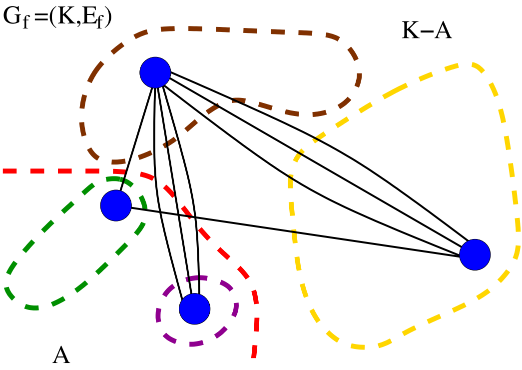

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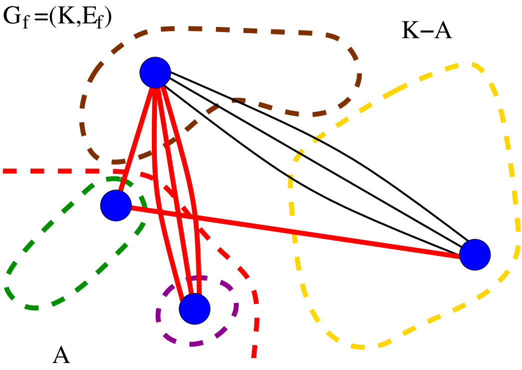

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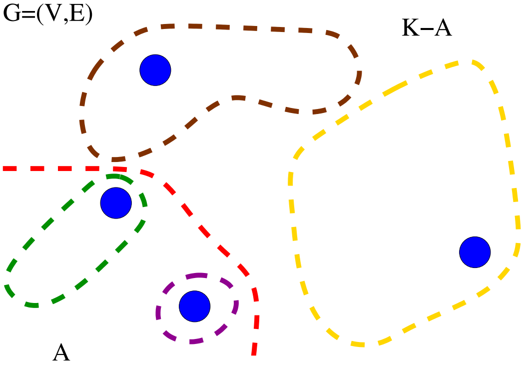

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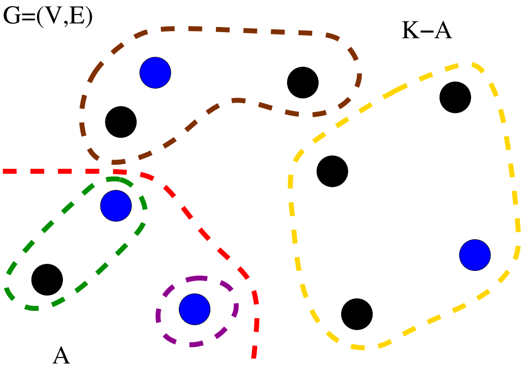

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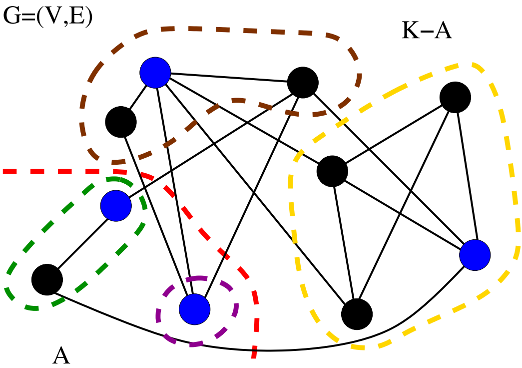

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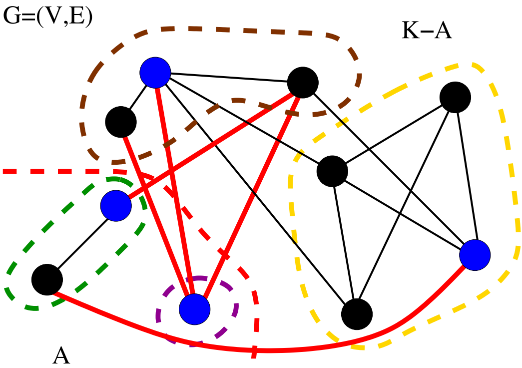

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Proof Outline

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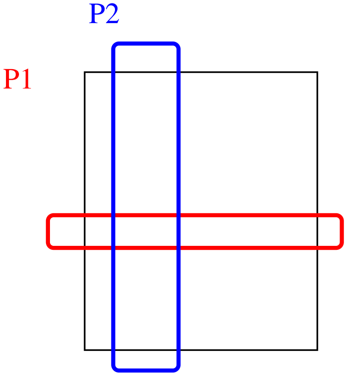
1. Define a **Zero-Sum Game**

The Extension-Cut Game



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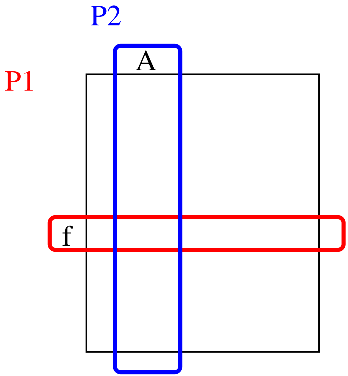

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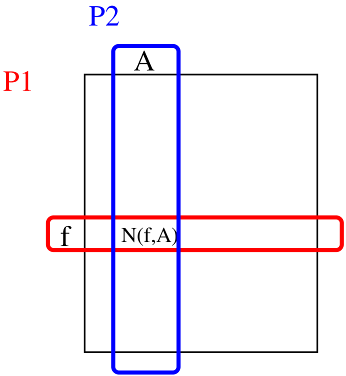

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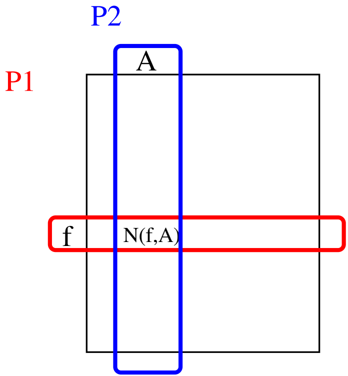

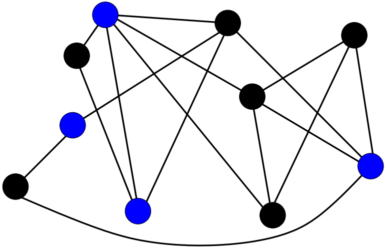

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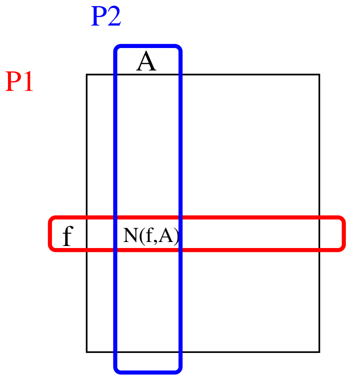

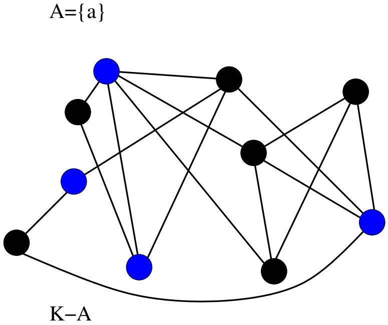

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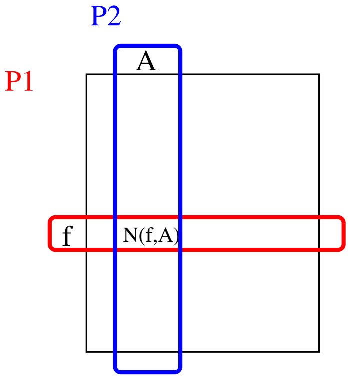

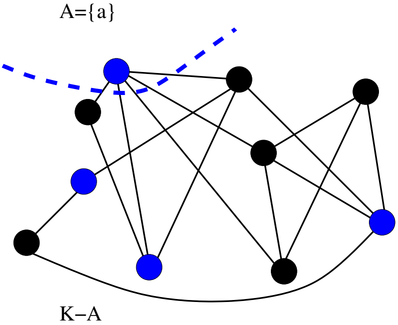

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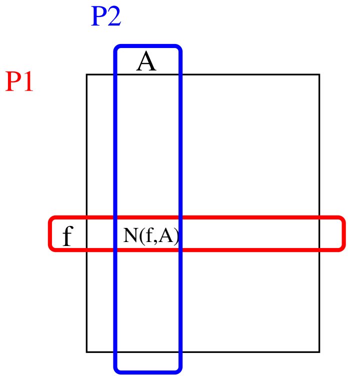

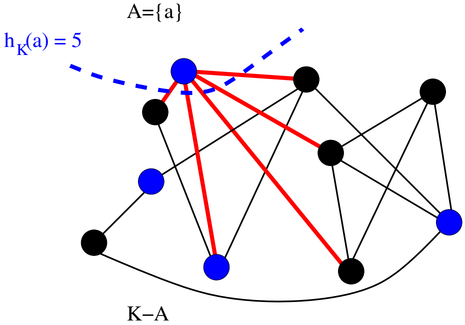

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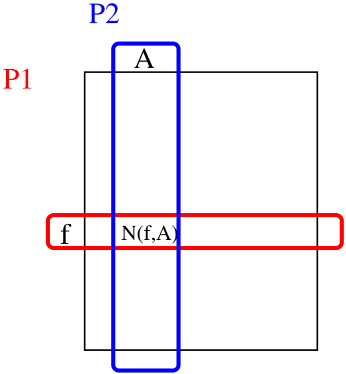

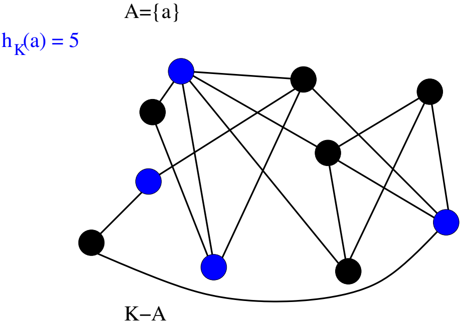

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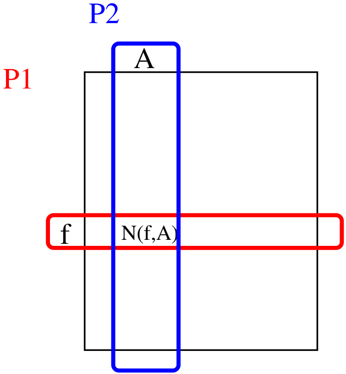

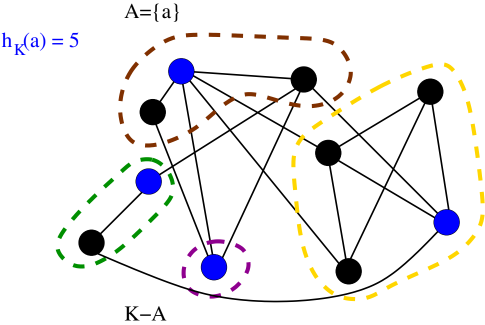

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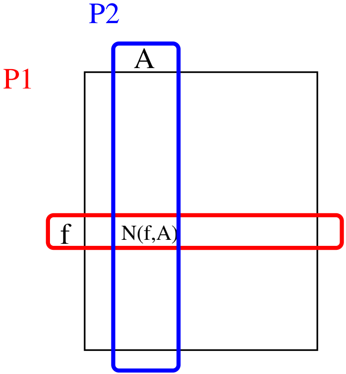

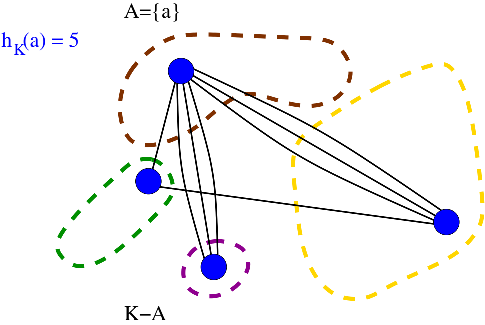

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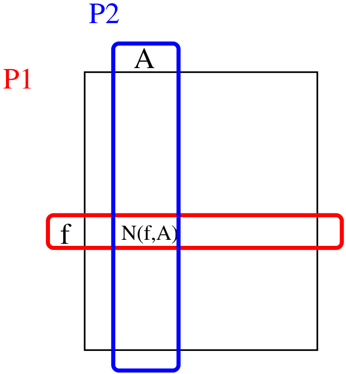

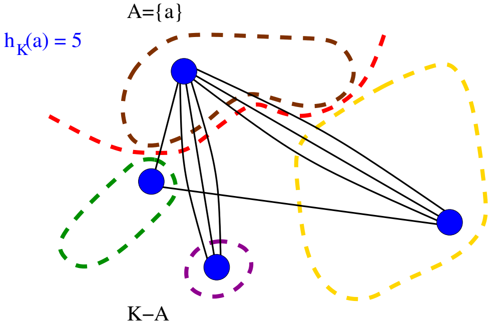

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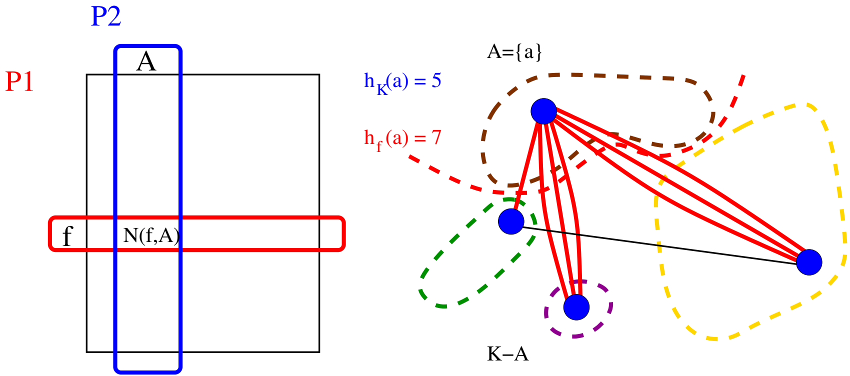

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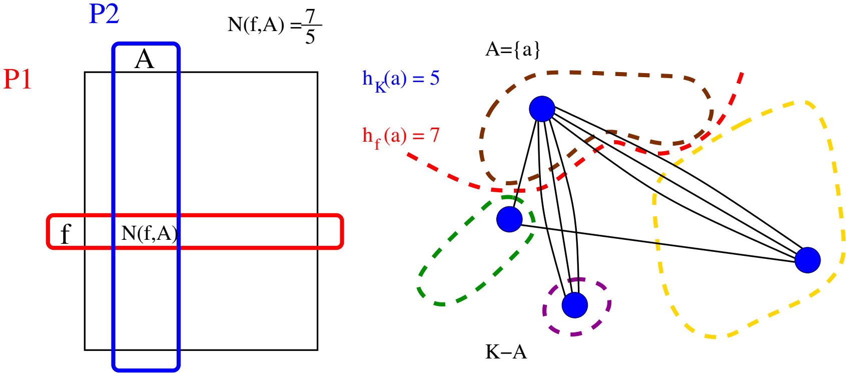

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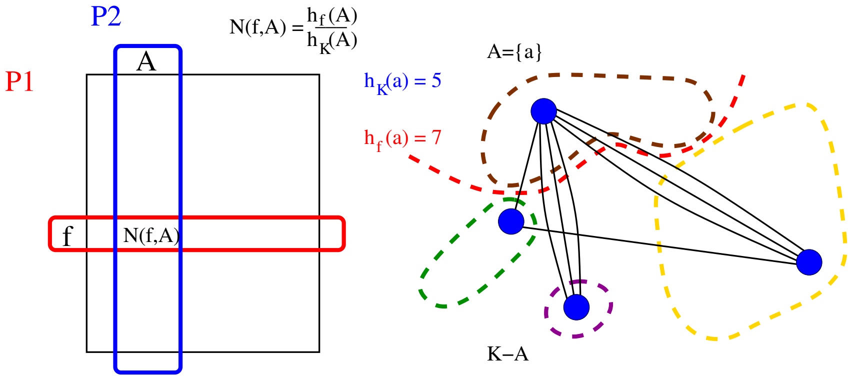

Minmax

Theorem (von Neumann)

$$\min_{\gamma} \max_{A} E_{f \leftarrow \gamma}[N(f, A)] = \max_{\lambda} \min_{f} E_{A \leftarrow \lambda}[N(f, A)]$$

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Bound on **game value** implies that good Cut Sparsifiers exist!

Minmax

Theorem (von Neumann)

$$\min_{\gamma} \max_{A} E_{f \leftarrow \gamma}[N(f, A)] = \max_{\lambda} \min_{f} E_{A \leftarrow \lambda}[N(f, A)]$$

Bound on **game value** implies that good Cut Sparsifiers exist!

Let $G' = \sum_f \gamma(f) G_f$ (**no good response for the cut player**)

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Question

For every distribution λ on $A \subset K$, is there a **good** *response f for the extension player?*

Proof Outline

1. Define a **Zero-Sum Game**

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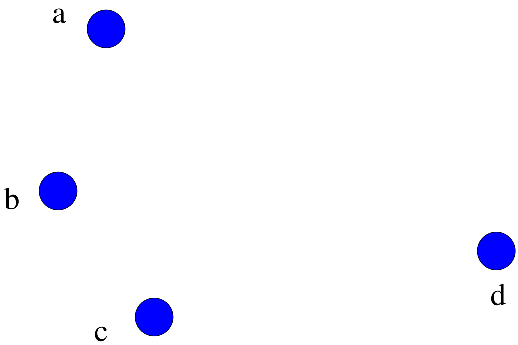
2. The **Best Response** is a 0-Extension Problem

Best Response?

Let $\mu = \frac{1}{2}\{a, b\} + \frac{1}{2}\{a, d\}$



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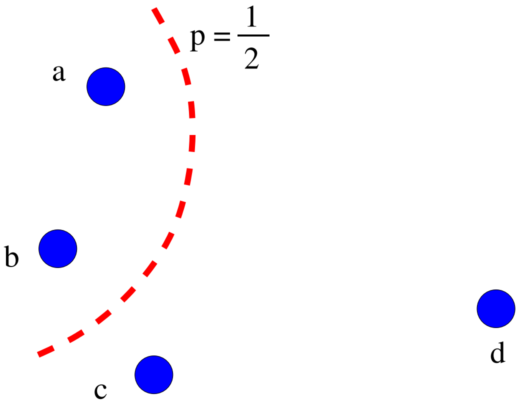

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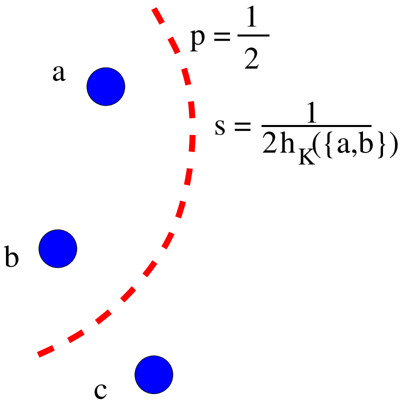

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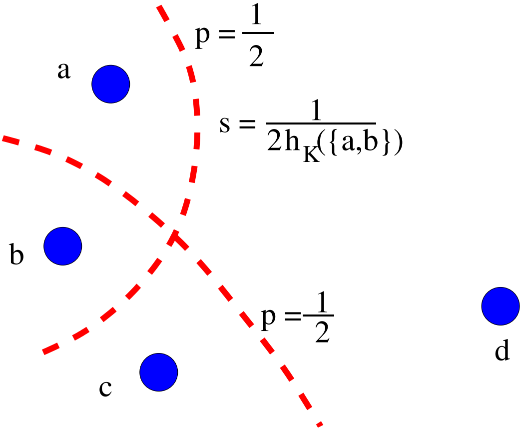

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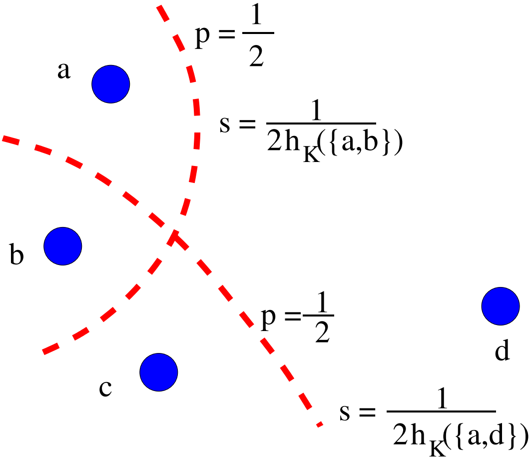

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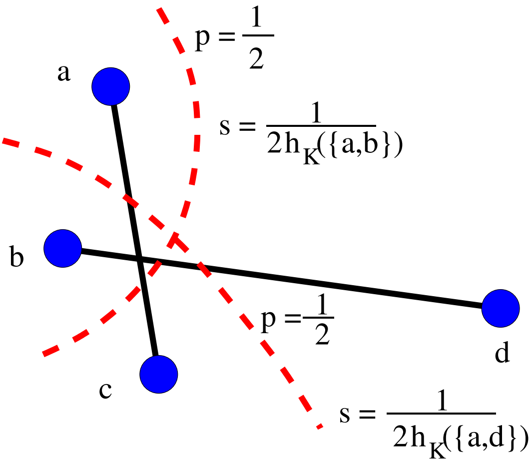

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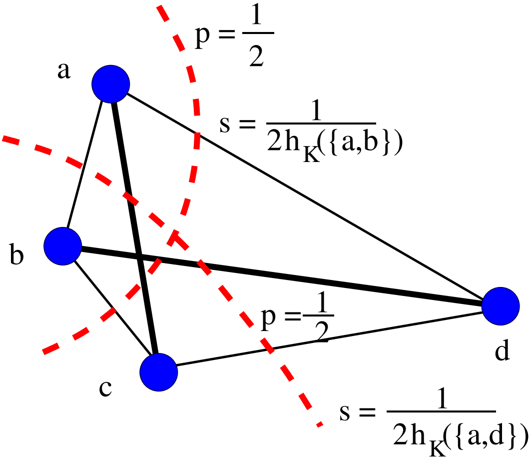

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   [Fakcharoenphol, Harrelson, Rao, Talwar '03]
   [Calinescu, Karloff, Rabani '01]

Summary, So Far

Non-constructive proof that good quality cut sparsifiers exist, through a zero sum game

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- Solve a sequence of routing problems as fast as solving just one!
- Reduce all your graph partitioning problems to trees!

Outline

- Introduction
  - Minimum Congestion Routing
  - Application: Routing
  - Application: Graph Partitioning
- Vertex Sparsification
  - Definitions
  - Zero-Sum Game
- Graph Partitioning, Revisited
- Learning via Polynomials

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Question

Can we use vertex sparsification to make the problem smaller and simpler?

e.g. can we **ROUND** the graph to a tree (on just the terminals)?

Fractional Graph Partitioning Problems

Definition

We call an optimization problem a Fractional Graph Partitioning Problem if it can be written as

$$\begin{array}{rl} \min & \sum_{(u,v)\in E} c(u,v)d(u,v) \\ \text{s.t.} & \\ & d : V \times V \to \Re^{+} \text{ is a semi-metric} \\ & \ldots \end{array}$$

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Theorem (Charikar, Leighton, Li, Moitra)

For any graph partitioning problem, the maximum integrality gap is at most $O(\log k)$ *times the max integrality gap restricted to trees*

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3. [Gupta, Nagarajan, Ravi]
4. ...

Thanks! Any Questions?

- Vertex sparsification
  existence via an exponential-sized zero-sum game
- Implications for routing
  save space and time, when solving a sequence of problems
- Implications for graph partitioning
  general case can be reduced to trees (on the set of terminals)
- Other: Learning, Lattices, Convex Geometry