Orbit Recovery

Setup: Unknown signal $x \in \mathbb{R}^d$ and group G with group action

$$\rho : G \to \mathbb{R}^{d \times d}$$

We get measurements of the form

$$y = \rho(g)x + n$$

noise

random group element from Haar

Goal: recover some $\hat{x}$ close to the orbit

$$\{\rho(g)x \mid g \in G\}$$

why only up to orbit?

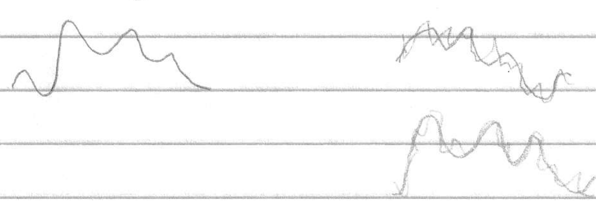
Examples

Multireference Alignment

group

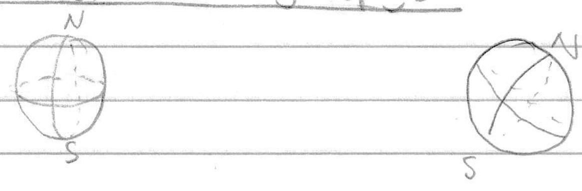
true signal

$\mathbb{Z}_n$ or $SO(2)$



Cryoelectron Tomography

$SO(3)$



+noise

Cryoelectron Microscopy: Cryo-ET + projection

[Perry, Weed, Bandeira, Rigollet, Singer]: Algorithms for discrete MRA with optimal sample complexity

Claim: we already know an algorithm

discrete MRA $\longleftrightarrow$ mixture of Gaussians where centers are cyclic shifts

Main Question: Can we exploit, rather than circumvent, group structure?

e.g. consider

$$T = \sum_{g \in G} (\rho(g) x)^{\otimes 3}$$

The rank of T is at most $|G|$. But what if the group is continuous/nonabelian?

Invariant Theory

definition: The ring of invariant polynomials consists of all polys q s.t.

$$q(\rho(g)x) = q(x) \quad \forall g \in G$$

Fact: For discrete MRA, the invariant ring is generated by

$$\prod_{j=1}^{s} \hat{x}_{i_j} \qquad \text{s.t. } i_1 + i_2 + \dots i_s = 0$$

fourier transform

When $s = 3$, this is called bispectrum

Theorem: If G is compact and acts continuously on every x, the invariant ring determines x up to orbit

just on generators

i.e. $\hat{x} \in \{ \rho(g) x \mid g \in G \} \iff \forall$ invariant q $q(\hat{x}) = q(x)$

This is the method of moments with group symmetries

How many moments suffice? $\iff$ At what degree do invariant poly's generate the full invariant ring?

Theorem [Kakarala] Provided that $\hat{x}_i \neq 0$ $\forall i$, the bispectrum determines x up to cyclic shift

This follows from Jennrich

Let d^* = degree at which you generate invariant ring generically

<u>Theorem</u> [Blum-Smith, Bandeira, Kileel, Perry, Weed, Wein] For orbit recovery over compact G, the number of samples you need for generic list recovery is $\Theta(\sigma^{-2d^*})$

<u>Caveats</u>: ① No efficient algorithms

② No bounds on d^*

③ Bounds hold in the limit as $\sigma \to \infty$, and hidden constants can depend on d, G

Main challenge: "orbit tensor decomposition

$$\text{Given } T = \int_{g \in G} (\rho(g)x)^{\otimes 3}\, dg$$

can we recover x up to orbit?

<u>Theorem</u> [Liu, Moitra] There is a quasipolynomial time algorithm for cryo-electron tomography in the smoothed analysis model

Open Questions

1. Solve orbit tensor decomposition

   Can we make the method of invariants algorithmic?

2. inverse problems in the sciences.

4. beyond worst-case analysis

   e.g. reinforcement learning

3. linear dynamical systems