# Topics in TCS: Problem Set # 1

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Due: April 2nd

If you work with other students, you must write-up your solutions by yourself and indicate at the top who you worked with!

## Problem 1 [Barvinok, page 19]

Give an example of an infinite family $\{A_i, i = 1, 2, ...\}$ of convex sets in $\mathbb{R}^d$ such that every $d+1$ sets have a common point but there are no points in common to all of the sets $A_i$. (*Hint*: Helly's theorem holds for infinite families of compact sets, so you will have to look for non-compact sets)

## Problem 2 [Matousek, page 12]

In the situation of Radon's lemma ($A$ is a $(d+2)$-point set in $\mathbb{R}^d$), call a point $x \in \mathbb{R}^d$ a *Radon point* of $A$ if it is contained in convex hulls of two disjoint subsets of $A$. Prove that if $A$ is in general position (no $d+1$ points affinely dependent), then its Radon point is unique.

## Problem 3 [Matousek, page 12] *Kirchberger's Theorem*

$(a)$ Let $X, Y \subset \mathbb{R}^2$ be finite point sets, and suppose that for every subset $S \subseteq X \cup Y$ of at most 4 points, $S \cap X$ can be linearly-separated from $S \cap Y$. Prove that $X$ and $Y$ are linearly-separable.

$(b)$ Extend $(a)$ to sets $X, Y \subset \mathbb{R}^d$, with $|S| \leq d+2$.

## Problem 4 [Barvinok, page 144]

Let $A \subset \mathbb{R}^d$ be a non-empty set such that $A^\circ = A$. Prove that $A$ is the unit ball, i.e.

$$A = \{x \in \mathbb{R}^d : \quad \|x\| \leq 1\}$$

## Problem 5 *Seidel's Algorithm*

Let $A = \{x \in \mathbb{R}^d : \quad \langle c_i, x \rangle \leq 1 \text{ for } i = 1, ...m\}$ be a polytope (namely it is bounded). Let $A'$ be obtained from $A$ by removing one of its constraints at random. Then prove:

$$\Pr[\max_{x \in A} u^T x < \max_{x \in A'} u^T x] \leq \frac{d}{m}$$

## Problem 6 [Barvinok, page 8] *Guass-Lucas Theorem*

Let $f(z)$ be a non-constant polynomial in one complex variable $z$ and let $z_1, ..., z_m$ be the roots of $f$ (that is, the set of all solutions to the equation $f(z) = 0$). Let us interpret a complex number $z = x + iy$ as a point $(x, y) \in \mathbb{R}^2$. Prove that each root of the derivative $f'(z)$ lies in the convex hull $\text{conv}(z_1, ..., z_m)$.

*Hint*: Without loss of generality we may suppose $f(z) = (z - z_1)...(z - z_m)$. If $w$ is a root of $f'(z)$, then $\sum_{i=1}^m \prod_{j \neq i} (w - z_j) = 0$, and, therefore, $\sum_{i=1}^m \prod_{j \neq i} \overline{(w - z_j)} = 0$ (where $\overline{z}$ is complex conjugate of $z$). Multiply both sides of the last identity by $(w - z_1)...(w - z_n)$ and express $w$ as a convex combination of $z_1....z_m$.