Perspectives on Phase Transitions

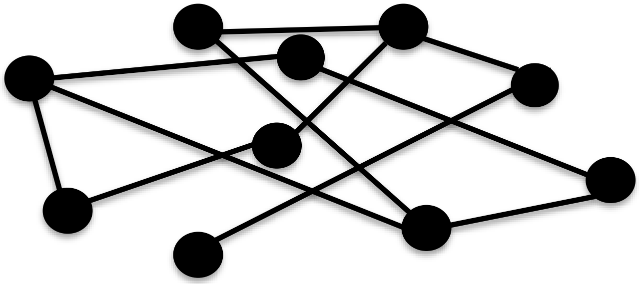
Ankur Moitra (MIT)

Swiss Winter School, Lecture #3.5

COUNTING INDEPENDENT SETS

Given a bounded degree graph (max degree Δ)

G = (V, E)



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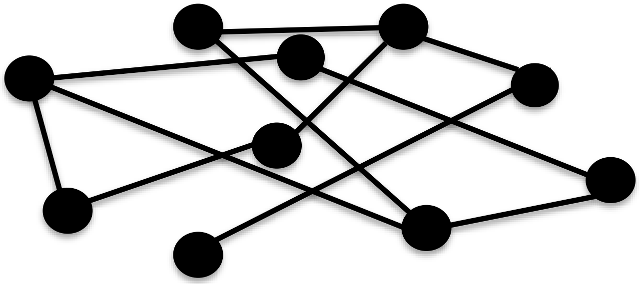

sample an independent set with probability

$$\Pr[I] = \frac{\lambda^{|I|}}{Z} \quad \text{where} \quad Z = \sum_I \lambda^{|I|}$$

partition function

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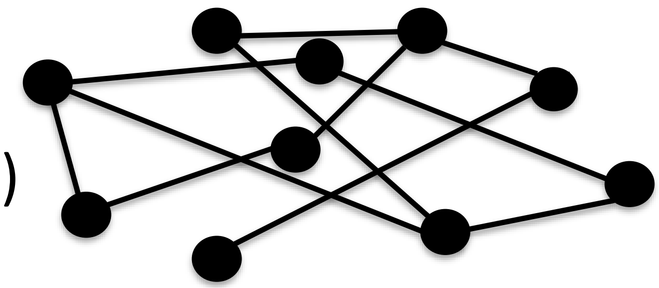

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Here λ is the inverse temperature, or fugacity

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When is it algorithmically possible to count/sample?

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Theorem [Weitz '06]: There is a deterministic algorithm to multiplicatively **approximate the partition function** i.e.

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$$\lambda \leq \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}$$

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Running time is polynomial in $1/\epsilon$ and n, exponential in Δ, yields randomized algorithm for **approximately sampling** too

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There are many ways to think about what is happening at this threshold, spanning **statistical physics**, **probability** and **analysis**

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Informally: No long-range dependencies, for any configuration

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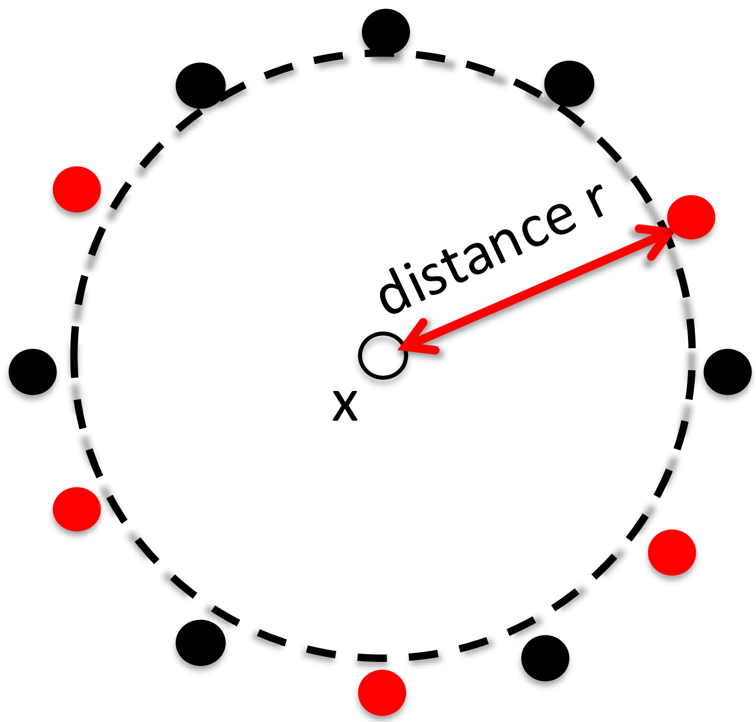

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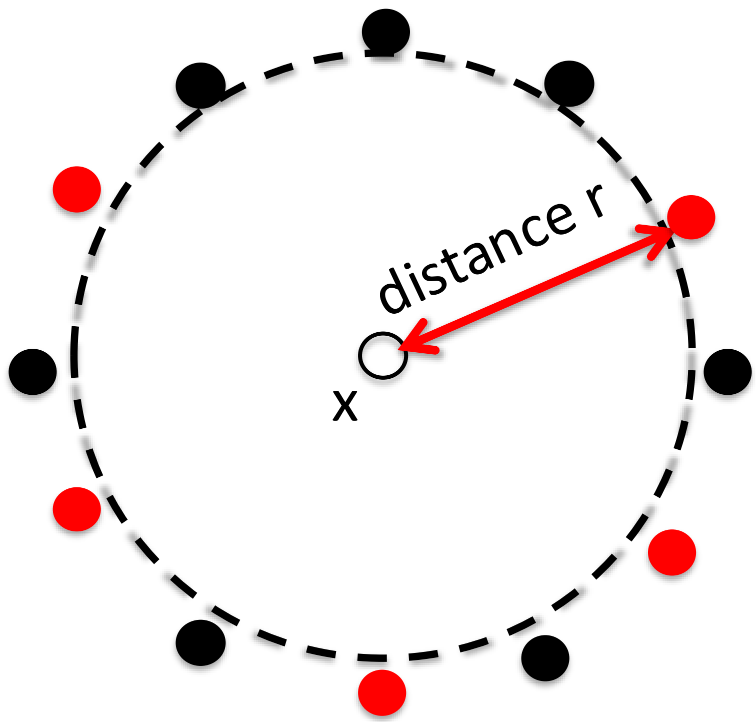

For any configuration on the boundary, change in marginal distribution at x goes to zero as r increases

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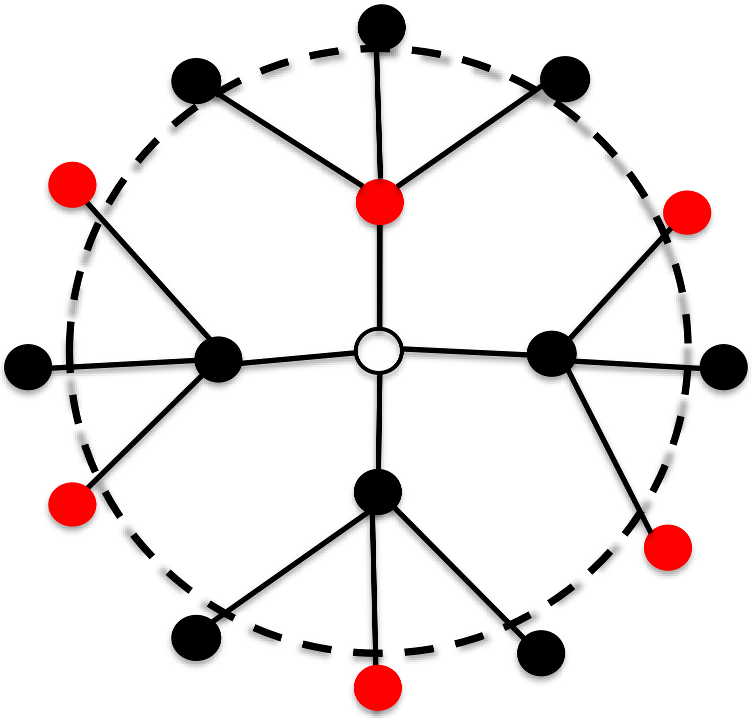

[Weitz '06]: Can reduce general case to trees (self-avoiding walks), approximate marginals by dynamic programming

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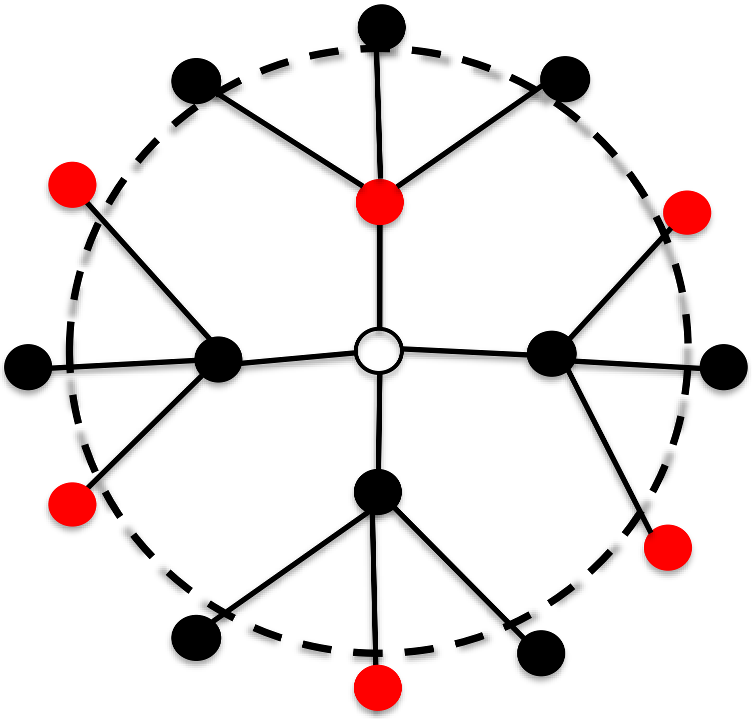

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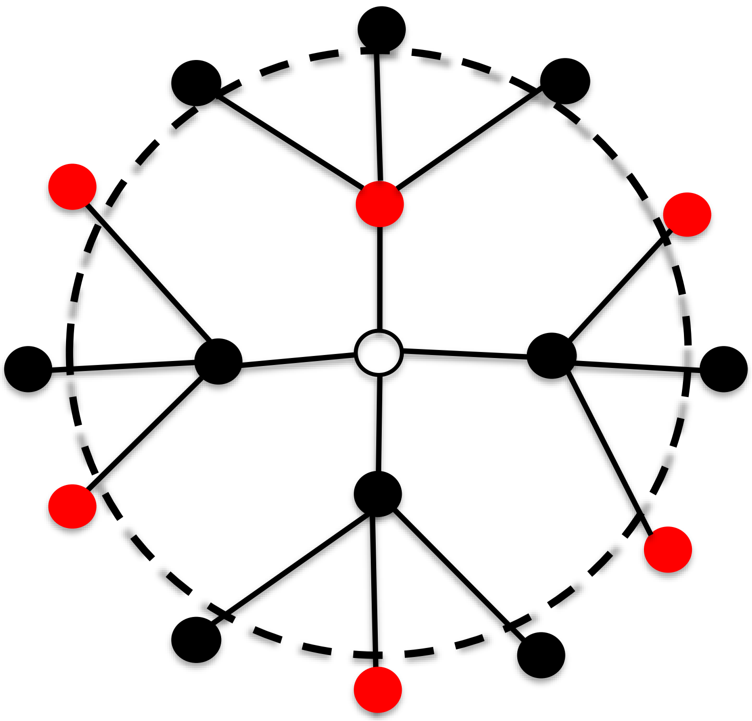
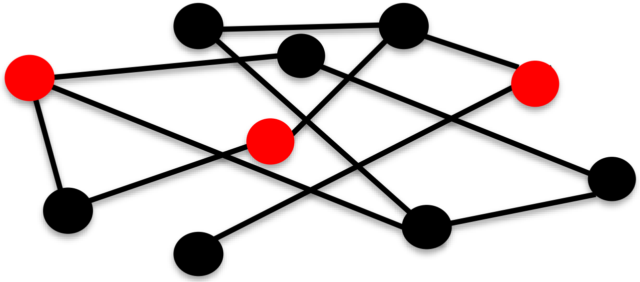


For bounded degree graphs, there is correlation decay iff

$$\lambda \leq \frac{(\Delta - 1)^{\Delta - 1}}{(\Delta - 2)^{\Delta}}$$

GLAUBER DYNAMICS

Create a Markov chain on the state space with local updates



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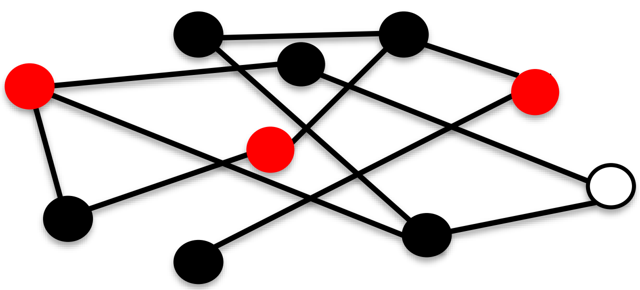

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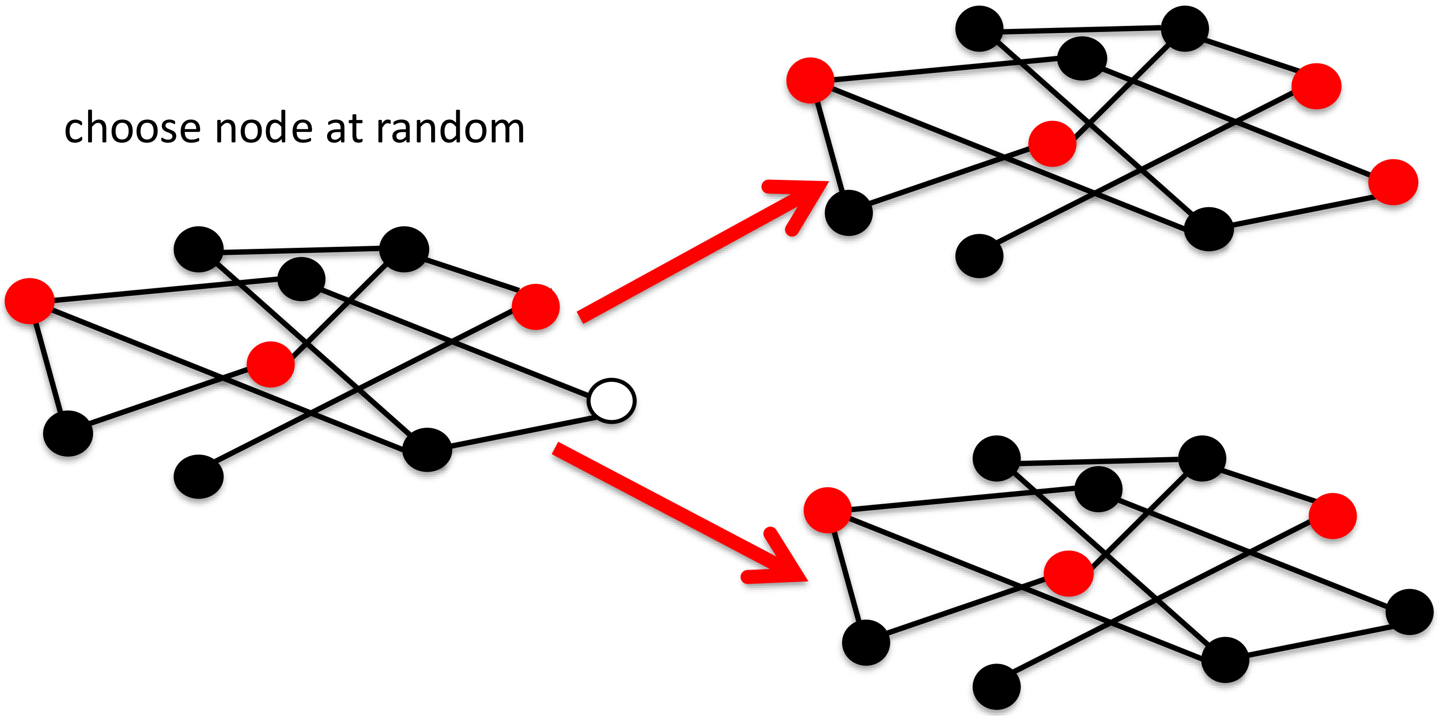

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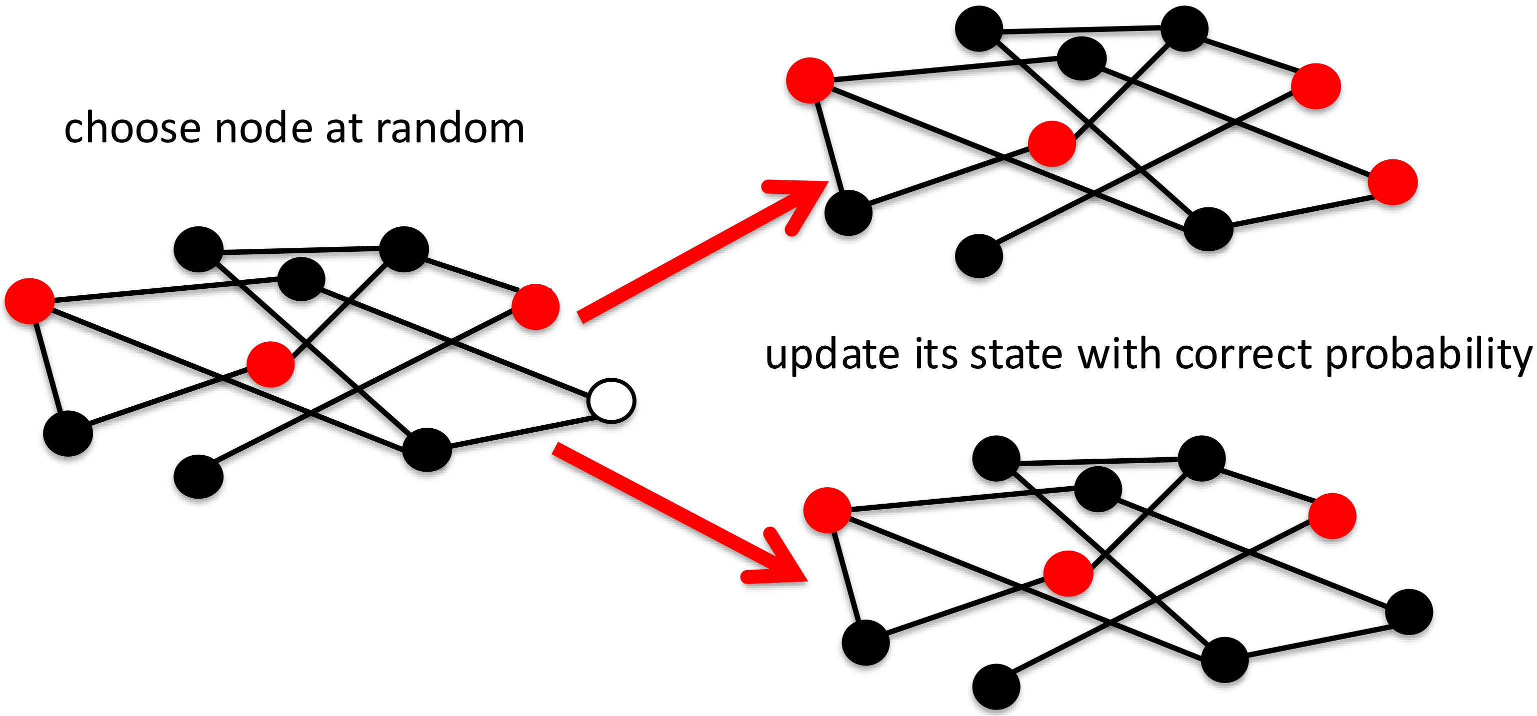

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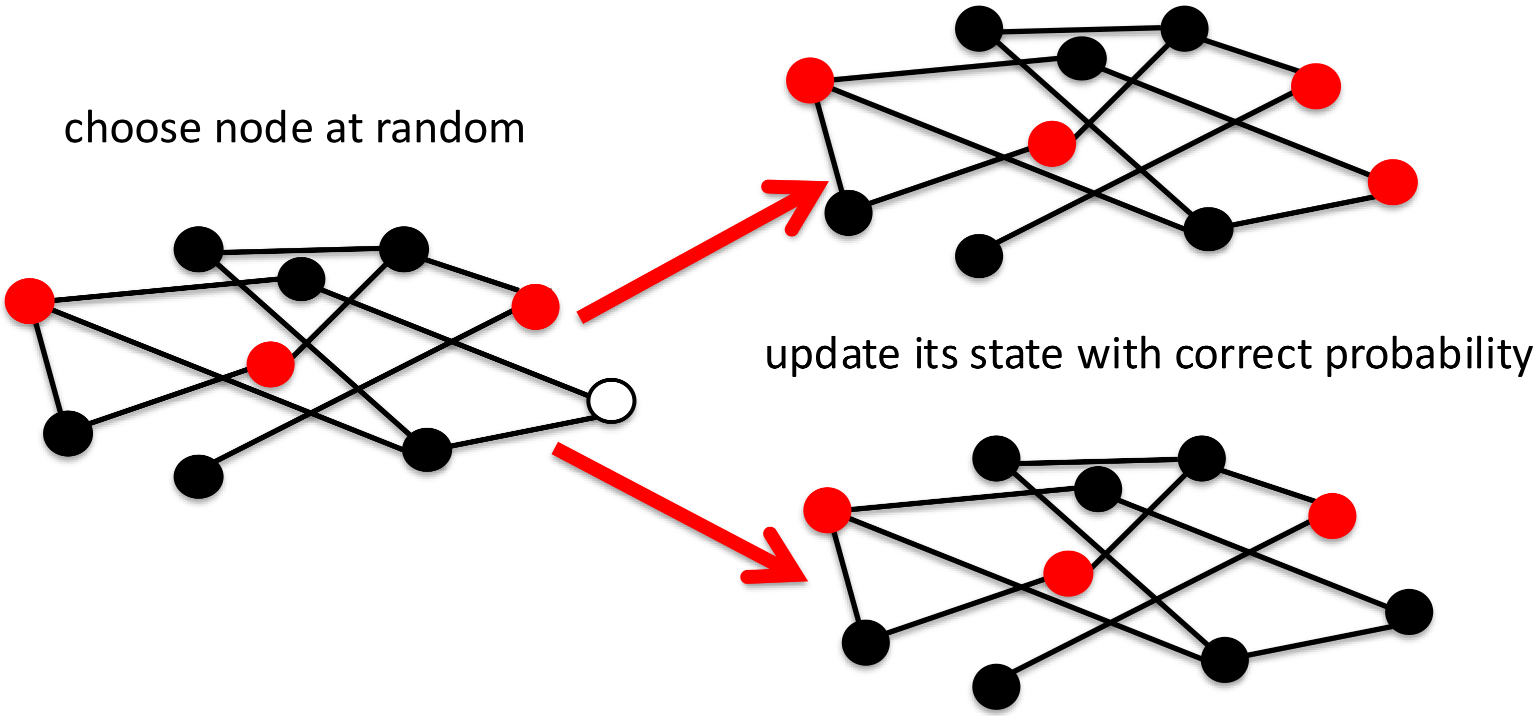

When does the Markov chain mix rapidly?

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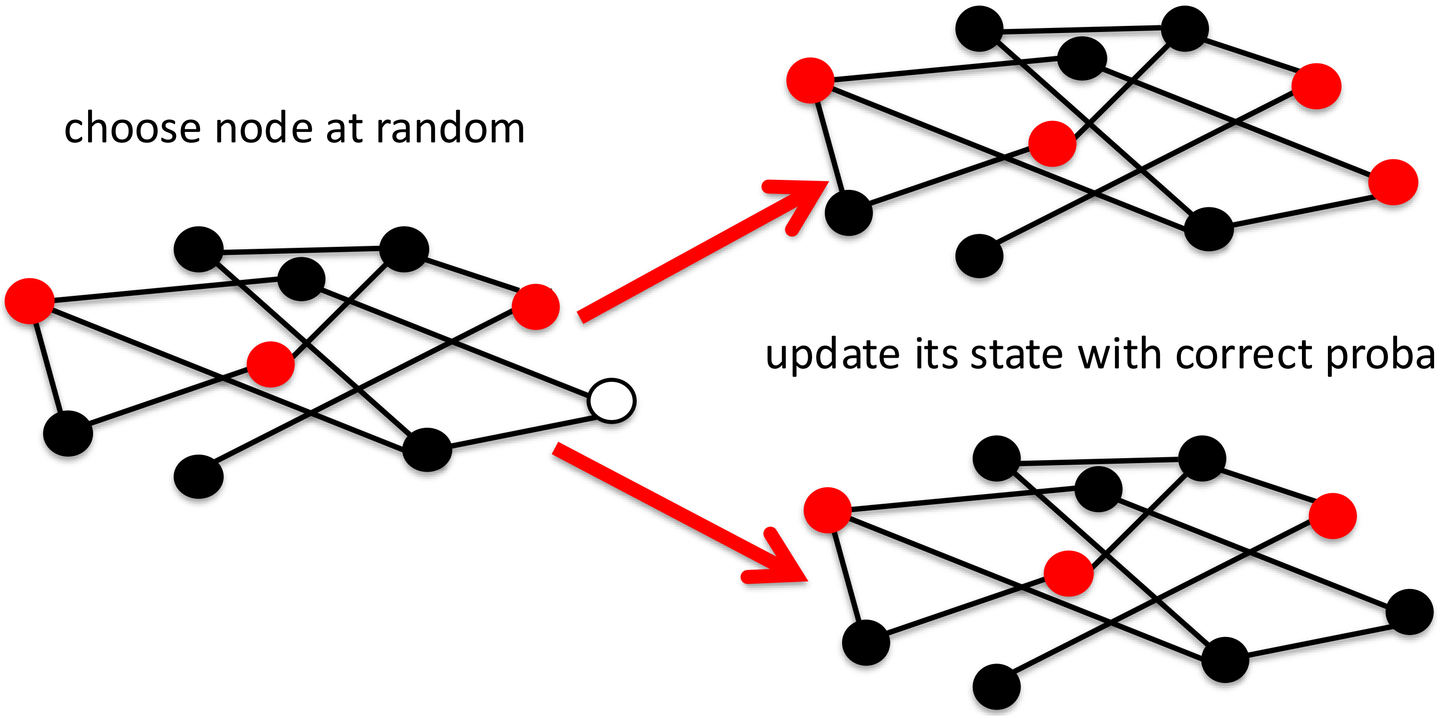

[Stroock, Zeglarinski '94], [Dyer et al '03]: For graphs of polynomial growth, if and only if there is correlation decay

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There are also connections to analysis (e.g. **Lee-Yang Theorems** relating phase transitions to zeros in complex plane)

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(4) **Computational:** When does approximate counting go from easy to hard?

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To Be Continued...

Any Questions?