Extended Formulations and Information Complexity

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The Permutahedron

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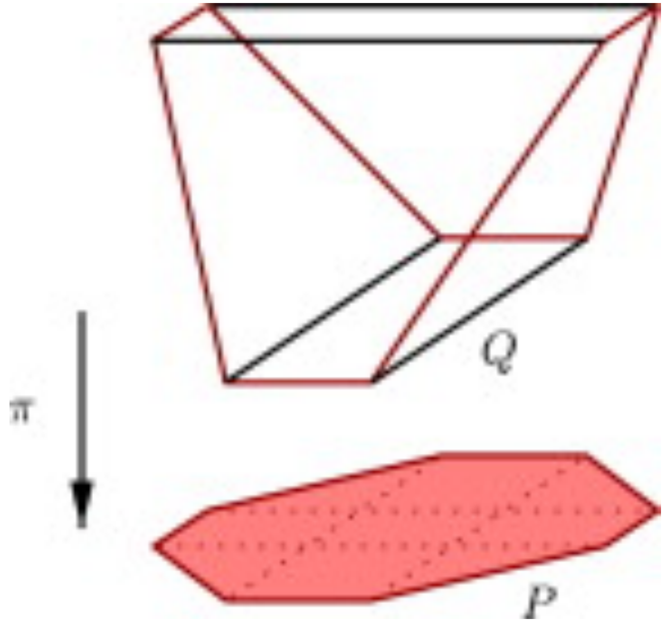
Yet Q has only $O(n^2)$ facets

Extended Formulations

The **extension complexity (xc)** of a polytope P is the minimum number of facets of Q so that P = proj(Q)



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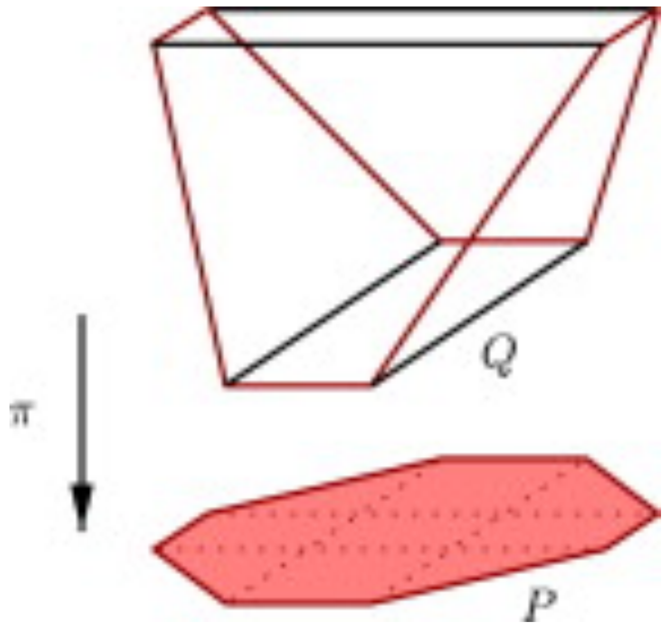

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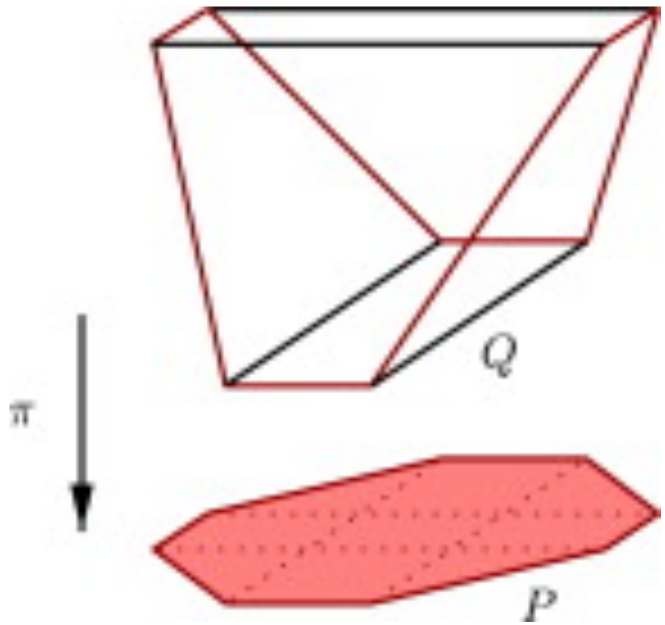

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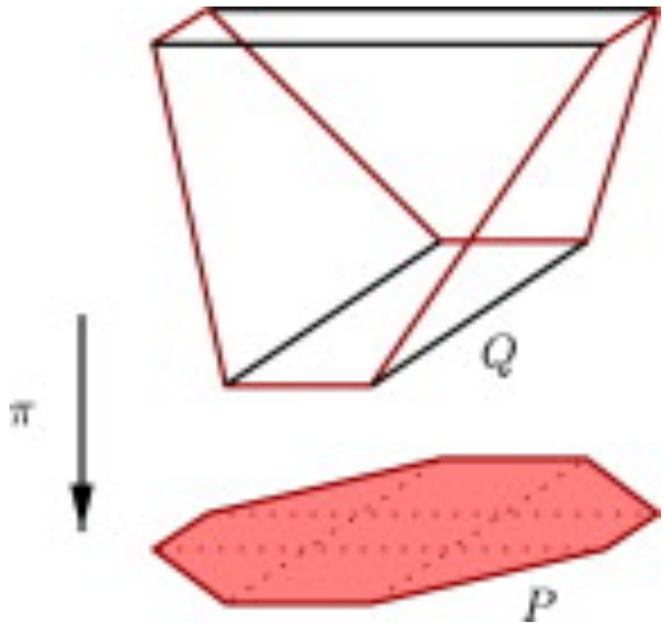

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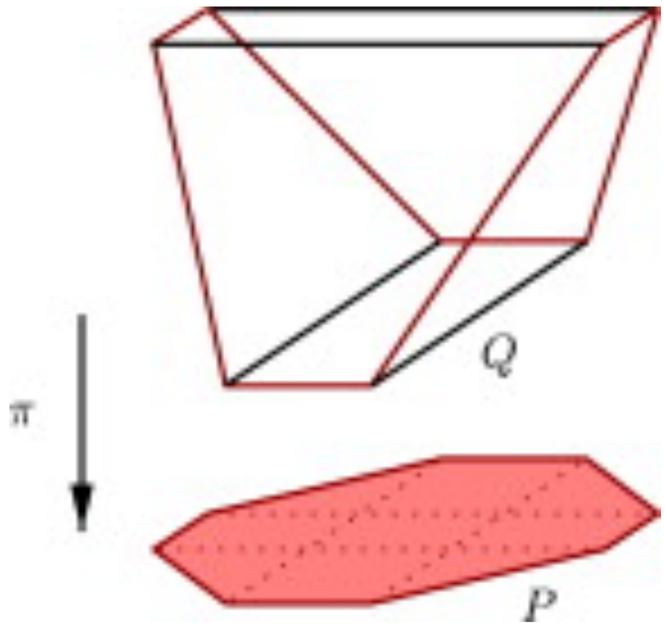

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…analogy with **quantifiers** in Boolean formulae

Applications of EFs

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e.g. prove there is low-cost object, through its polytope

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[Yannakakis '90]: Yes, through the **nonnegative rank**

An Abridged History

Theorem [Yannakakis '90]: Any symmetric EF for TSP or matching has size $2^{\Omega(n)}$

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Theorem [Fiorini et al '12]: Any EF for TSP has size $2^{\Omega(\sqrt{n})}$ (based on a $2^{\Omega(n)}$ lower bd for clique)

Approach: connections to non-deterministic CC

An Abridged History II

Theorem [Braun et al '12]: Any EF that approximates clique within $n^{1/2-eps}$ has size $exp(n^{eps})$

Approach: Razborov's rectangle corruption lemma

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Approach: information complexity

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Approach: information complexity

see also **[Braun, Pokutta '13]:** reformulation using common information, applications to avg. case

An Abridged History III

Theorem [Chan et al '12]: Any EF that approximates MAXCUT within 2-eps has size $n^{\Omega(\log n / \log\log n)}$

Approach: reduction to Sherali-Adams

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Theorem [Rothvoss '13]: Any EF for perfect matching has size $2^{\Omega(n)}$ (same for TSP)

Approach: hyperplane separation lower bound

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Definition of the **slack matrix**…

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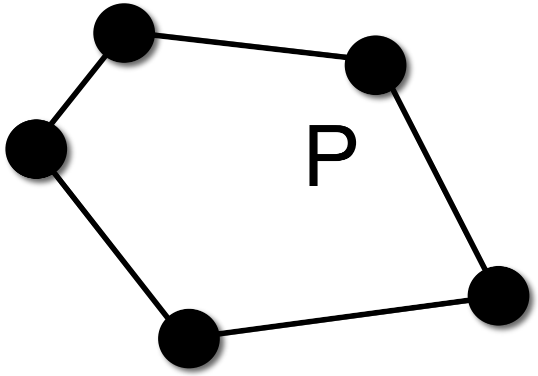

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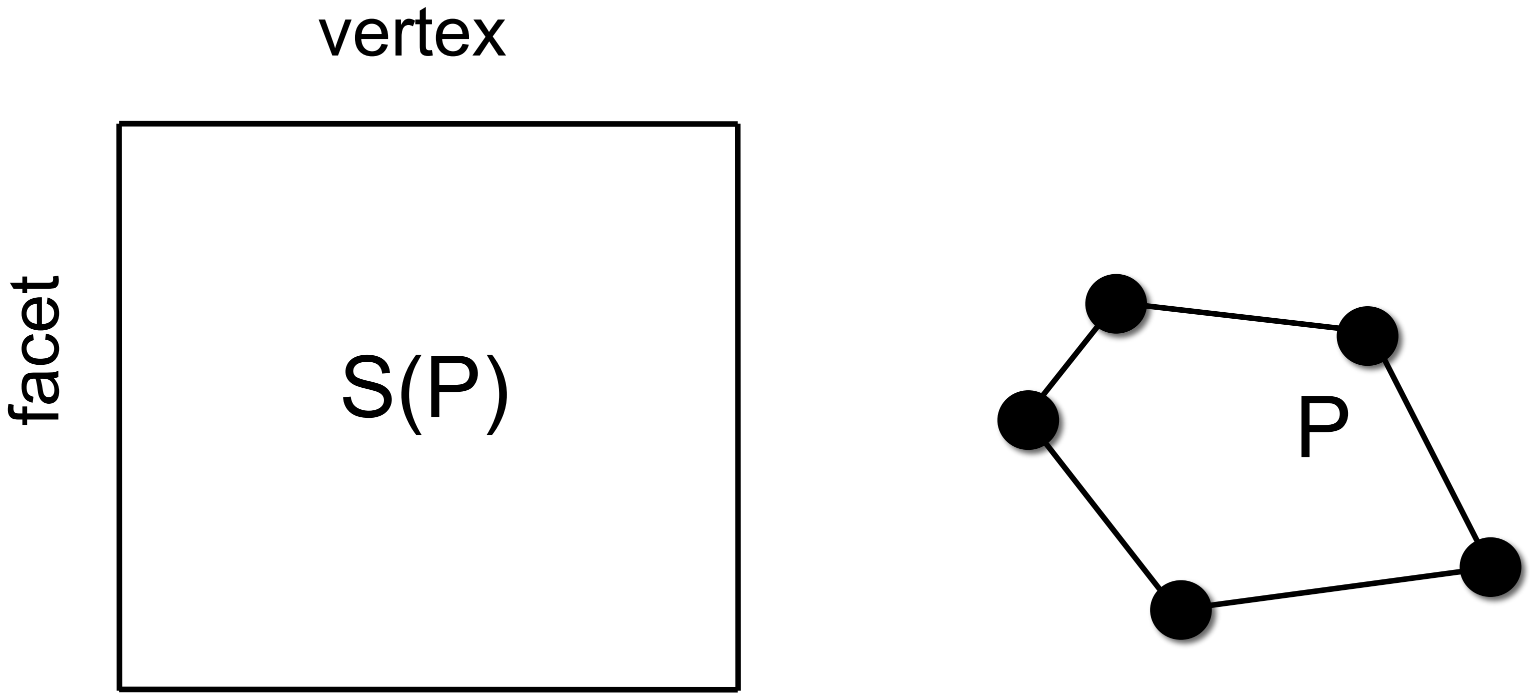

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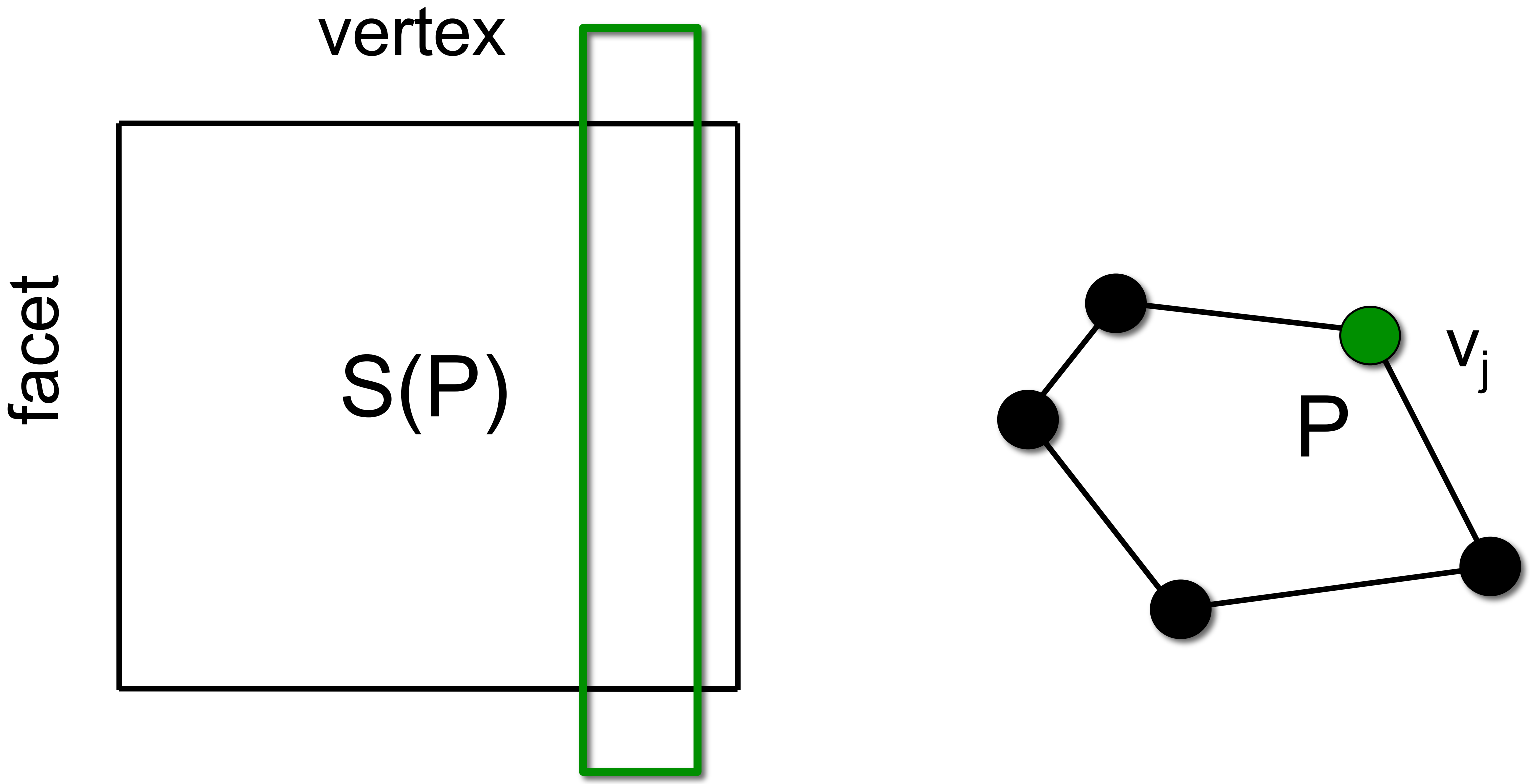

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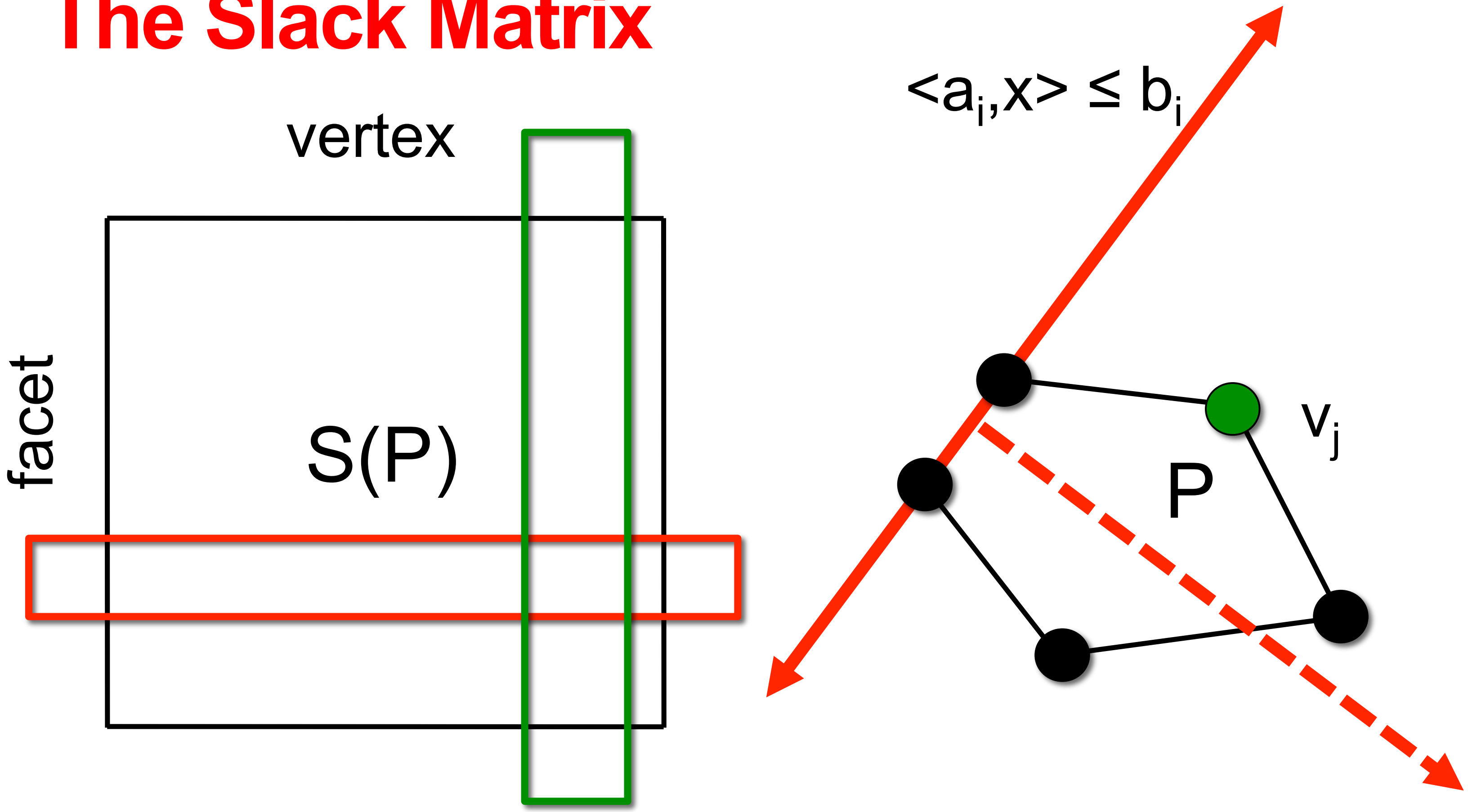

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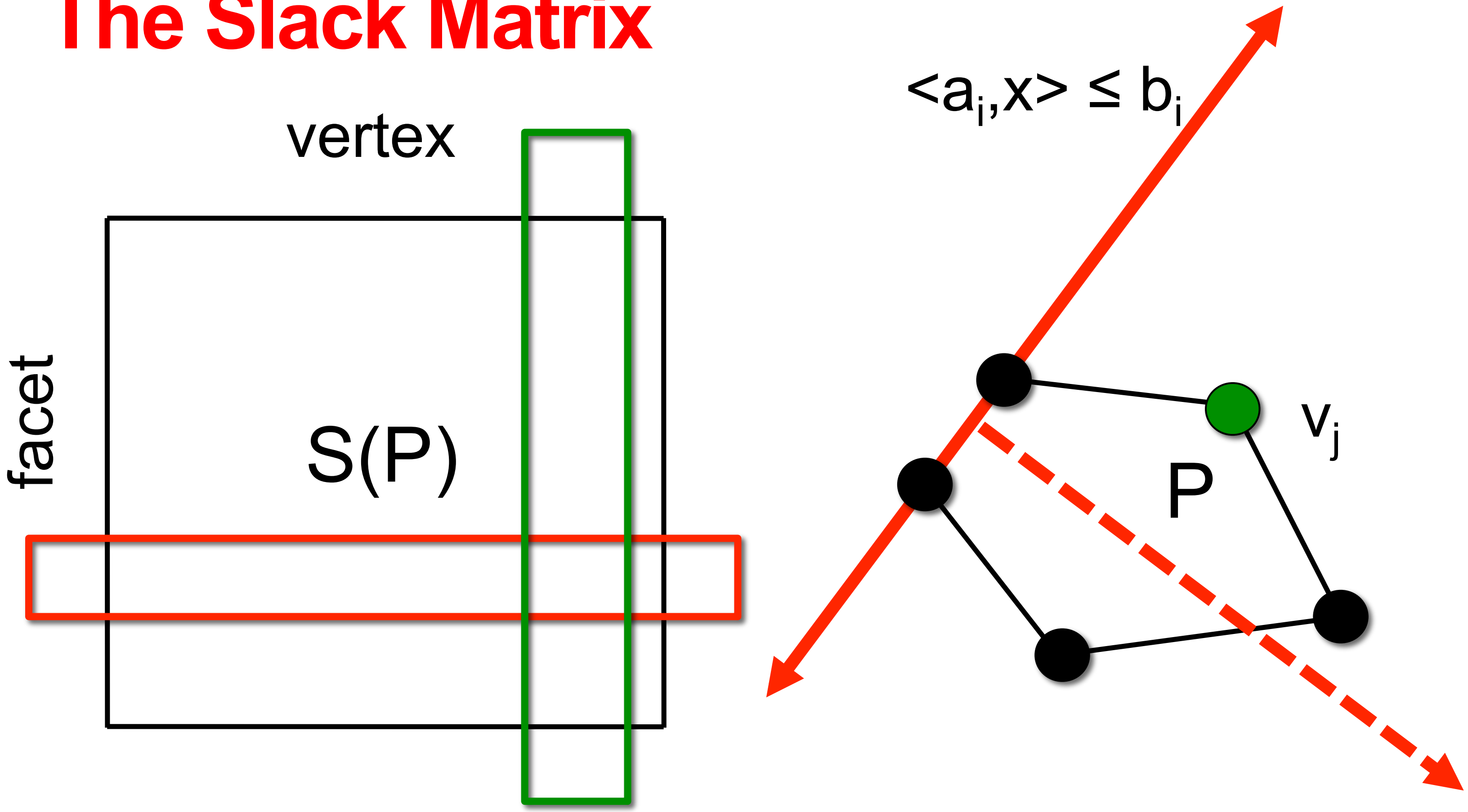

The entry in row i, column j is how *slack* the j^{th} vertex is on the i^{th} constraint

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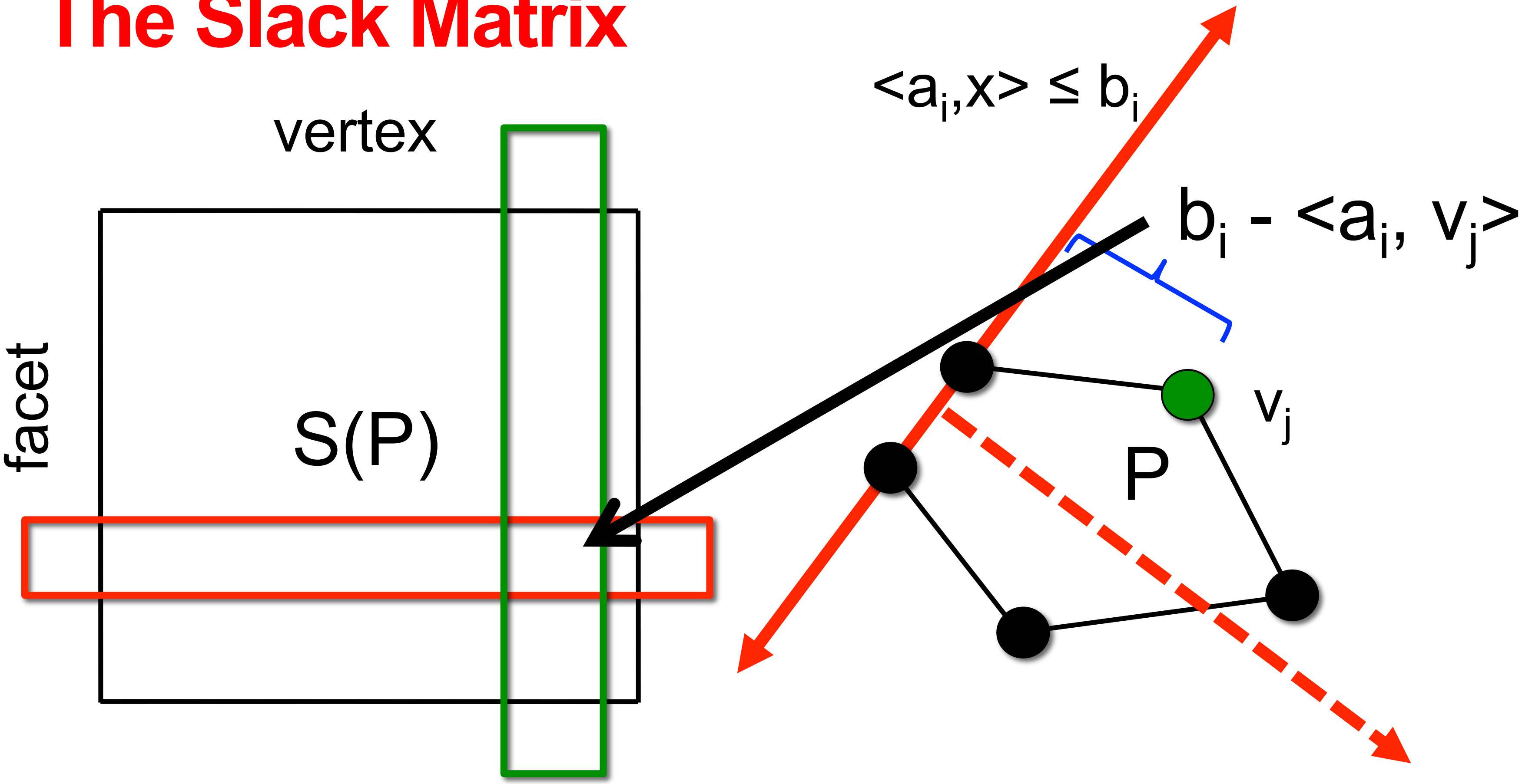

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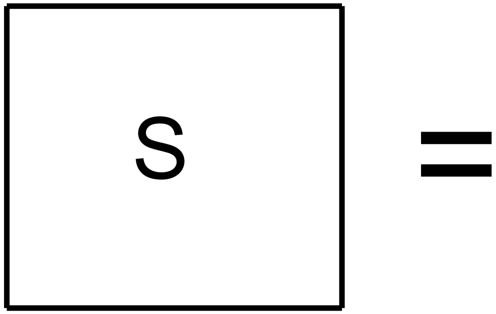
Definition of the **nonnegative rank**…

Nonnegative Rank



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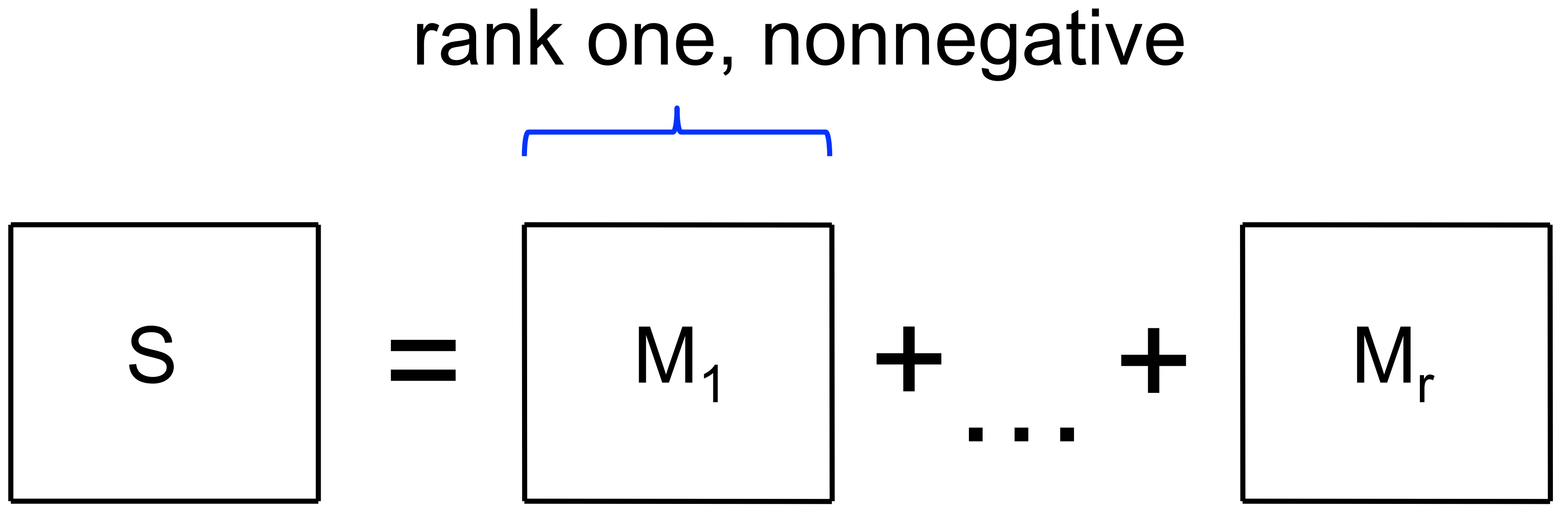

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rank one, nonnegative

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Note: rank$^+$(S) $\geq$ rank(S), but can be much larger too!

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Intuition: the factorization gives a change of variables that preserves the slack matrix!

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Next we will give a method to lower bound $rank^+$ via **information complexity**…

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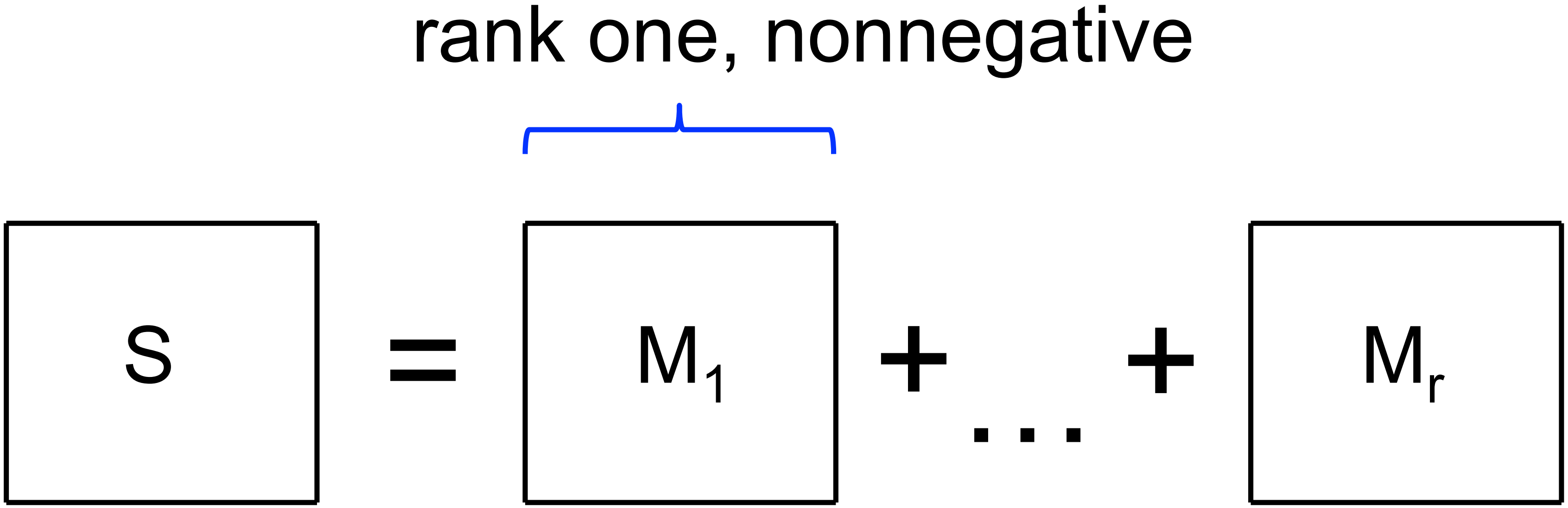

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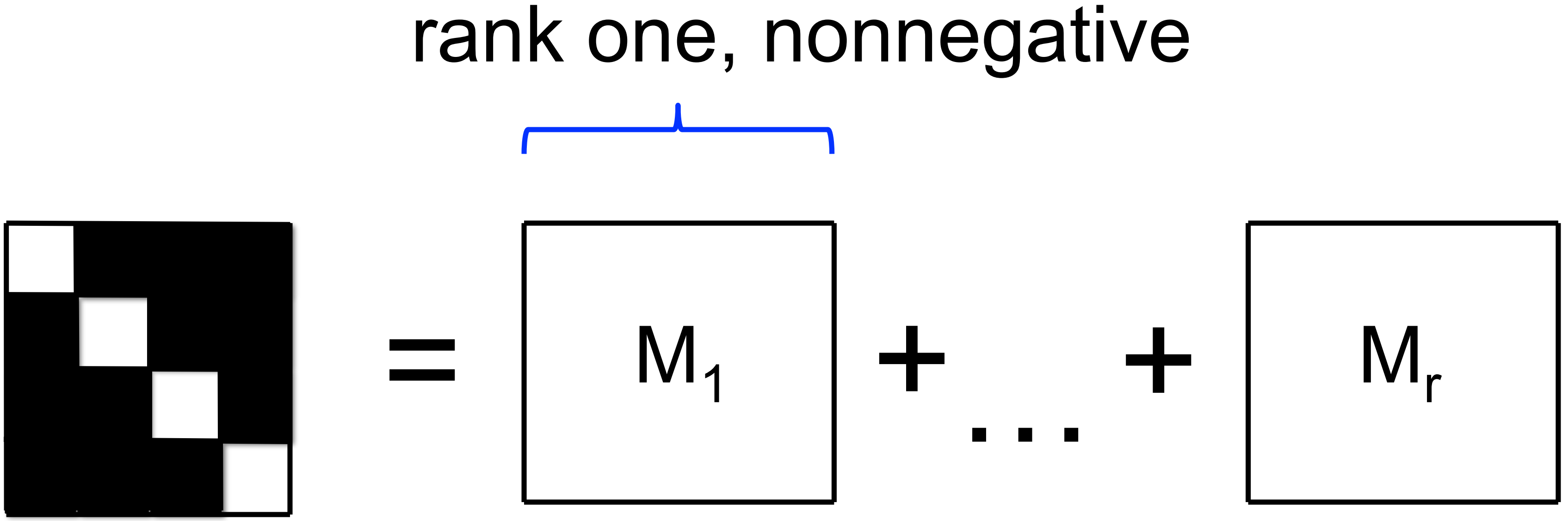

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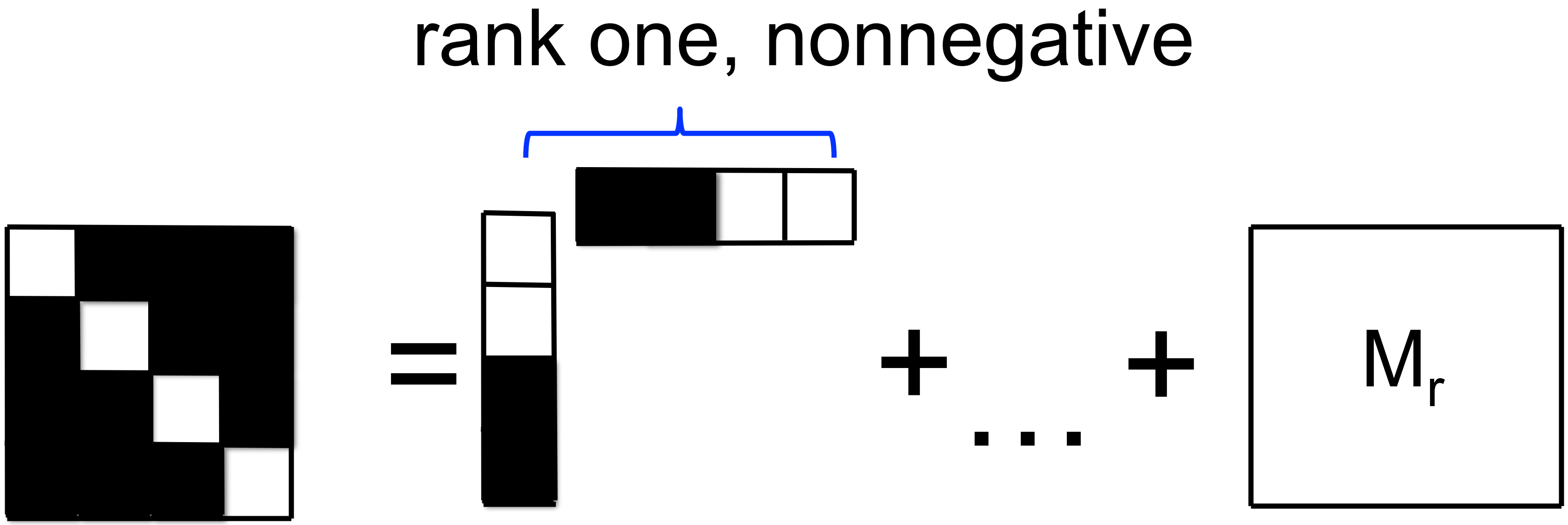

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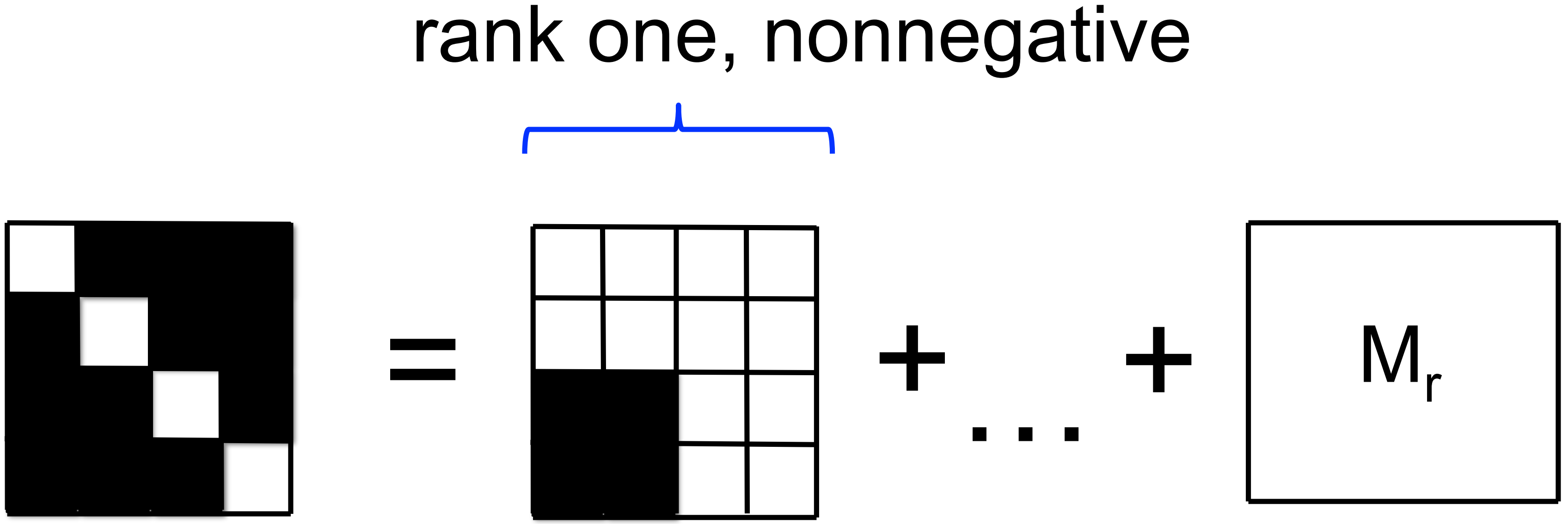

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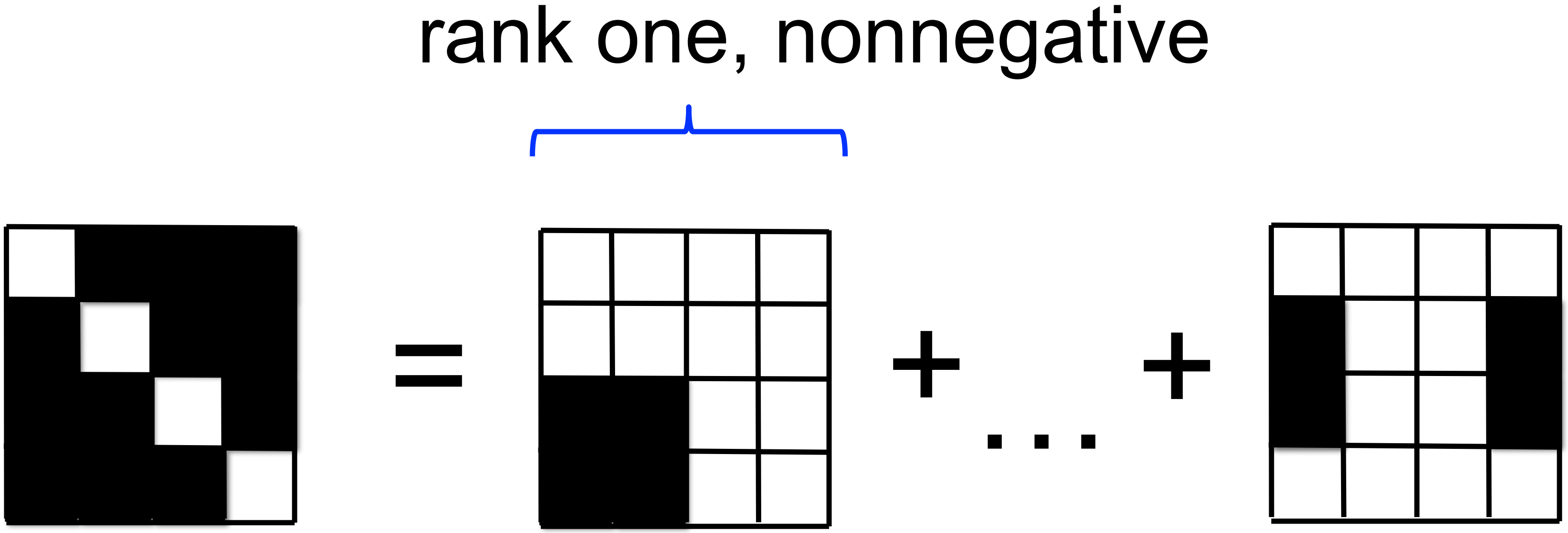

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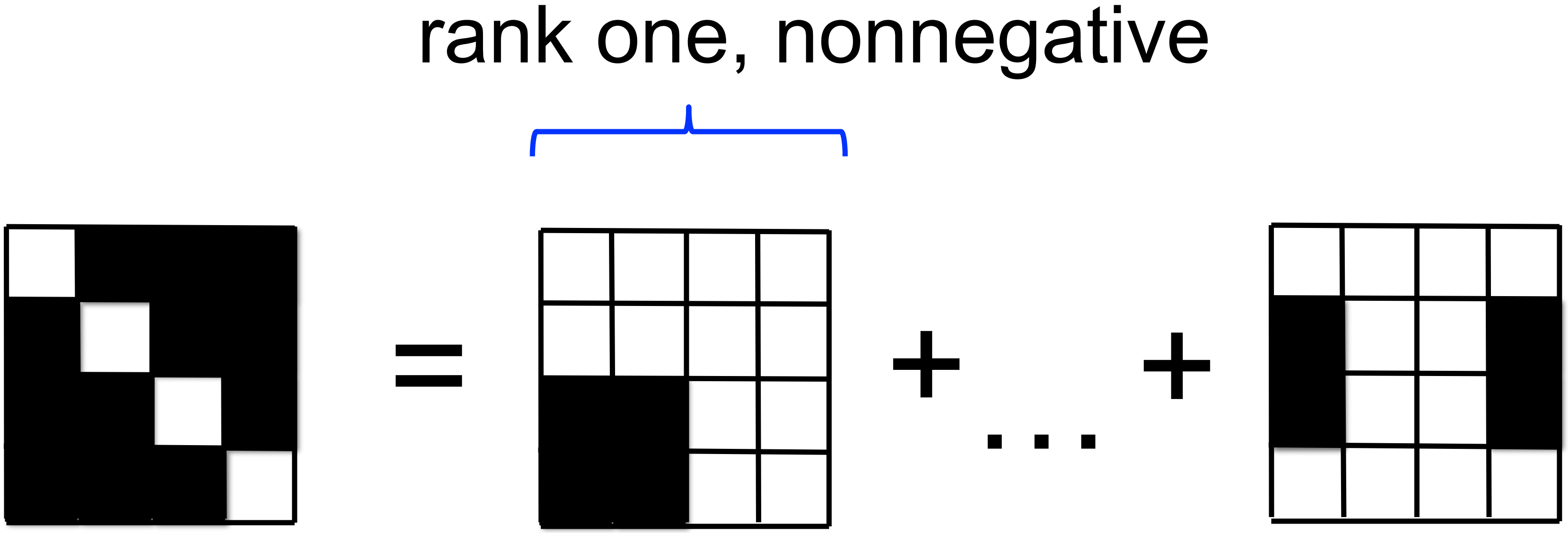

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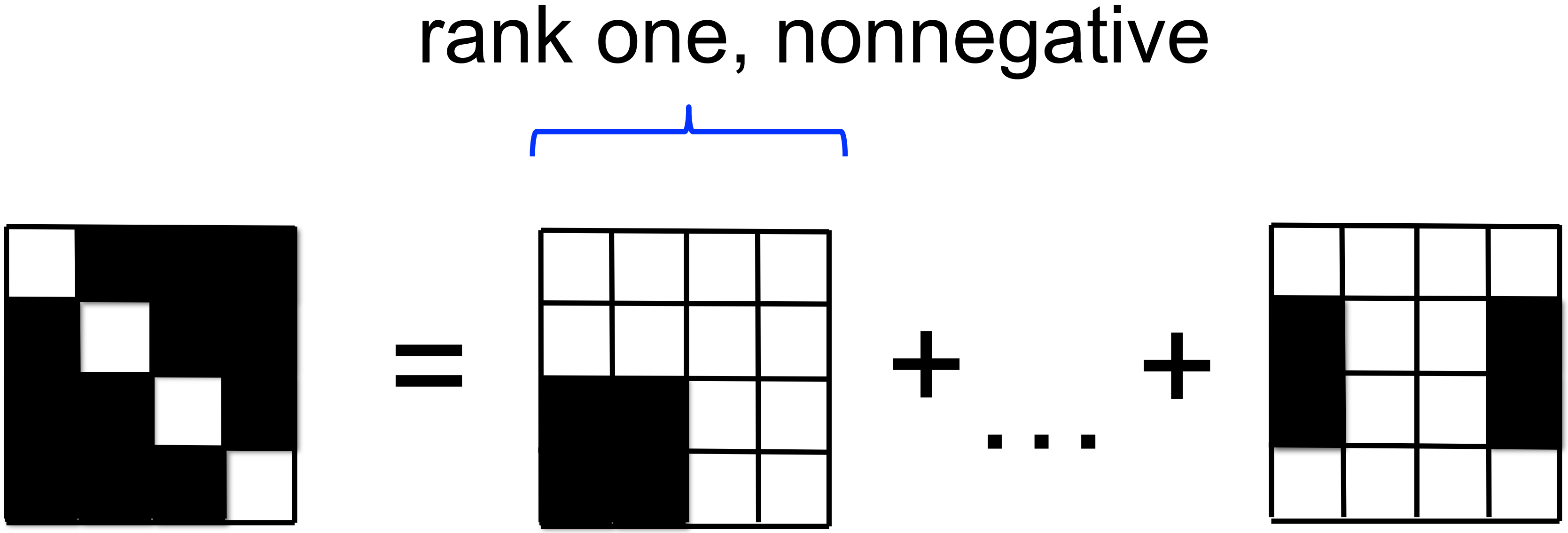

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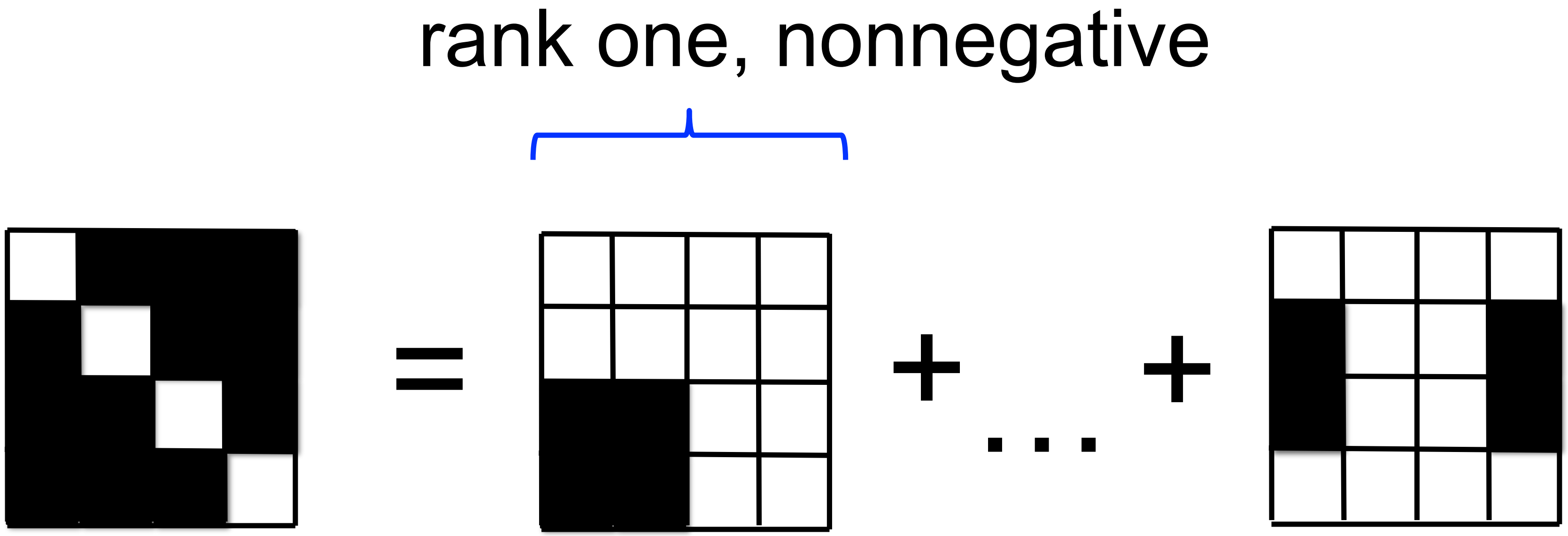

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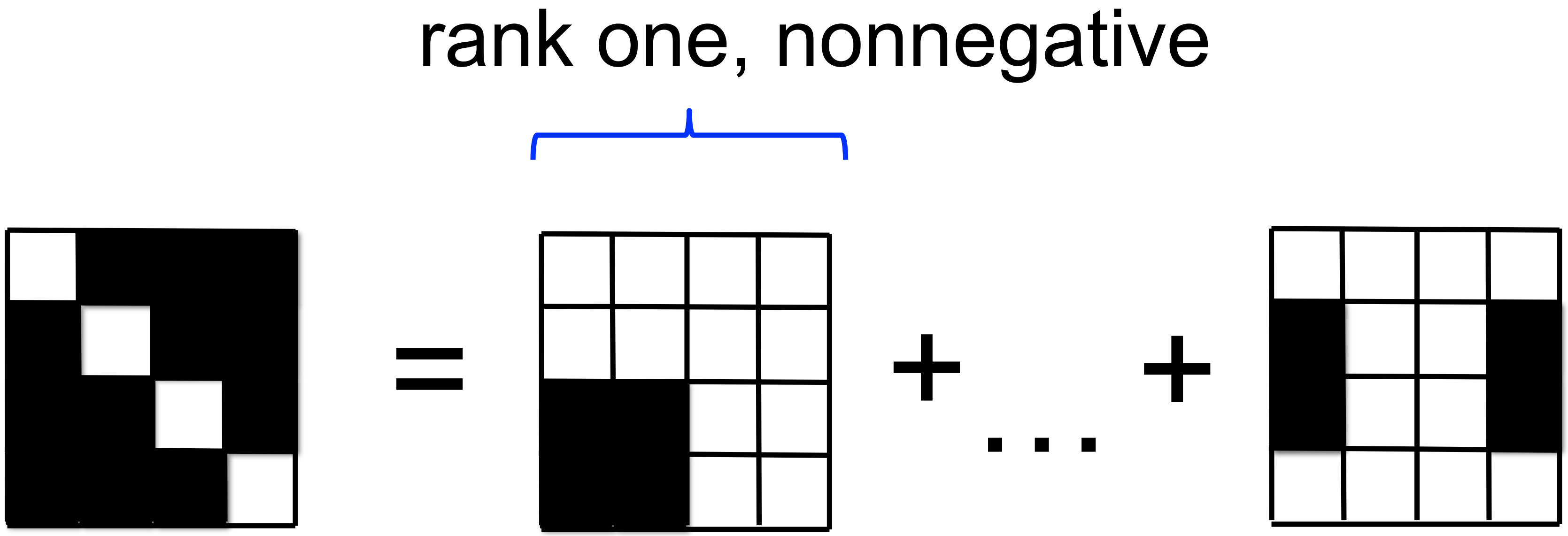

Non-deterministic Comm. Complexity

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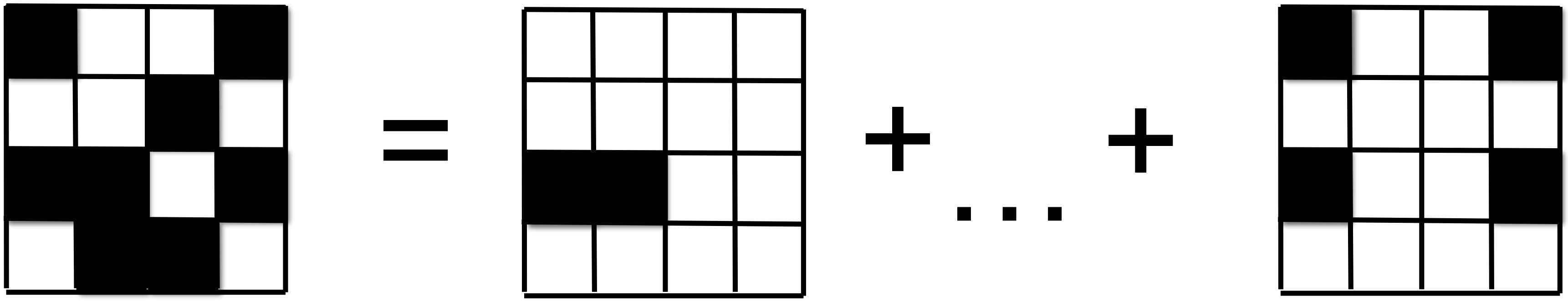

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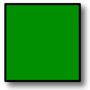
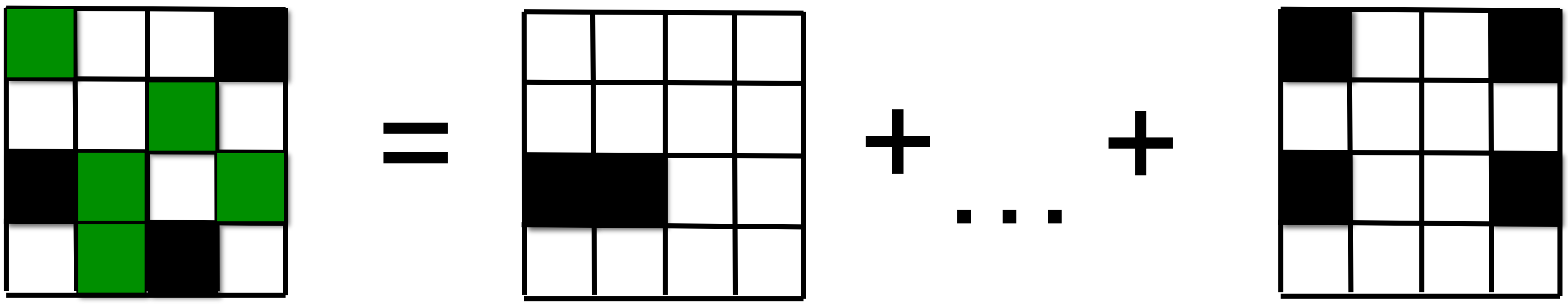
T = {}, set of entries in S with same value



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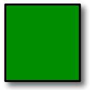
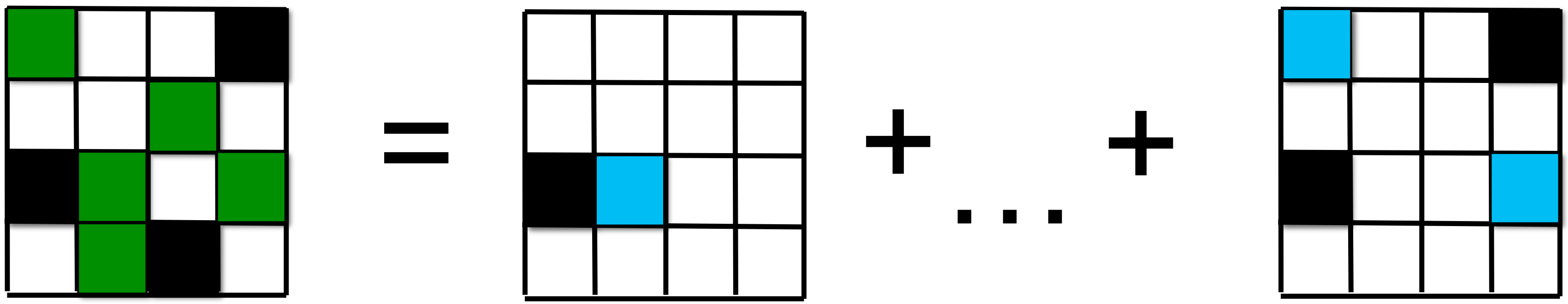
T = {}, set of entries in S with same value



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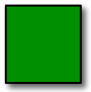
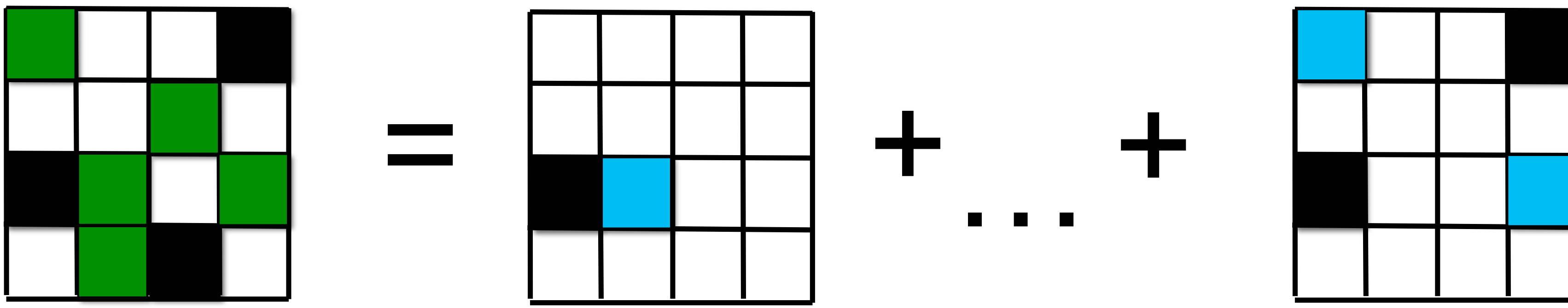
T = {}, set of entries in S with same value



Choose M_i proportional to total value on T

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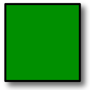
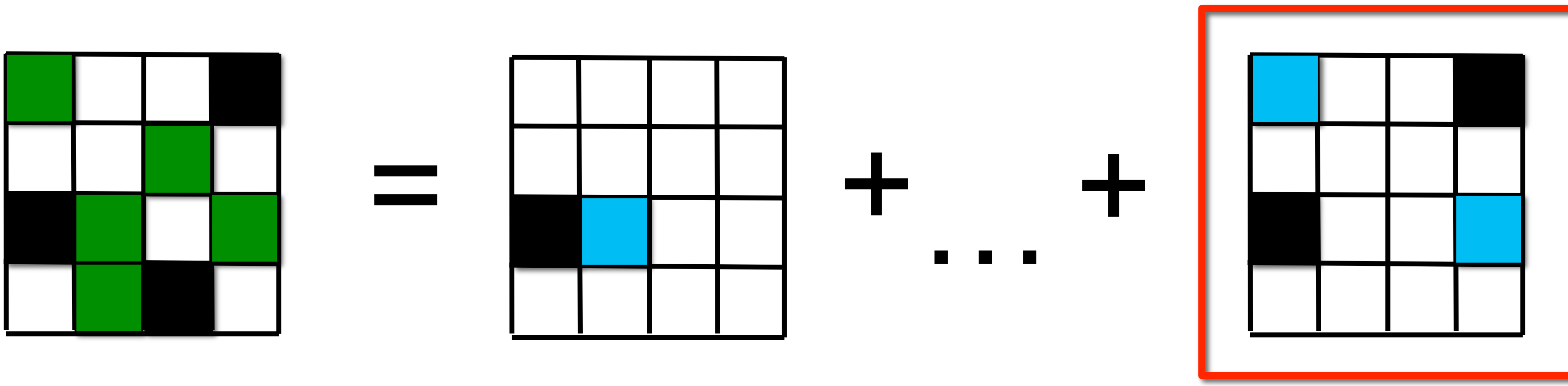
T = {}, set of entries in S with same value



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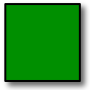
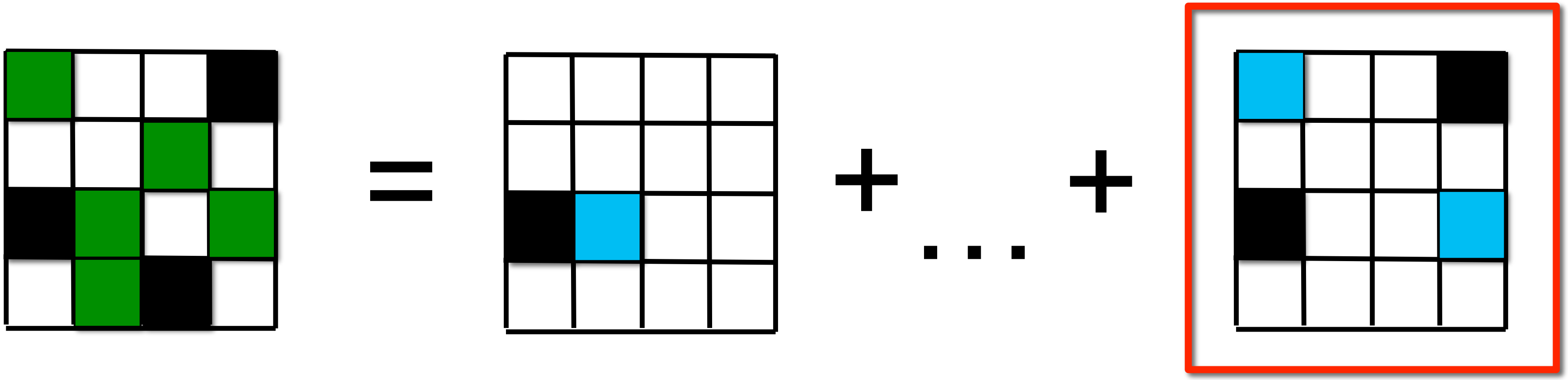
T = {}, set of entries in S with same value



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Choose (a,b) in T proportional to relative value in M_i

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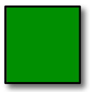
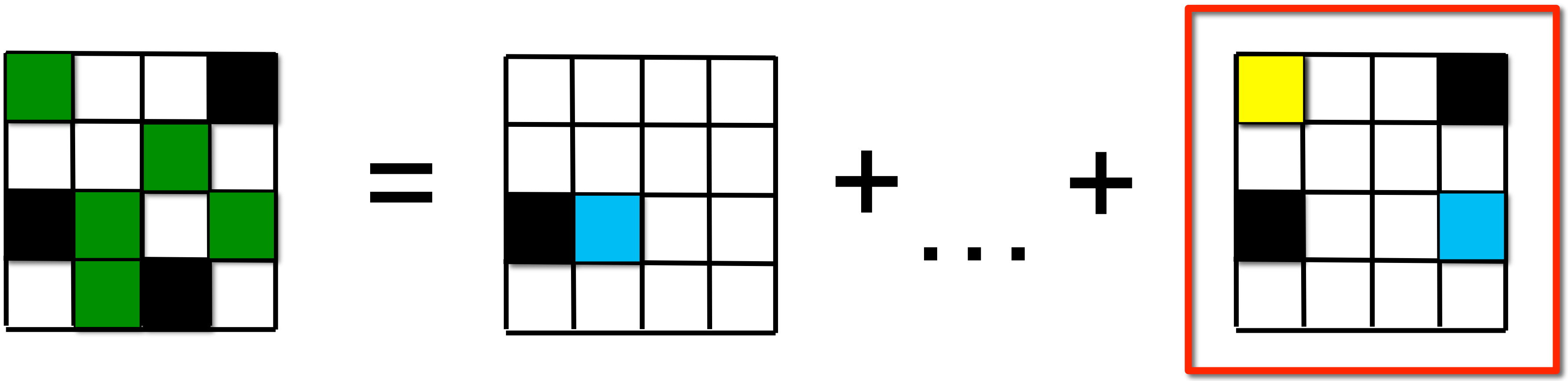
T = {}, set of entries in S with same value



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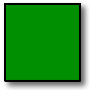
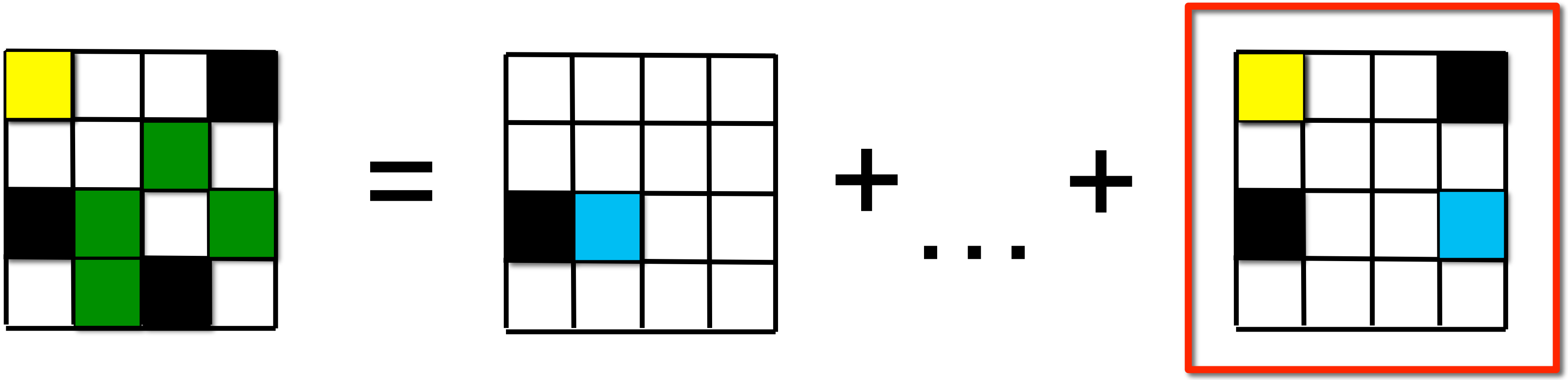
T = {}, set of entries in S with same value



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Choose (a,b) in T proportional to relative value in M_i

A Sampling Argument

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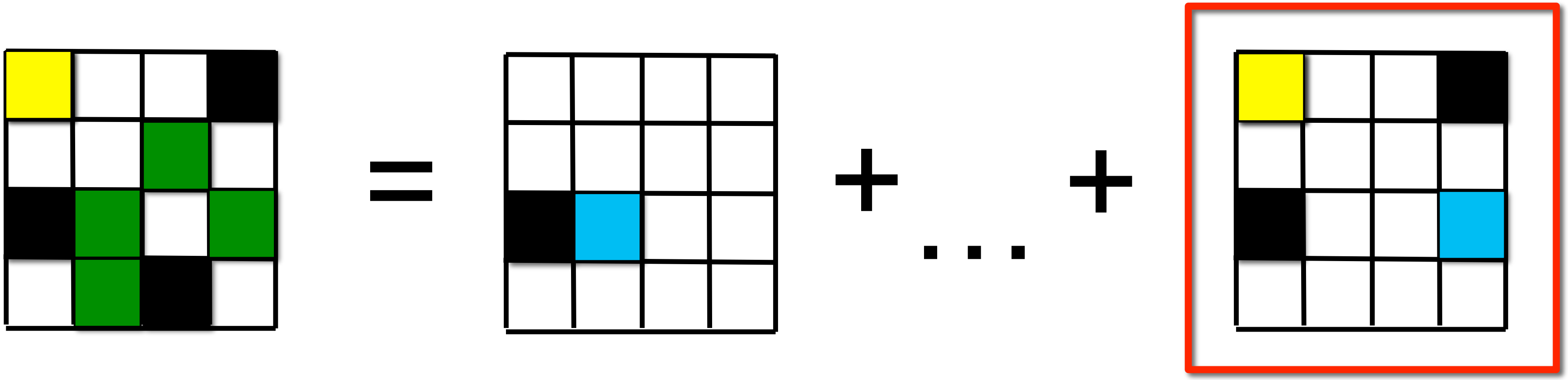

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This outputs a uniformly random sample from T

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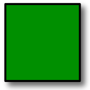
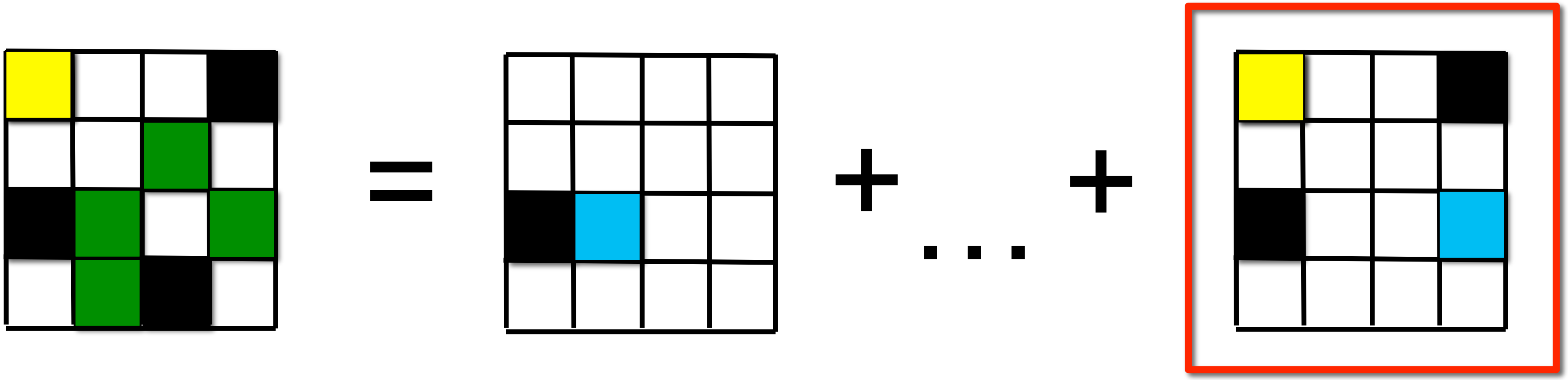
T = {  }, set of entries in S with same value



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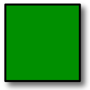
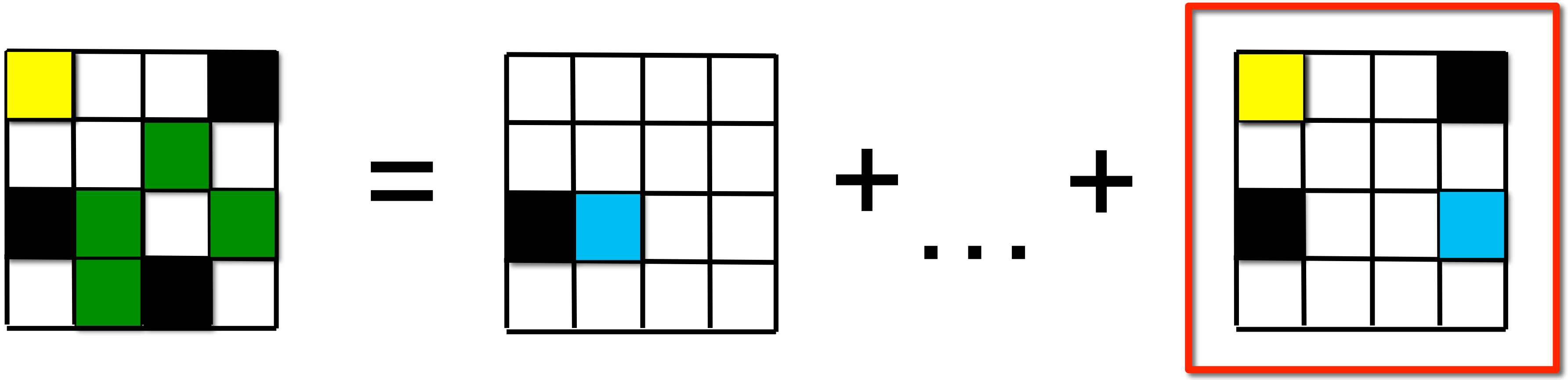
T = {}, set of entries in S with same value



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If r is too small, this procedure uses too little entropy!

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Part I: Tools for Extended Formulations

- Yannakakis's Factorization Theorem
- The Rectangle Bound
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Part II: Applications

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- Approximating the Correlation Polytope
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The Construction of [Fiorini et al]

correlation polytope: $P_{corr} = conv\{aa^T | a \text{ in } \{0,1\}^n \}$

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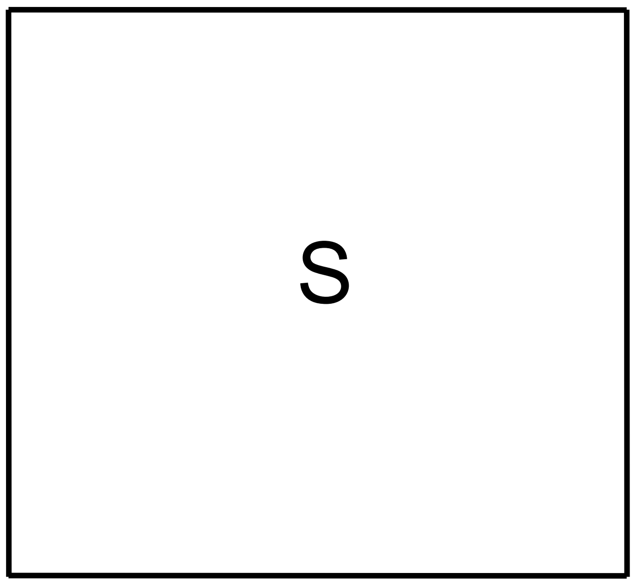

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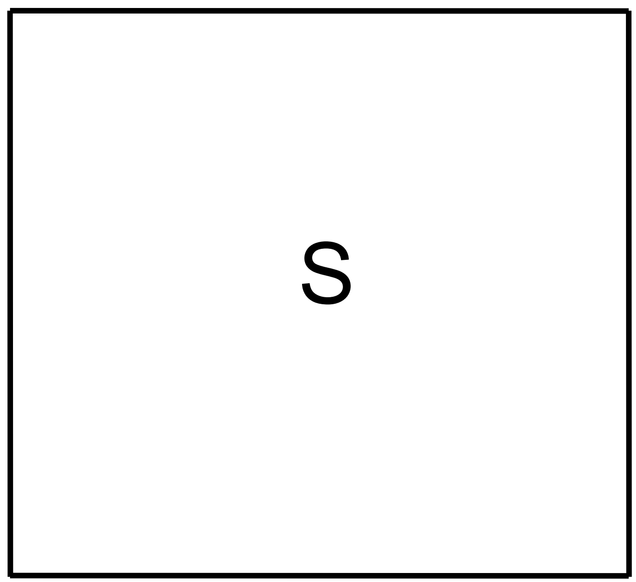

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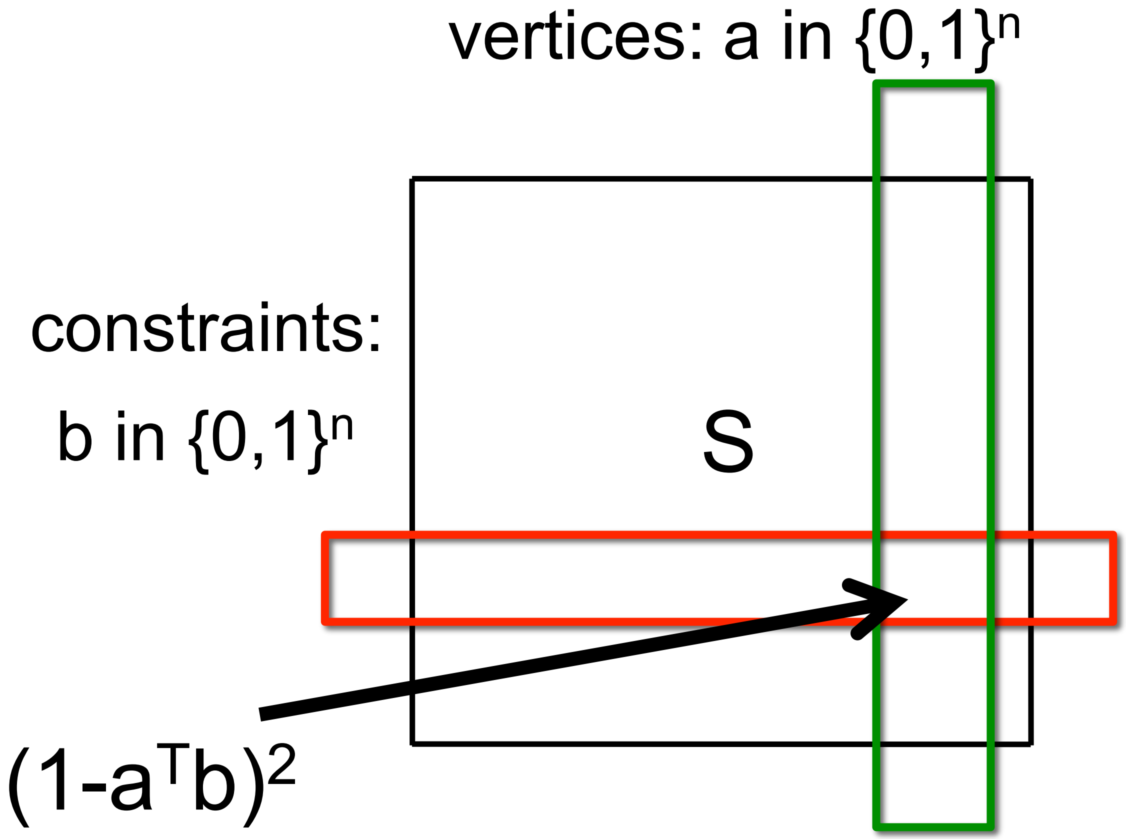

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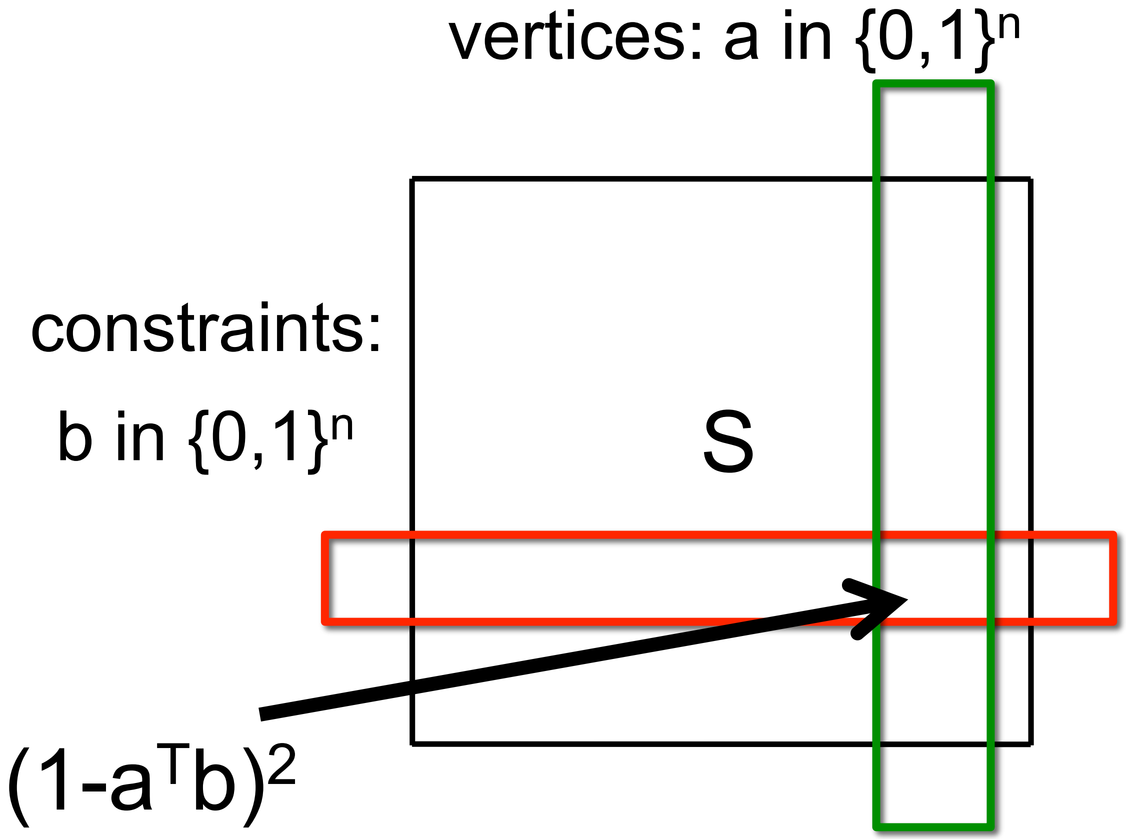

UNIQUE DISJ.
Output 'YES' if a and b as sets are disjoint, and 'NO' if a and b have one index in common

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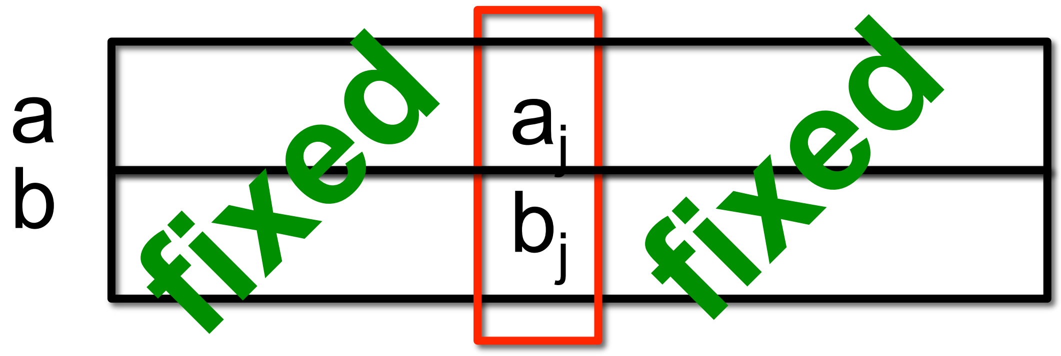
Suppose that a_{-j} and b_{-j} are **fixed**



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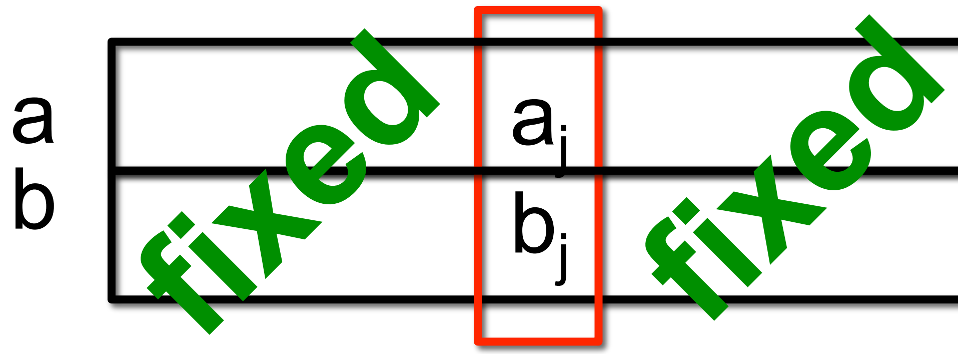

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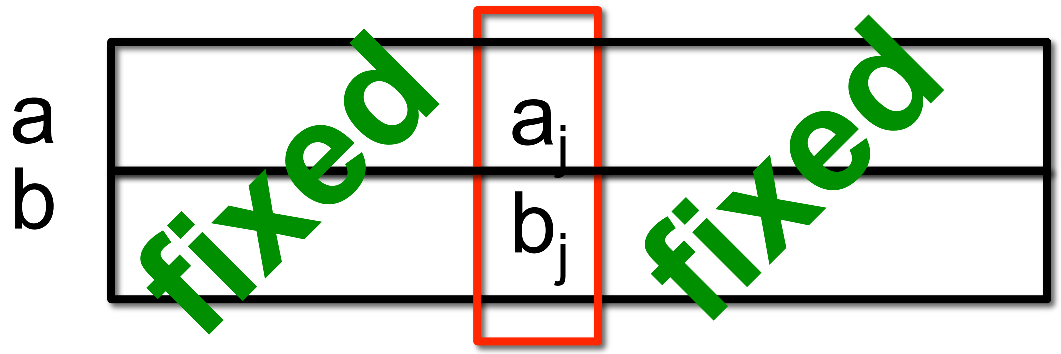

M_i restricted to (a_{-j},b_{-j})

	$(\ldots b_j=0\ldots)$	$(\ldots b_j=1\ldots)$
$(a_{1..j-1},a_j=0,a_{j+1\ldots n})$	$M_i(a,b)$	$M_i(a,b)$
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$H(a_j,b_j \mid i, a_{-j}, b_{-j}) \leq 1 < \log_2 3$

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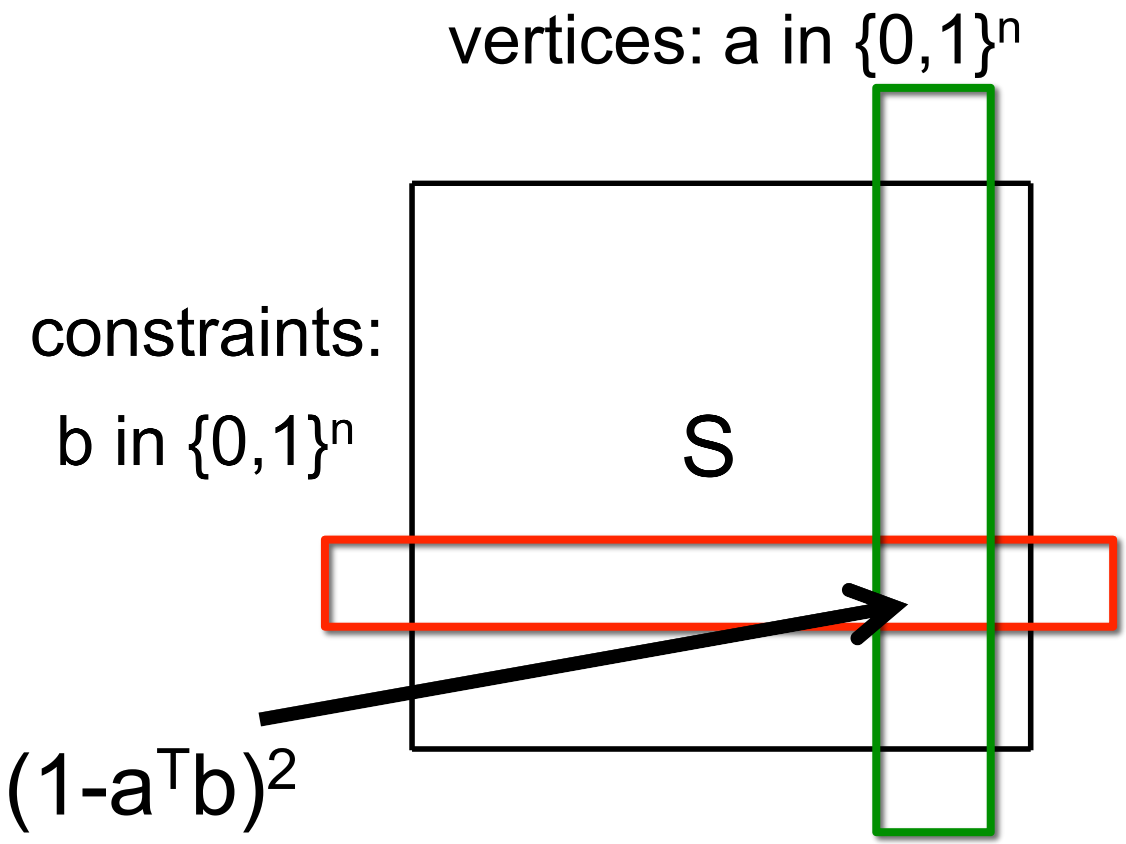
Approximate EFs [Braun et al]



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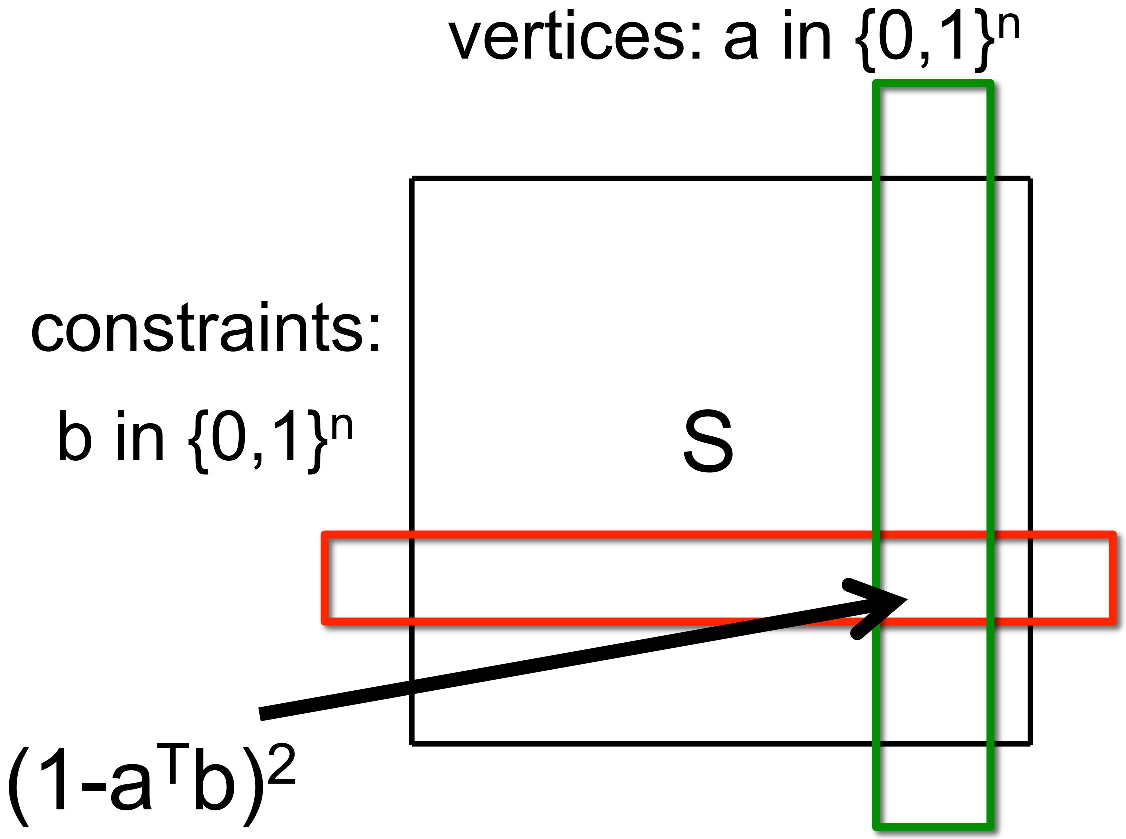

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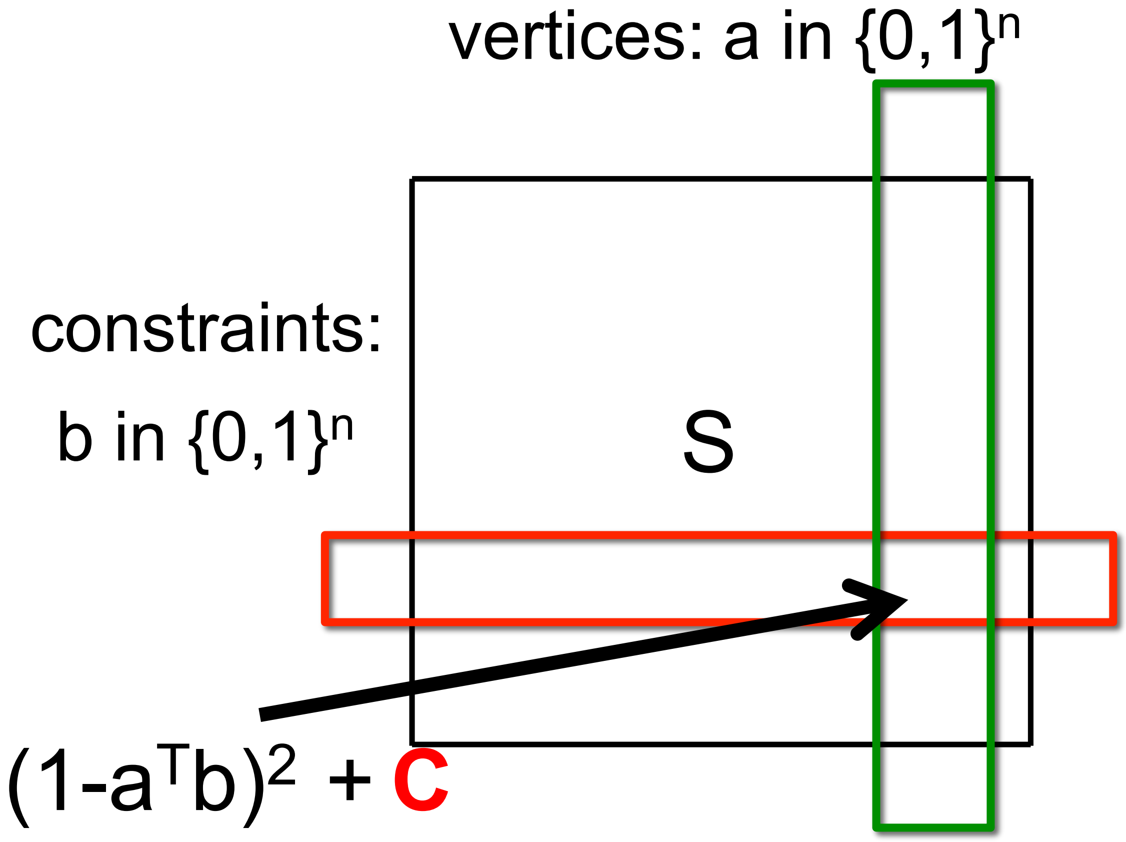

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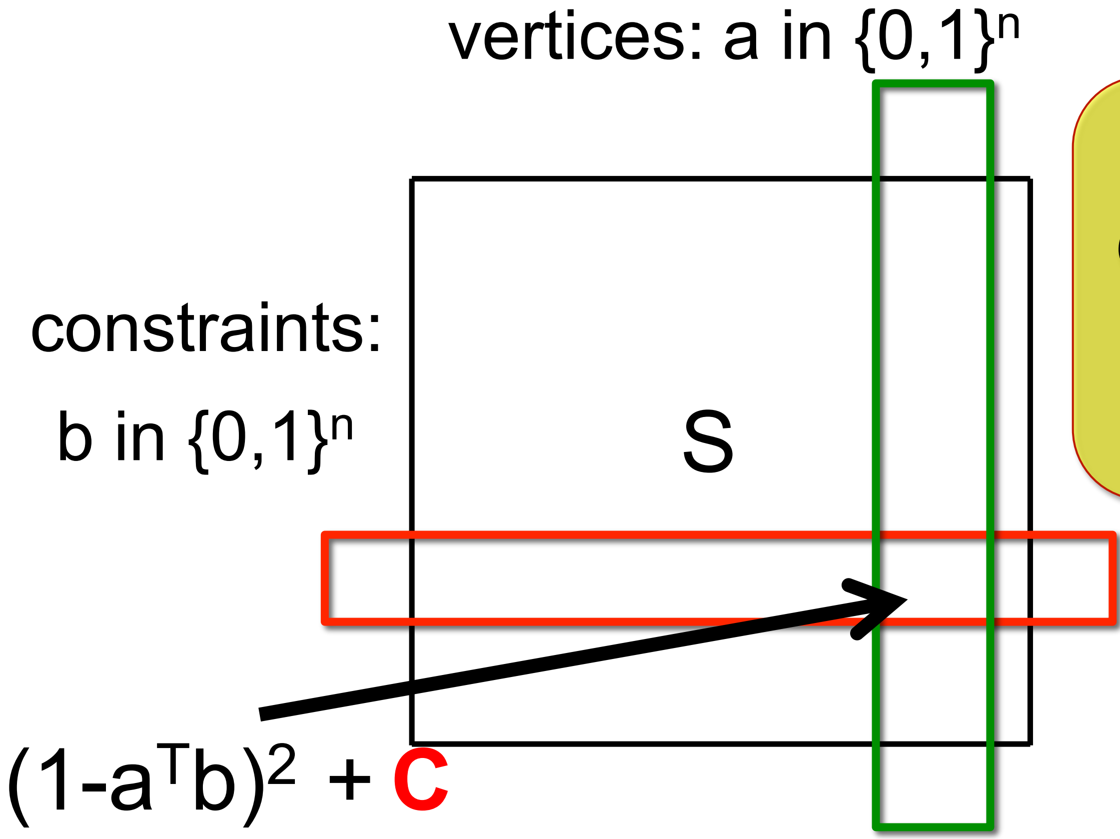

New Goal:
Output the answer to UDISJ with prob. at least ½ + 1/2(C+1)

Is the correlation polytope hard to approximate for large values of C?

Analogy: Is UDISJ hard to compute with prob. $\frac{1}{2}+1/2(C+1)$ for large values of C?

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Claim: If UDISJ can be computed with prob. $\frac{1}{2}+1/2(C+1)$ using $o(n/C^2)$ bits, then UDISJ can be computed with prob. $\frac{3}{4}$ using $o(n)$ bits

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Claim: If UDISJ can be computed with prob. $\frac{1}{2}+1/2(C+1)$ using $o(n/C^2)$ bits, then UDISJ can be computed with prob. ¾ using $o(n)$ bits

Proof: Run the protocol $O(C^2)$ times and take the majority vote

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Theorem [B-M]: Any EF that approximates clique within n^{1-eps} has size $\exp(n^{eps})$

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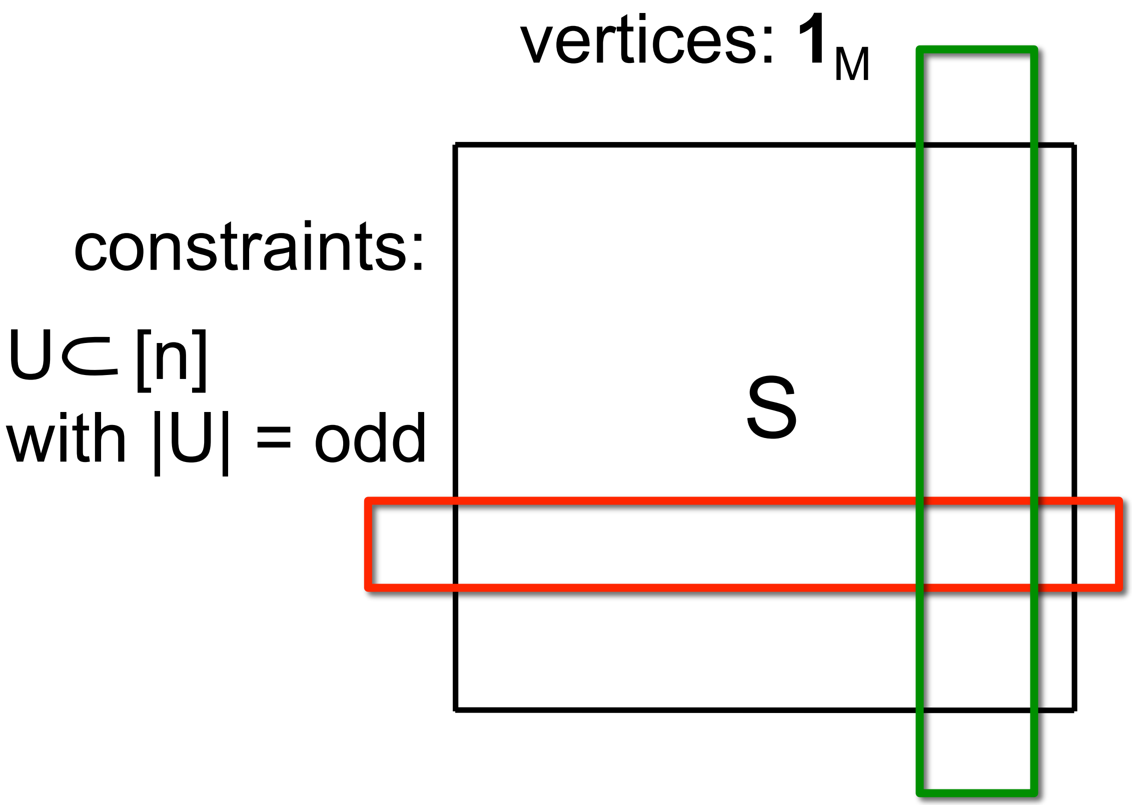

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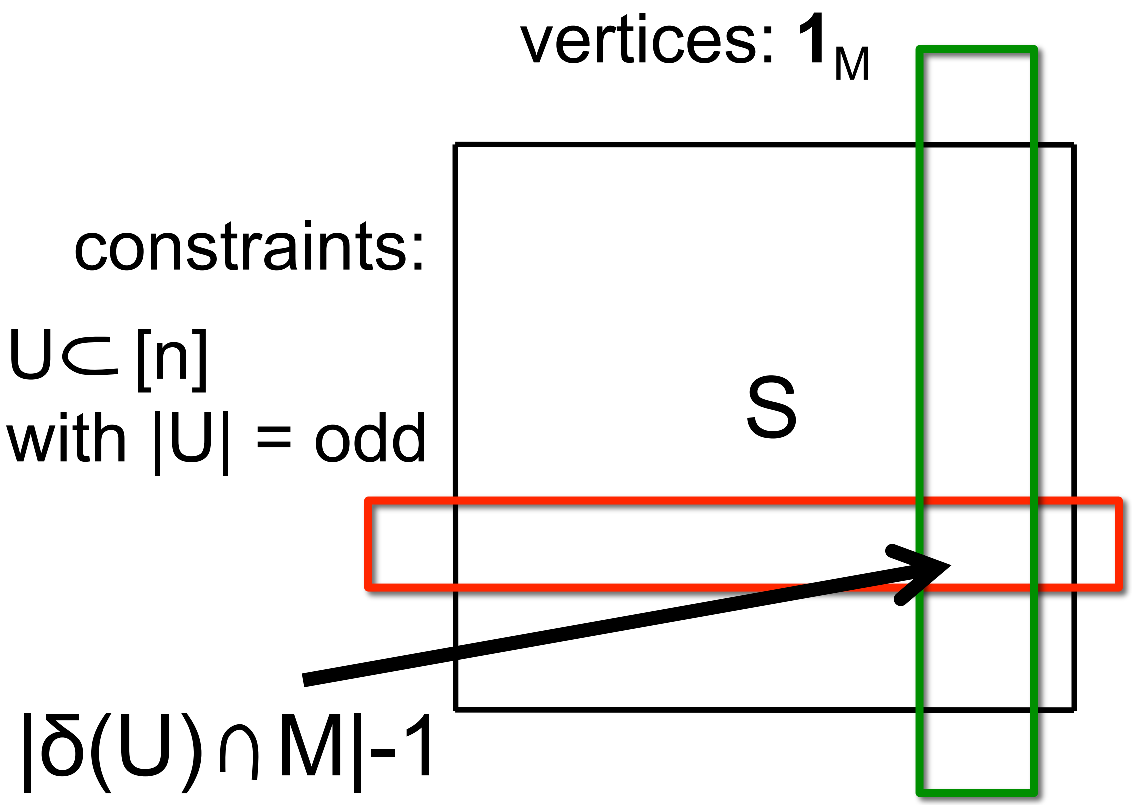

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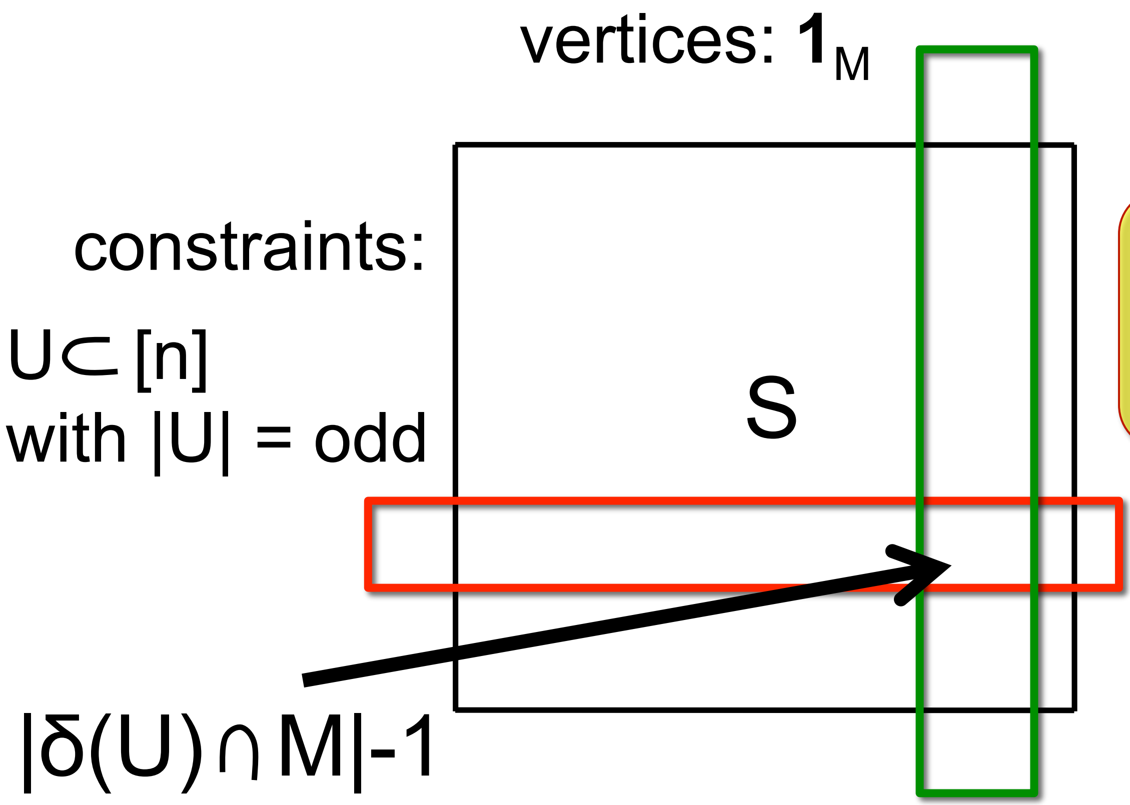

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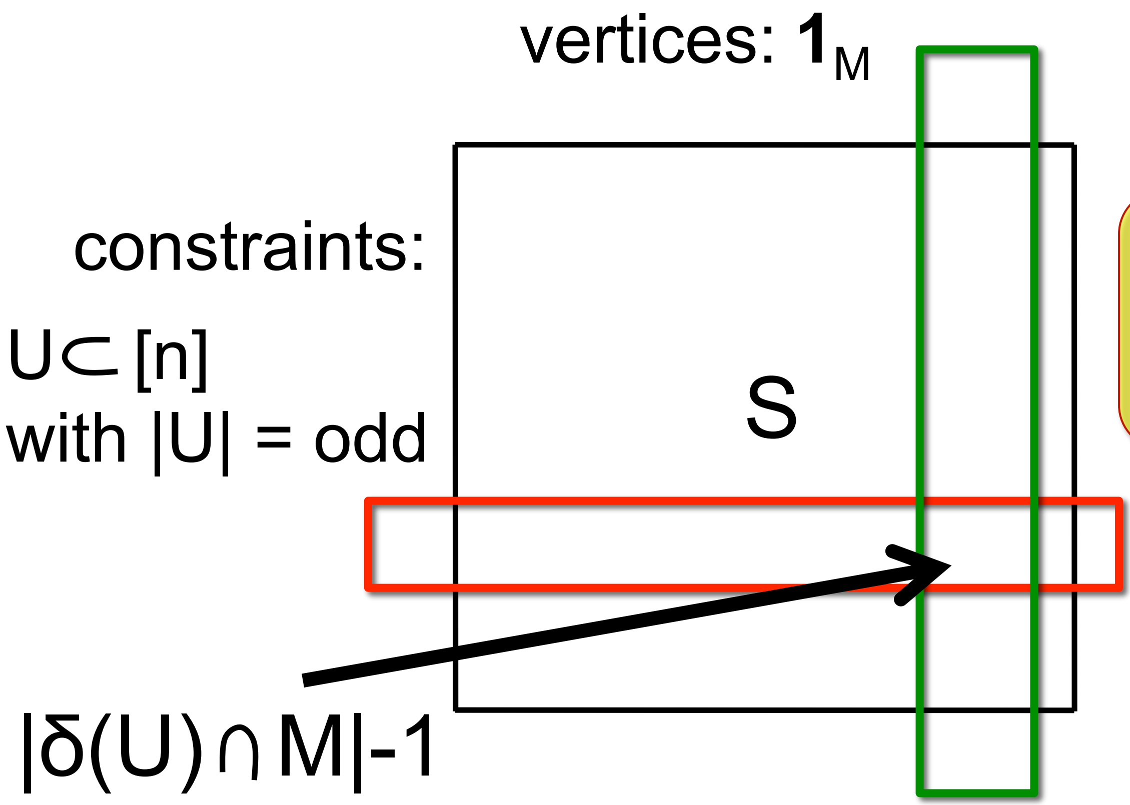

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Yes! Just guess two edges in M, crossing the cut

Hyperplane Separation Lemma

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Lemma: For slack matrix S, any matrix W:

$$\text{rank}^+(S) \geq \frac{\langle S, W \rangle}{\|S\|_\infty \, \alpha}$$

where $\alpha = \max \langle W, R \rangle$ s.t. R is rank one, entries in [0,1]

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Proof:

$$\langle W, S \rangle = \Sigma \, \|R_i\|_\infty \langle W, R_i / \|R_i\|_\infty \rangle \leq \alpha \, r \, \|S\|_\infty$$

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$$W_{U,M} = \begin{cases} -\infty & \text{if } |\delta(U) \cap M| = 1 \\ 1/Q_3 & \text{if } |\delta(U) \cap M| = 3 \\ -1/Q_k & \text{if } |\delta(U) \cap M| = k \\ 0 & \text{else} \end{cases}$$

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Proof is a substantial modification to Razborov's rectangle corruption lemma

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Any Questions?

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Thanks!