Problem 1

Part (a)

I argue that at any vertex M of the polytope, at least $m - 2n + 1$ of the x_{ij} must be equal to 0.

Since a vertex is a basic feasible solution, I know that at any vertex, (i) all equalities are tight and (ii) m linearly independent constraints are tight. There are $2n$ equalities, but only $2n - 1$ of them are independent, because $\sum_i \sum_{j \in N(i)} x_{ij} = \sum_j \sum_{i \in N(j)} x_{ij}$. So that leaves $m - 2n + 1$ tight inequalities (of the form $x_{ij} \geq 0$). Hence, $m - 2n + 1$ of the x_{ij} are 0.

Part (b)

In M, there are potentially m nonzero entries (x_{ij}), one for each edge ij. Because of the equality constraint, each row must have at least one nonzero entry. Suppose that each row has at least 2 nonzero entries, for a total of at least $2n$ nonzero entries. Then, there can be at most $m - 2n$ zero entries. This contradicts part (a), in which I showed that at least $m - 2n + 1$ of the entries are 0. So at least one row of M must contain a single nonzero entry. Because of the equality constraint on this row, this nonzero entry must be 1 with all other entries 0. Because of the equality constraint on the column of this single nonzero entry, all other entries in that column must be 0.

Part (c)

Let the single entry of value 1 from part (b) be x_{ij}, so that it involves row i and column j. Let M' be the matrix obtained by deleting row i and column j. The sum of each remaining row and column of M' is still 1, because only zero entries have been deleted from these rows and columns. Therefore, M' is doubly stochastic. Suppose I loose k variables when going from M to M'. I lost two equality constraints and $k - 1$ tight inequality constraints, only $k - 2$ of which are independent (since I can determine one entry by the sum constraint from part (a)). Therefore, I loose as many constraints as I loose variables, so that M' is still a vertex of the resulting LP. Hence, M' satisfies part (b), so it has at least one row / column containing a single 1 with all other entries 0, and it can be reduced further. When $n = 1$, the matrix has a single entry with 1. I have thus shown that any vertex M is an integer doubly stochastic matrix (i.e. a perfect matching). Because any feasible point of the polytope is a convex combination of vertices, any solution is a convex combination of perfect matchings.

Let λ be the maximum of (i) the largest rate at which packets arrive on an input line and (ii) the maximum rate at which packets want to depart from an output line. By the result above, the switch can decompose any demand into a convex combination of matchings, in which the sum of the weights of the matchings is $\leq \lambda$. So as long as the switch can deliver matchings at rate λ, it can deliver the specified traffic.

Problem 2

Farkas' lemma states that *exactly one* of the following two systems has a solution:

1. $Ax = b$ and $x \geq 0$
2. $yA \geq 0$ but $yb < 0$

In this problem, I prove Farkas' lemma.

Part (a)

I give a direct proof that the two systems cannot both be feasible, by showing that if 1 is feasible then 2 is unfeasible.

Suppose 1 is feasible, so $Ax = b$ and $x \geq 0$. Suppose that $yA \geq 0$. Then $yb = y(Ax) = (yA)x \geq 0$ since $yA \geq 0$ and $x \geq 0$. So 1 is feasible implies that 2 is unfeasible.

Part (b)

I shows that the linear program $\min yb \mid yA \geq 0$ can only have two possible answers: 0 and $-\infty$.

First, notice that the linear program has the trivial solution $y = 0$, which gives $yb = 0$. So the minimum $yb \leq 0$.

Suppose $\neg\exists y$, s.t. $yb < 0$ and $yA \geq 0$. So the minimum is $yb = 0$.

Suppose $\exists y$, s.t. $yb < 0$ and $yA \geq 0$. Then, I can multipliy y by some $\lambda > 0$, $y' = \lambda y$. Then, $y'b < yb$ and $y'A > yA$. So y' satisfies the linear program constraint and gives a smaller $y'b$. With $\lambda \to \infty$, $y'b = \lambda yb \to -\infty$. So the minimum is $-\infty$.

Part (c)

I show that if 1 is unfeasible, then 2 is feasible. Combined with part (a), this proves Farkas' lemma.

Consider the dual of $\min yb \mid yA \geq 0$: $\max 0x \mid Ax = b, x \geq 0$. If 1 is unfeasible, then the dual is unfeasible. If the dual is unfeasible, by strong duality, the primal is either unfeasible or unbounded. By part (b), the primal cannot be unfeasible. Therefore it is unbounded. Therefore, $\exists y$, s.t. $yA \geq 0$ but $yb < 0$, i.e. 2 is feasible. So 1 is unfeasible implies that 2 is feasible.

Problem 3

I suppose I am given two polyhedra $P = \{x|Ax \leq b\}$ and $Q = \{x|Dx \leq e\}$.

Part (a)

Using duality, I prove that if polyhedra P and Q have empty intersections, then there are $y, z \geq 0$ s.t. $yA + zD = 0$ but $yb + ze < 0$.

Consider $R = \{x|Ax \leq b, Dx \leq e\}$. If $R = \emptyset$, then the linear program $\max 0x \mid x \in R$ is unfeasible. Its primal counterpart, $\min yb + ez \mid yA + zD = 0, y \geq 0, z \geq 0$, is either unbounded or unfeasible. However, the primal is clearly not unfeasible since $y = z = 0$ is a solution. So it must be unbounded. Thus, $\exists y, z \geq 0$ s.t. $yA + zD = 0$ but $yb + ze < 0$.

Part (b)

I conclude that if polyhedra P and Q have empty intersections, then there is a separating hyperplane for P and Q (i.e., a vector c s.t. $cx < cw$ for all $x \in P$ and $w \in Q$).

By part (a), if P and Q have empty intersections, $\exists y, z \geq 0$ s.t. $yA + zD = 0$ but $yb + ze < 0$. Consider $c = yA$. Then, for $x \in P$, $cx = yAx \leq yb$ since $Ax \leq b$. So for $x \in P$, $cx \leq yb$. Similarly, for $w \in Q$, $cw = yAw = -zDw$, using $yA + zD = 0$. Since $Dw \leq e$, $cw = -zDw \geq -ze$. So for $w \in Q$, $-cw \leq ze$. Adding the inequalities, $cx - cw \leq yb + ze < 0$. So for $x \in P$ and $w \in Q$, $cx < cw$. In other words, there is a separating plane between P and Q.

Part (c)

I conclude that given two polyhedra P and Q, there is a quickly verifiable answer as to whether or not the two polyhedra have a point in common.

If the two polyhedra have a point x in common, simply give the point x. It is easy to verify that $x \in P$ by checking $Ax \leq b$ and that $x \in Q$ by checking $Dx \leq e$.

If the two polyhedra have no point in common, simply give y and z, s.t., $y \geq 0$, $z \geq 0$, $yA + zD = 0$ but $yb + ze < 0$ (these constraints are all easily verifiable). By part (a), if the two polyhedra have no point in common, then there exists such a y and z. By part (b), if such a y and z exists, there is a separating plane between the two polyhedra, so they have no point in common.

Problem 4

I argue that weak duality holds for an arbitrary linear program, with its dual taken by the cookbook method.

Let A be a matrix with rows a'_i and columns A_j.

The arbitrary primal linear program is: minimize $c'x$, subject to

$$\begin{aligned} a'_i x &\geq b_i,\ i \in M_1 \\ a'_i x &\leq b_i,\ i \in M_2 \\ a'_i x &= b_i,\ i \in M_3 \\ x_j &\geq 0,\ j \in N_1 \\ x_j &\leq 0,\ j \in N_2 \\ x_j &\text{ free},\ j \in N_3 \end{aligned}$$

The dual linear program is: maximize $y'b$, subject to

$$\begin{aligned} y_i &\geq 0,\ i \in M_1 \\ y_i &\leq 0,\ i \in M_2 \\ y_i &\text{ free},\ i \in M_3 \\ y'A_j &\leq c_j,\ j \in N_1 \\ y'A_j &\geq c_j,\ j \in N_2 \\ y'A_j &= c_j,\ j \in N_3 \end{aligned}$$

PRIMAL	minimize	maximize	DUAL
constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	variables
variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	constraints

Let x and y be vectors for the primal and dual feasible. Define,

$$\begin{aligned} u_i &= y_i(a'_i x - b_i) \\ v_j &= (c_j - y'A_j)x_j \end{aligned}$$

The definition of the dual requires that the sign of y_i equals the sign of $a'_i x - b_i$ and that the sign of $c_j - y'A_j$ equals the sign of x_j. Thus, the primal and dual feasibility imply that:

$$\begin{aligned} u_i &\geq 0,\ \forall i \\ v_j &\geq 0,\ \forall j \end{aligned}$$

Notice:

$$\sum_i u_i = y'Ax - y'b$$
$$\sum_j v_j = c'x - y'Ax$$

Adding these two equalities and using non-negativity of u_i, v_j:

$$0 \leq \sum_i u_i + \sum_j v_j = c'x - y'b$$

Thus $yb \leq cx$ or OPT(dual) $\leq$ OPT(primal). So I have just shown that weak duality holds for an arbitrary linear program.

Problem 5

I consider this linear program formulation of the max-flow problem as the dual:

$$
\begin{aligned}
z = \max \sum f_P \\
\sum_{P \ni e} f_P \leq u_e \\
f_P \geq 0
\end{aligned}
$$

I derive the primal of the dual using the cookbook method:

$$
\begin{aligned}
z = \min \sum u_e x_e \\
\sum_{e \in P} x_e \geq 1 \\
x_e \geq 0
\end{aligned}
$$

I interpret the variable x_e as representing the saturation of an edge (0 for unsaturated, 1 for saturated, though the linear program allows arbitrary non-negative values). Using this interpretation, an English explanation of the objective and constraints is: minimize the sum of the capacities of saturated edges subject to the constraint that there is at least one saturated edge per path.

Still in other words, using network flow concepts, the linear program is looking for the s-t cut of minimum value: the objective function defines the value of an s-t cut as the sum of the capacities of cross-cut edges and the constraints require that, in an s-t cut, at least one edge in each s-t path is across the cut. In this interpretation, x_e is 1 for cross-cut edges and 0 for other edges (though, again, the linear program doesn't constrain the values to be 0 or 1).

Problem 6

Part (a)

I argue that any LP optimization problem can be transformed into a problem of the form $\min 0x | Ax = b, x \leq 0$, i.e., the objective is to minimize $0x$ and the constraint on x is in standard form. This form is only concerned with the feasibility of the LP, as it has optimium value 0 if it is feasible, and ∞ if it is unfeasible.

Let's start with a LP optimization problem in standard form: $\min cx | Ax = b, x \geq 0$. Its dual is $\max yb | yA \leq c$. By strong duality, if the problem is feasible and bounded, then $cx = yb$. So here are the constraints on a LP program that decides whether the optimization problem is feasible and bounded:

$$\begin{aligned} Ax &= b \\ x &\geq 0 \\ yA &\leq c \\ cx &= yb \end{aligned}$$

I can put these constraints in standard form and use the objective $0x + 0y$ to end up with a problem of the required form. The program returns 0 if the original program is feasible and bounded and ∞ if it is unfeasible or unbounded.

Part (b)

The dual of the linear program $\min 0x | Ax = b, x \leq 0$ is $\max yb | yA \leq 0$.

Part (c)

If the primal is feasible, then its solution is 0. By strong duality, 0 is also the optimum of the dual. In the dual, I can obviously get $yb = 0$ by letting $y = 0$. So if the primal is feasible, the dual has the obvious optimum solution $y = 0$.

Part (d)

Given an algorithm that could optimize an LP with an $m \times n$ constraint matrix in $O((m+n)^k)$ time given an optimal solution to the dual LP, I can build an LP algorithm that will solve any LP without knowing a dual solution, in the same asymptotic time bounds. I assume that, if given a wrong optimal solution to the dual, the given algorithm might not return a correct answer, but will stop nevertheless (I can always stop it after it runs for more than $c(m + n)^k$ steps where c is the constant in its running time).

I call the algorithm on the same LP with the objective function modified to 0 and use an obvious dual solution similar to that of part (b). If the algorithm returns with a feasible solution, then I know that the LP is feasible. Otheriwse, it is unfeasible.

If the LP is feasible, I transform the LP into the form of part (a), call it LP'. I call the algorithm on LP' using the obvious optimum solution $y = 0$ for the dual. If LP' is feasible, the algorithm should return an x that satisfies the constraints of LP'. If it doesn't, I can assume LP' is unfeasible, which means that the original LP is unbounded (since I know it is feasible). If the LP is feasible and bounded, I can easily transform the solution of LP' into a solution to the original LP.

The asymptotic time bounds remain $O((m+n)^k)$ because, LP' has a $O(m+n) \times O(m+n)$ constraint matrix. Transforming the LP' solution back into LP is straightforward, as it just involves reading the variables of interest.