1 Definitions

DT convolution $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$

CT convolution $y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$

CFS synthesis $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$

CFS analysis $a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-j(2\pi/T)t}dt$

CFT synthesis $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$

DFT analysis $X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-jkt}dt$

DFS synthesis $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$

DFS analysis $a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=<N>} x[n]e^{-j(2\pi/N)n}$

DFT synthesis $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$

DFT analysis $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jkn}$

LCC differential equation $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \Longleftrightarrow H(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$

LCC difference equation $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \Longleftrightarrow H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$

2 Summary of Bode Plotting Rules

First, normalize so that $H(s) = c\frac{(s+z_1)(s+z_2)\cdots(s+z_n)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$.

2.1 Plotting Magnitude $20\log|H(j\omega)|$

1. Identify locations and order (how many) of all poles and zeros - these are the breakpoints.
2. Draw axes. Note that it is impossible to include $\omega = 0$ on a log scale. Start with a small ω, like 1, 0.00001, or whatever is appropriate. It is also useful to draw vertical dashed lines at breakpoints.
3. Starting at the left, the magnitude plot starts flat unless there is a pole at $s = 0$ (start plot with slope of -20dB/dec for each pole at origin) or there is a zero at $s = 0$ (start with a plot of $+20$dB/dec for each zero at origin).
4. Continue drawing asymptote in a straight line until you reach a breakpoint (pole/zero).
5. For each pole, *decrease* slope of asymptote by 20dB/dec. For each zero, *increase* slope by 20dB/dec. Go to step 4 unless there are no more breakpoints left.
6. Label one point on the y-axis by plugging in a value of ω into $H(j\omega)$ from any flat region of the y-axis using slopes of asymptotes as guides.
7. Round corners by ±3dB for a more accurate magnitude plot.

2.2 Plotting Phase $\angle H(j\omega)$

1. Identify locations and order (how many) of all poles and zeros - these are the breakpoints.
2. Draw axes and vertical dashed lines at breakpoints.
3. Starting at the left, the phase plot starts at $\angle H(j\omega = 0)$ (usually 0^o). Plot starts at $+90^o$ for each zero at origin and -90^o for each pole at origin. A leading minus sign will add 180^o to the phase. Plug in a very small imaginary number for $j\omega$ and evaluate the phase manually if you're confused. Also, remember, shifting the phase curve up or down by 360^o doesn't change anything.
4. The phase plot continues as a flat line until reaching $0.1\times$breakpoint.
5. Each pole *substracts* 90^o from the phase, spread over a distance of $0.1\times$pole location to $10\times$pole location. At the pole location, the phase has dropped by 45^o (halfway there). The situation is the same for zeros, but this time phase is *added*. Watch out for multiple poles/zeros. Go back to step 4 unless there are no more breakpoints left.
6. Round all corners to ressemble an arctan curve (that's how phase is calculated) for more accurate plotting; the phase round by about 6^o at $0.1\times$breakpoint and at $10\times$breakpoint.

3 Partial Fraction Expansion

3.1 linear, non-repeated factors

$$X(s) = \frac{k_1}{s - p_1} + frack_2 s - p_2 + \dots$$
$$k_i = X(s)(s - p_i)|_{s=p_i}$$

3.2 linear, repeating factors

The formula is inhuman, but reduces to the one for non-repeated roots for the residual of the highest power term. So find that, then substract that term from the equation and keep going.

4 Quadratic Formula

$$ax^2 + bx + c = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

5 Sampling

5.1 sampling period

The sampling rate (ω_s) must be greater than twice the maximum frequency present in the input signal (ω_M).

$$\omega_s > 2\omega_M$$
$$\omega_s = \frac{2\pi}{T}$$

5.2 low-pass filter

gain T

cutoff between ω_M and $\omega_s - \omega_M$

5.3 c/d conversion

If $x_p = x_c$ sampled with period T, then $X_p(j\omega)$ is $X_c(j\omega)$ repeated every $\frac{2\pi}{T}$. If $x_d[n] = x_c(nT)$, then $X_d(e^{jw})$ is $X_c(j\omega)$ repeated every 2π. In short, to convert a sampled CT signal into a DT signal, in the frequency domain, multiply the ω-axis by the sampling period.

$$\omega_{ct} = T\omega_{dt}$$

6 C Systems Properties

6.1 Causality

- LTI system is causal $\leftrightarrow h(t) = 0$ for $t < 0$.
- system is causal $rightarrow$ ROC is a right-half plane.
- rational system is causal $\leftrightarrow$ ROC is a rightmost right-half plane.

6.2 Stability

- LTI system is stable $\leftrightarrow \int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$.
- LTI system is stable $\leftrightarrow$ ROC includes $j\omega$-axis ($\Re e\{s\} = 0$).

6.3 Causality and Stability

Rational system is causal and stable $\leftrightarrow$ all poles lie in the left-half of the s-plane.

7 D Systems Properties

7.1 Causality

- LTI system is causal $\leftrightarrow h[n] = 0$ for $n < 0$.
- system is causal $\rightarrow$ ROC is the exterior of a circle including infinity.
- rational system is causal $\leftrightarrow$ ROC is the exterior of a circle outside the outermost pole.

7.2 Stability

- LTI system is stable $\leftrightarrow \sum_{k=-\infty}^{+\infty} |h[k]| < \infty$.
- LTI system is stable $\leftrightarrow$ ROC includes the unit circle, $|z| = 1$.

7.3 Causality and Stability

Rational system is causal and stable $\leftrightarrow$ all poles lie inside the unit circle – i.e., thez must all have magnitude smaller than 1.

8 Checking if All Poles Are in LHP

Calculate all the roots and see! or apply Routh-Hurwitz – without having to solve for roots.

8.1 Routh-Hurwitz

	Polynomial	Condition so that all roots are in the LHP
1st order	$s + a_0$	$a_0 > 0$
2nd order	$s^2 + a_1 s + a_0$	$a_1 > 0, a_0 > 0$
3rd order	$s^3 + a_2 s^2 + a_1 s + a_0$	$a_2 > 0, a_1 > 0, a_0 > 0$ $and\ a_0 < a_1 a_2$

9 Bode Plots from Pole-Zero Plot

For a rational system,

$$X(s) = M \frac{\Pi_{i=1}^{R}(s - \beta_i)}{\Pi_{j=1}^{P}(s - \alpha_j)}$$

The magnitude of $X(s_1)$ is then the magnitude of the scale factor M, *times* the product of the lengths of the zero vectors (i.e., the vectors from the zeros to s_1) *divded* by the product of the lengths of the pole vectors (i.e., the vectors from the poles to s_1).

The angle of the complex number $X(s_1)$ is the *sum* of the angles of the zero vectors *minus* the sum of the angles of the pole vectors. If the scale factor M is negative, an additional angle of π would be included.

10 Root Locus

$K = \frac{1}{|G(s_0)H(s_0)|}$

- For $K = 0$, poles of $G(s)H(s)$.
- For $K = \infty$, zeroes of $G(s)H(s)$.
- For $K > 0$, *odd* number of real poles and zeros of $G(s)H(s)$.
- For $K < 0$, *even* number of real poles and zeros of $G(s)H(s)$.
- Branches of the root locus between two real poles must break off into complex plane for $|K|$ large enough.