

6.003 Lab 3

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Basic Problems

(a)

Given an unstable system $H(s)$ which satisfies

$$\frac{dy(t)}{dt} = 2y(t) + x(t)$$

I determine the closed-loop system function $Q(s)$ as a function of K

$$Q(s) = \frac{H(s)}{1 + G(s)H(s)}$$

when $G(s)$ is a simple proportional feedback: $G(s) = K$.

From the differential equation of $H(s)$, I find that

$$H(s) = \frac{1}{s - 2}$$

$H(s)$ has one pole, at $s = 2$. Since the system is unstable, the ROC must be $\Re\{s\} > 2$ so as not to include $\Re\{s\} = 0$.

Now,

$$Q(s) = \frac{1}{s - 2} \frac{1}{1 + \frac{K}{s - 2}}$$

$$Q(s) = \frac{1}{s - 2 + K}$$

(b)

$Q(s) \rightarrow \infty$ when $s - 2 + k = 0$, thus the pole of the closed-loop occurs at $s = 2 - k$.

For the system to be stable, $\Re\{s\} = 0$ needs to be part of the ROC. Assuming the system is causal, implying a right-sided ROC, the system will be stable if the pole occurs before 0. Thus,

$$\begin{aligned} s &< 0 \\ k - 2 &< 0 \\ -k &< -2 \\ k &> 2 \end{aligned}$$

For the system to be stable, $k > 2$ is required.

(c)

See Figure 1.

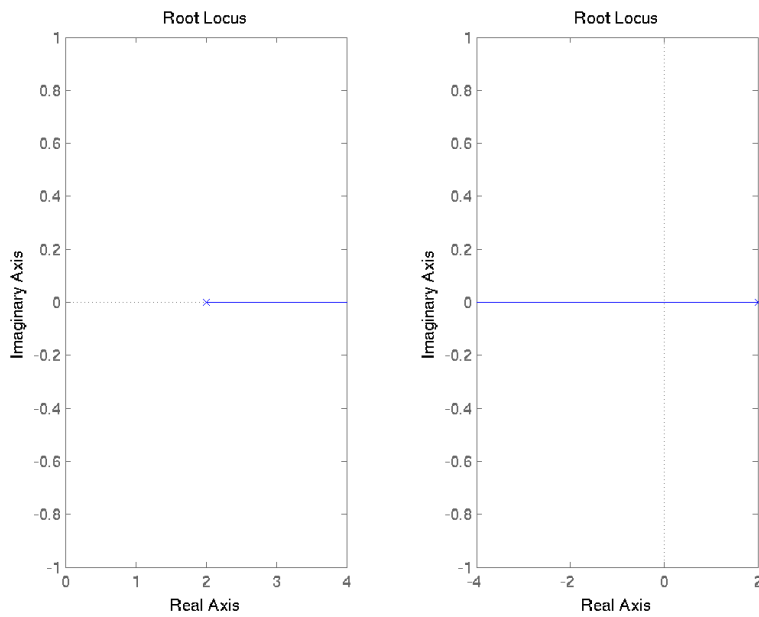


Figure 1: Plots of the root locus for $K \leq 0$ and $K \geq 0$.

(d)

See Figure 2.

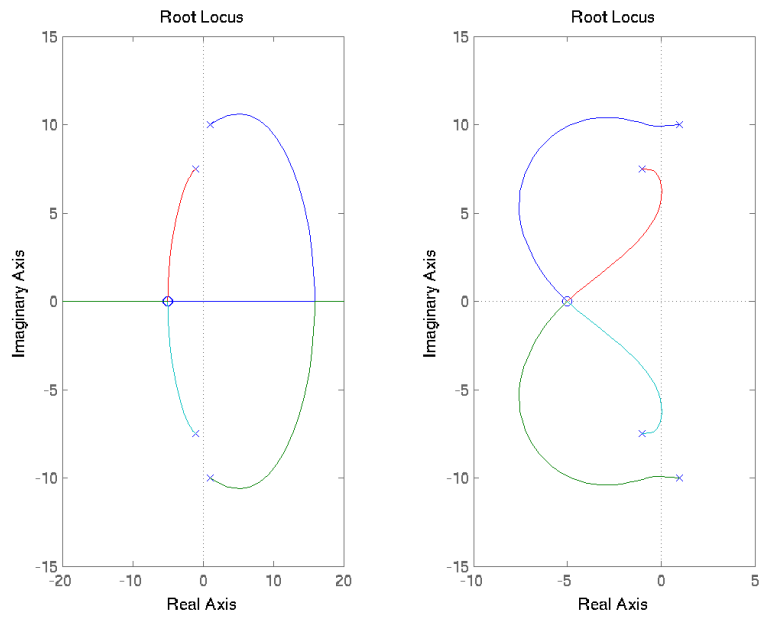


Figure 2: Plots of the root locus for $K \leq 0$ and $K \geq 0$.

(e)

The ranges for K that I found such that all poles p_k satisfy $\Re\{p_k\} < 0$ and $-5 < \Im\{p_k\} < 5$ are $[-\infty, -6.943]$ and $[17.05, \infty]$.

(f)

See Figure 3.

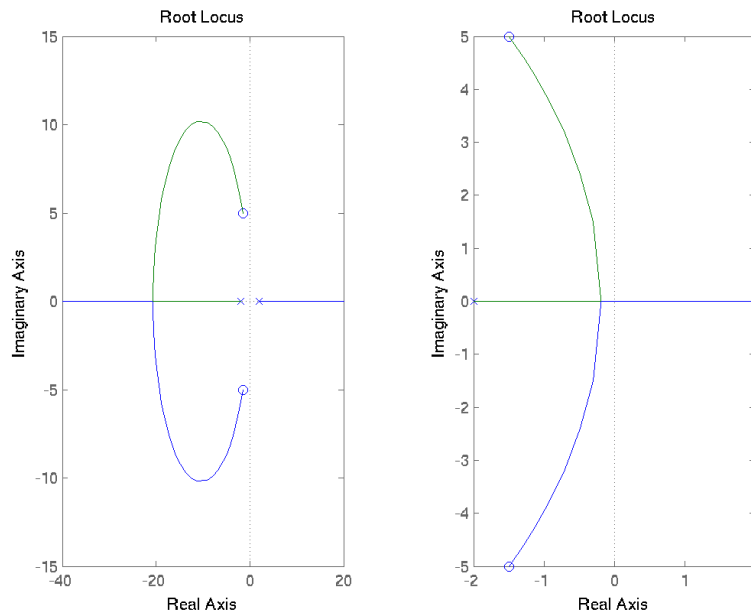


Figure 3: Plots of the root loci for $K \leq 0$ and $K \geq 0$.

(f)

$Q(s)$ will be stable if all poles have negative real parts, assuming the system is causal. The poles occur when the denominator is null. Thus,

$$\begin{aligned}
 1 + G(s)H(s) &= 0 \\
 1 + KH(s) &= 0 \\
 s^2 - 4 + K(s^2 + 3s + 27.25) &= 0 \\
 (K + 1)s^2 + 3Ks + (27.25K - 4) &= 0
 \end{aligned}$$

Solving for s , I find that,

$$s = \frac{-3K \pm \sqrt{-100K^2 - 93K + 16}}{2K + 2}$$

Setting $s = 0$ and solving for K ,

$$\begin{aligned} -3K \pm \sqrt{-100K^2 - 93K + 16} &= 0 \\ -109K^2 - 93K + 16 &= 0 \end{aligned}$$

Thus,

$$K = -1 \qquad K = 0.1468$$

From this, I conclude that the ranges for K , for which the closed-loop system is stable, are $[-\infty, -1[$ and $[0.1468, \infty]$. This is indeed confirmed by the plot of the root loci.

(g)

I found that $K = 1$ gives me values close to -0.75 for the real parts of the close-loop poles. Let b be the numerator of $H(s)$ and a be the denominator. I have

$$Q(s) = \frac{H(s)}{1 + G(s)H(s)}$$

$$Q(s) = \frac{H(s)}{1 + KH(s)}$$

$$Q(s) = \frac{H(s)}{1 + H(s)}$$

$$Q(s) = \frac{b}{a} \frac{1}{1 + \frac{b}{a}}$$

$$Q(s) = \frac{b}{a + b}$$

Let b_K be the numerator of $Q(s)$ and a_K be the denominator of $Q(s)$ for some value of K . I just showed that for $K = 1$, $b_1 = b$ and $a_1 = a + b$.

(f)

$$H_c(s) = M(s)H(s)$$

$$H_c(s) = \frac{s^2 + 3s + 27.25}{2s^2 + 2\sqrt{2} + 23}$$

See Figure 4.

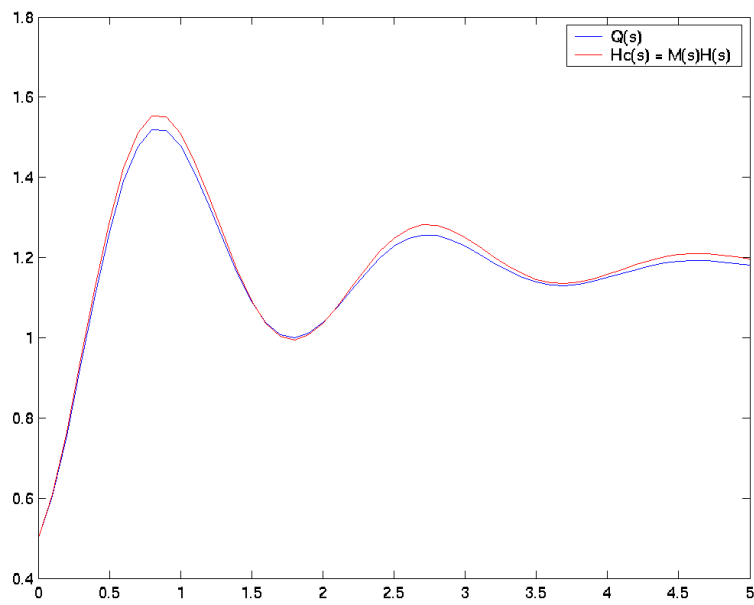


Figure 4: Plots for the step response of $Q(s)$ when $K = 1$ and $H_c(s)$ for $0 \leq t \leq 5$.

(l)

The poles occur when the denominator goes to 0. So I set the denominator to 0 and solve for s in terms of ϵ .

$$s^2 - \epsilon s - (4 + 2\epsilon) = 0$$

I find,

$$s = -2$$

$$s = \epsilon + 2$$

(m)

When $\epsilon = 0.01$, $H_\epsilon(s) = \frac{s^2 + 3s + 27.25}{s^2 - 0.01s - 4.02}$.

(n)

See Figure 5.

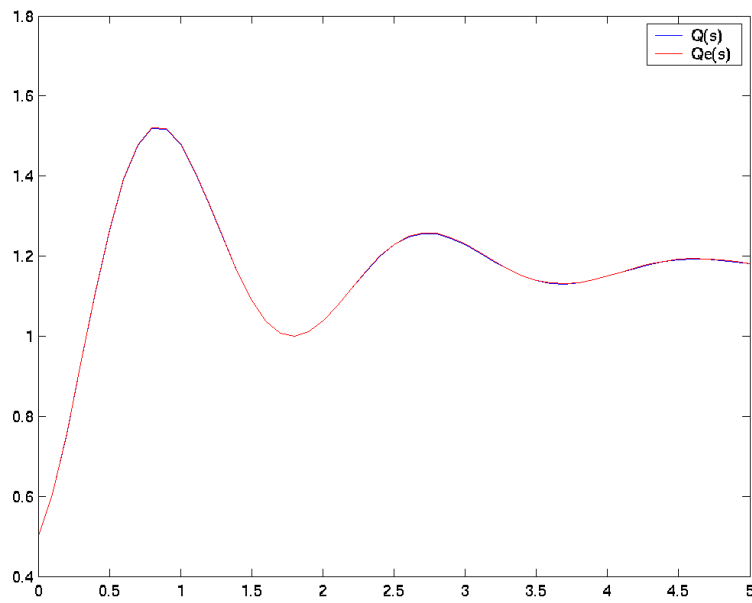


Figure 5: Plots of $Q(s)$ and $Q_\epsilon(s)$

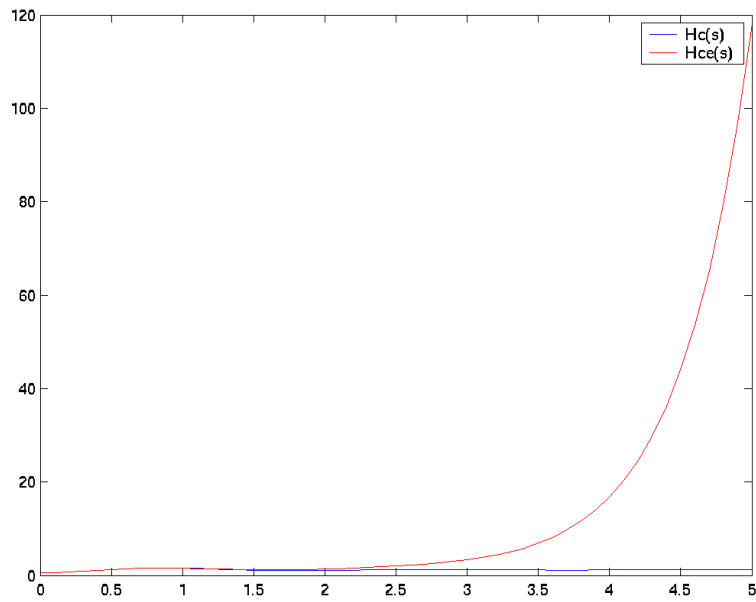


Figure 6: Plots of $H_c(s)$ and $H_{ce}(s)$

(o)

See Figure 6.

(p)

The feedback system is unarguably more robust, because it doesn't require a complete knowledge of the transfer function. Since the compensated system depends on zero-pole cancellation, the compensating zero must exactly match the pole to cancel, or else the system won't stabilize. This is why the compensated system will usually diverge in the face of even small uncertainties.

MATLAB Code

This script demonstrates how I completed the lab.

```

% (c)
b = [1];
a = [1 -2];
sys = tf(b, a);
% plotting the root locus
subplot(121), rlocus(-sys); % for K<=0
subplot(122), rlocus(sys); % for K>=0
% verifying the values that are stable
r = rlocus(sys, [-2 0 2 4 6]);
actual_signs = sign(r);
wanted_signs = [-1 -1 0 1 1];
if actual_signs ~= wanted_signs
    error('unexpected_root_locus');
end

% (d)
% deg 4 3 2 1 0
b = [1 20 150 500 625];
a = [1 0 154.25 87.5 5782.85];
sys = tf(b, a);
% plotting the root locus
subplot(121), rlocus(-sys); % for K<=0
subplot(122), rlocus(sys); % for K>=0

% (e)
[r,k]=rlocus(sys);
indices = find(real(r)<0 & -5<imag(r) & imag(r)<5)
[i,j] = ind2sub(size(r), indices);
j
% i look for the first j that appears for all values of i

```

```

% then I do a binary search between r(j-1) and r(j) to see
% where the prop starts to hold
% from |K|=17.05 to infinity for K>=0
% (fine, this was before I discovered I could just click on the
% plot)

```

```

[r,k]=rlocus(-sys);
indices = find(real(r)<0 & -5<imag(r) & imag(r)<5)
[i,j] = ind2sub(size(r), indices);
j
% from |K|=6.943 to infinity for K<=0

```

```

% (f)
hb = [1 3 27.25];
ha = [1 0 -4];
H = tf(hb, ha);
mb = ha;
ma = [2 2*sqrt(2) 23];
M = tf(mb, ma);
% plotting the root locus
subplot(121), rlocus(-H); % for K<=0
subplot(122), rlocus(H); % for K>=0

```

```

% (g)
[r,k] = rlocus(-H);
% |K|>1 for K<=0
[r,k] = rlocus(H);
% from |K|=0.147 to infinity for K>=0

```

```

% (h)
% K=1 gives poles with real part -0.75
bk = hb;
ak = ha+hb;
Qk = tf(bk, ak);

```

```

% (i)
t = [0:0.1:5];
sq = step(Qk, t);
% (j)
% the compensated system is M(s)H(s)
%C = M*H;
%[bc, ac] = tfdata(C);
bc = hb;
ac = ma;
C = tf(bc, ac);
sc = step(C, t);

```

```

% (k)
figure;
plot(t, sq);
hold on;
plot(t, sc, 'r');
legend('Q(s)', 'Hc(s) = M(s)H(s)');
% (m)
be = [1 3 27.25];
ae = [1 -0.01 -4.02];
% (n)
He = tf(be, ae);
bke = be;
ake = ae+be;
Qke2 = tf(bke, ake);
Qke = He/(1+He);
sqe = step(Qke, t);
figure;
plot(t, sq, 'b');
hold on;
plot(t, sqe, 'r');
legend('Q(s)', 'Qe(s)');
% (o)
Ce = M*He;
[bce, ace] = tfdata(Ce, 'v');
sce = step(Ce, t);
figure;
plot(t, sc, 'b');
hold on;
plot(t, sce, 'r');
legend('Hc(s)', 'Hce(s)');

```