Reversibility &

Computing
Quantum

Quantum Computing

Reversibility &

Mandate

"Why do all these Quantum Computing guys use reversible logic?"

Material

- Logical reversibility of computation

  Bennett '73

- Elementary gates for quantum computation

  Berenco et al '95

- […] quantum computation using teleportation

  Gottesman, Chuang '99

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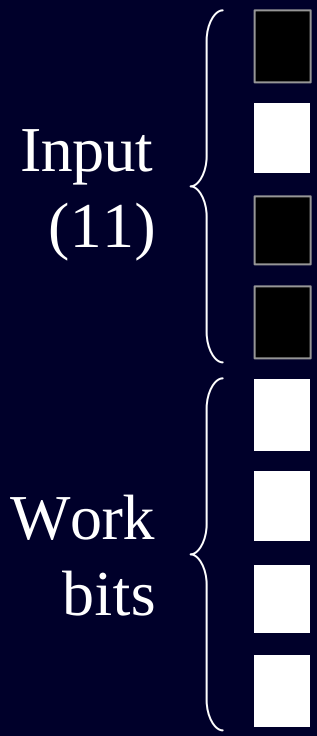
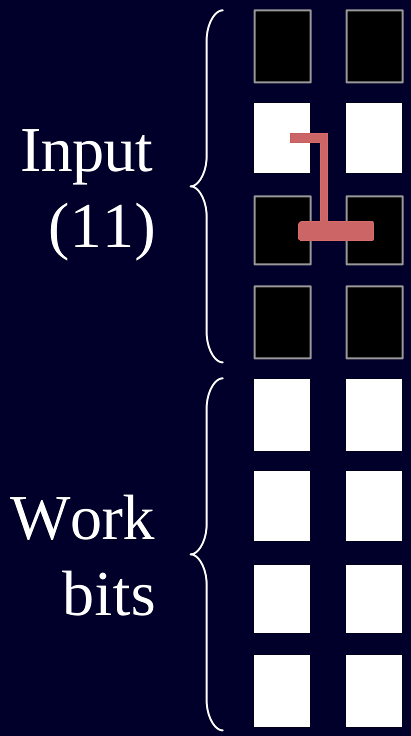
Heat Generation in Computing

- Landauer’s Principle
  - Want to erase a random bit? It will cost you
  - Storing unwanted bits just delays the inevitable
- Bennett’s Loophole
  - Computed bits are not random
  - Can uncompute them if we’re careful

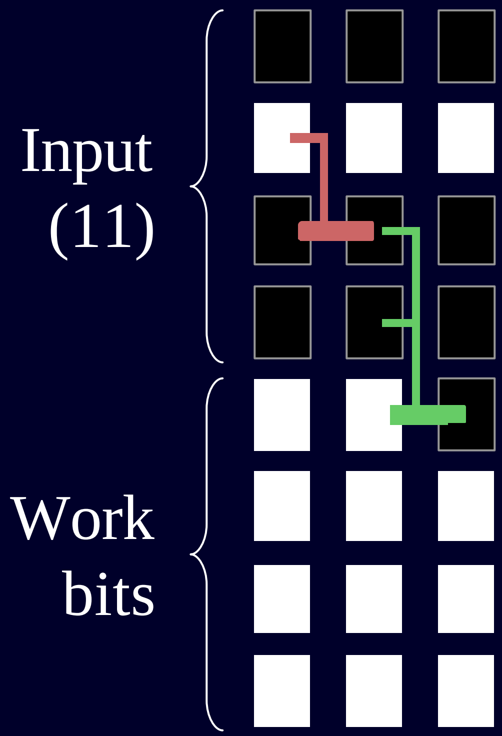
Example



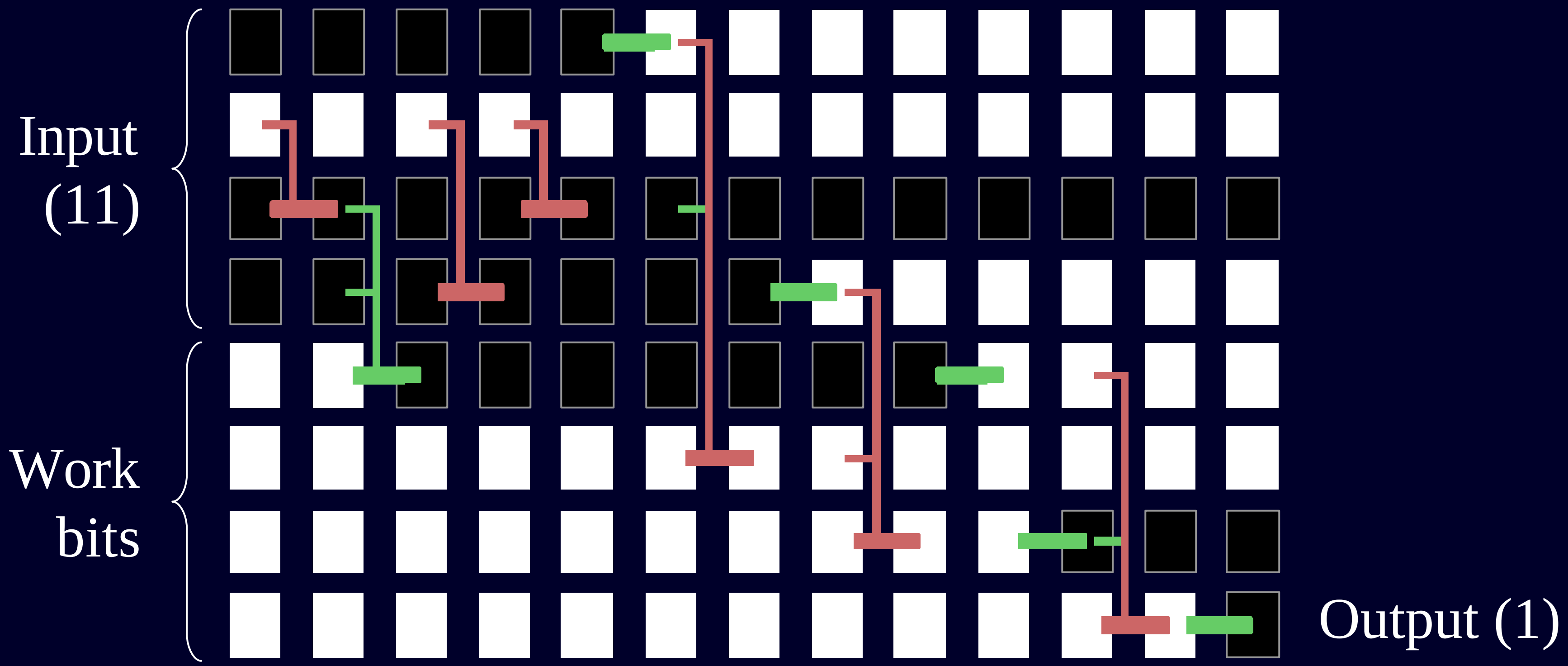
Example



Example



Example



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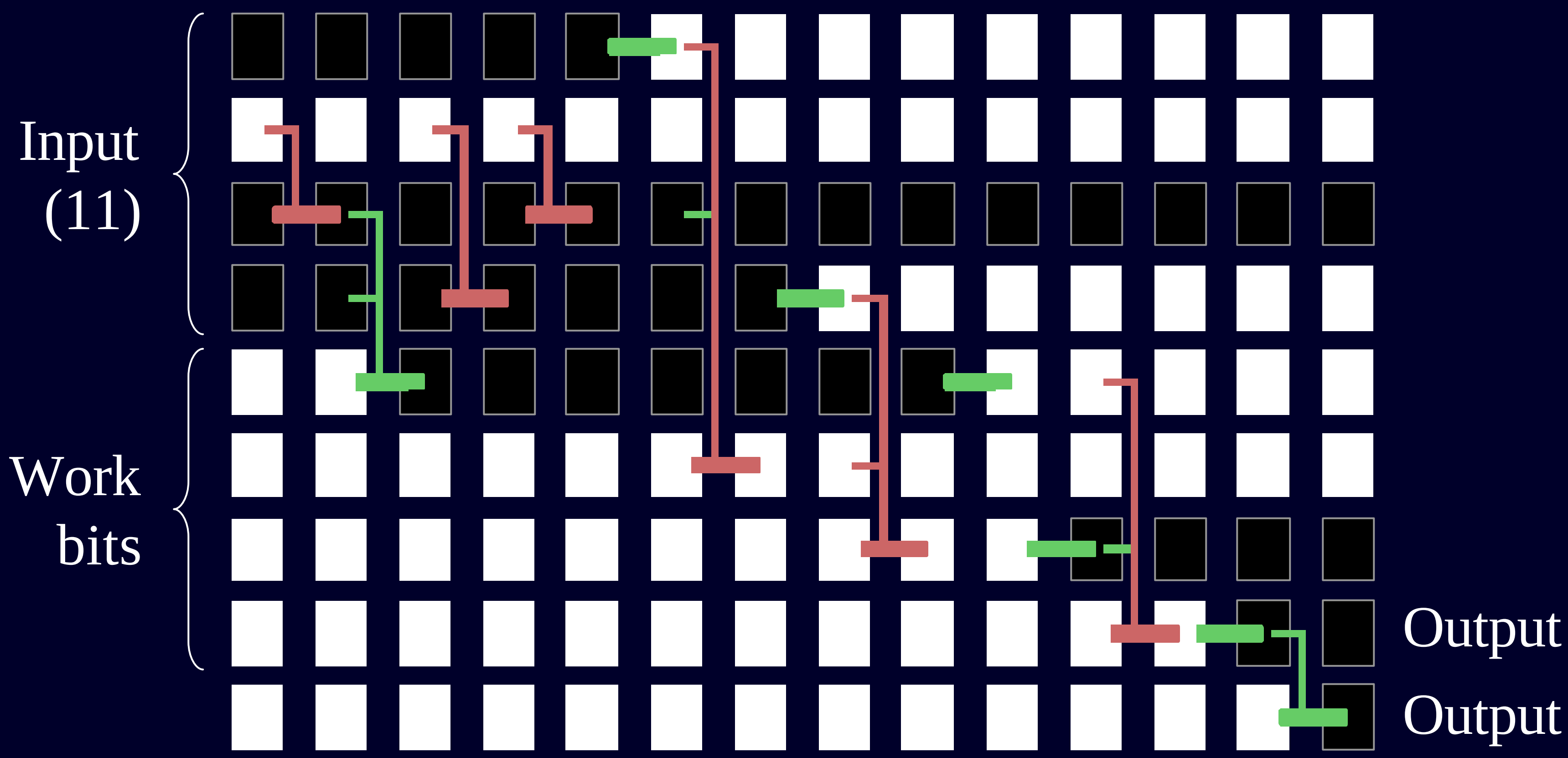

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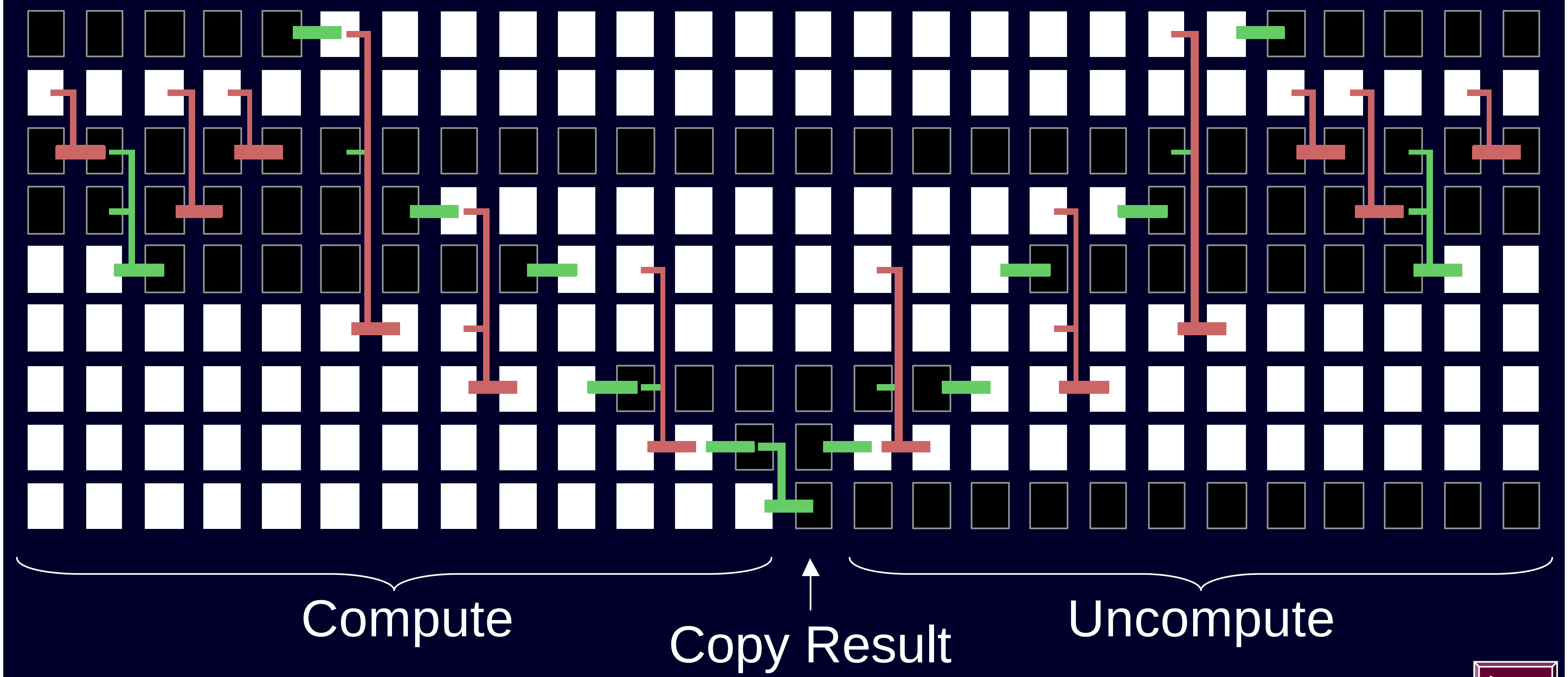
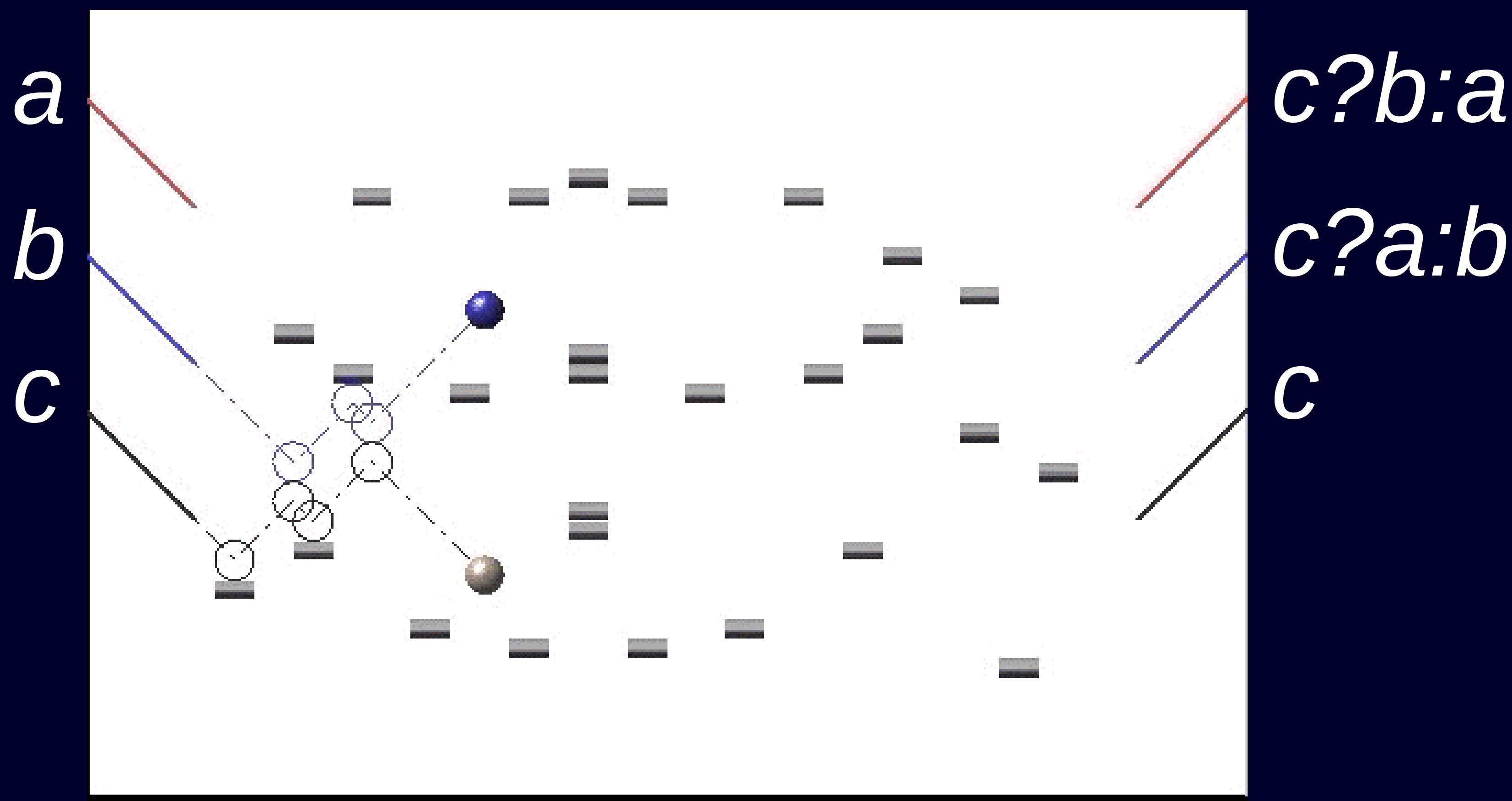


Thermodynamic Reversibility



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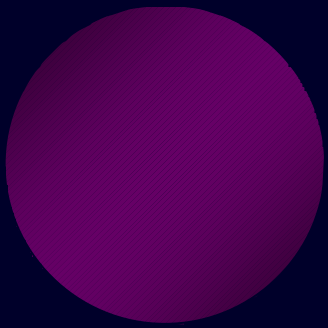
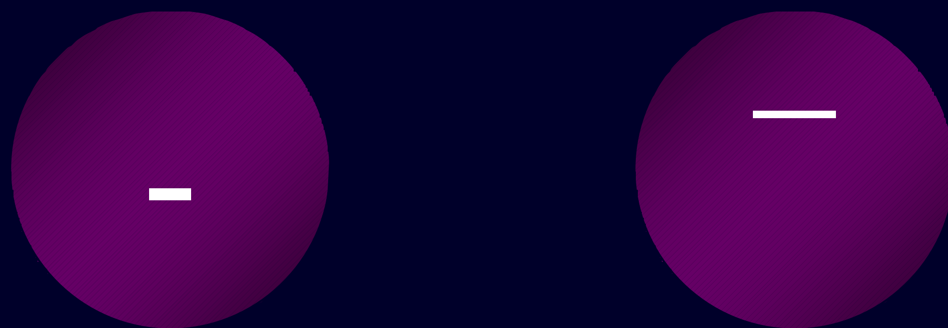
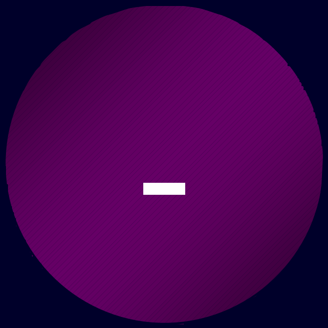
Quantum State

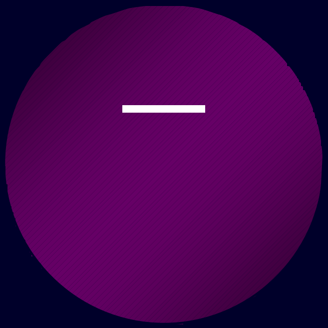


Two Distinguishable States



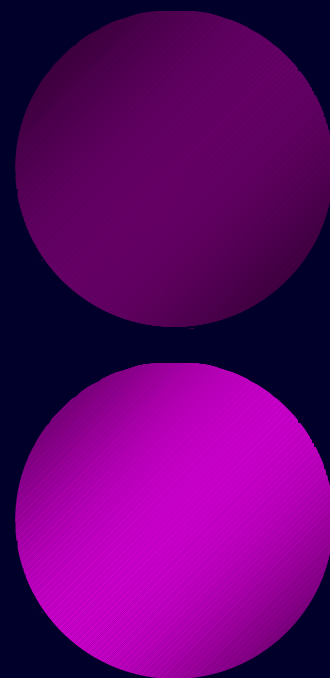
Continuous State Space

a 

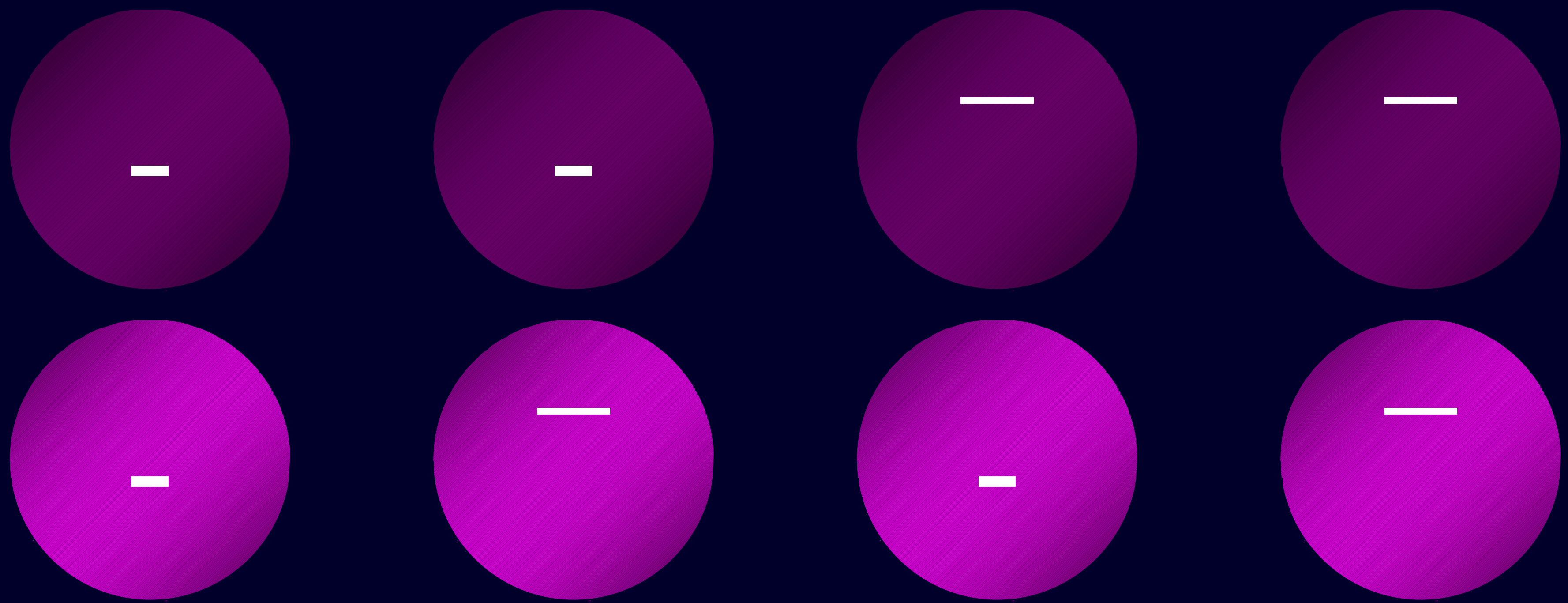
+ b 

$$|a|^2 + |b|^2 = 1 \quad a, b \in C$$

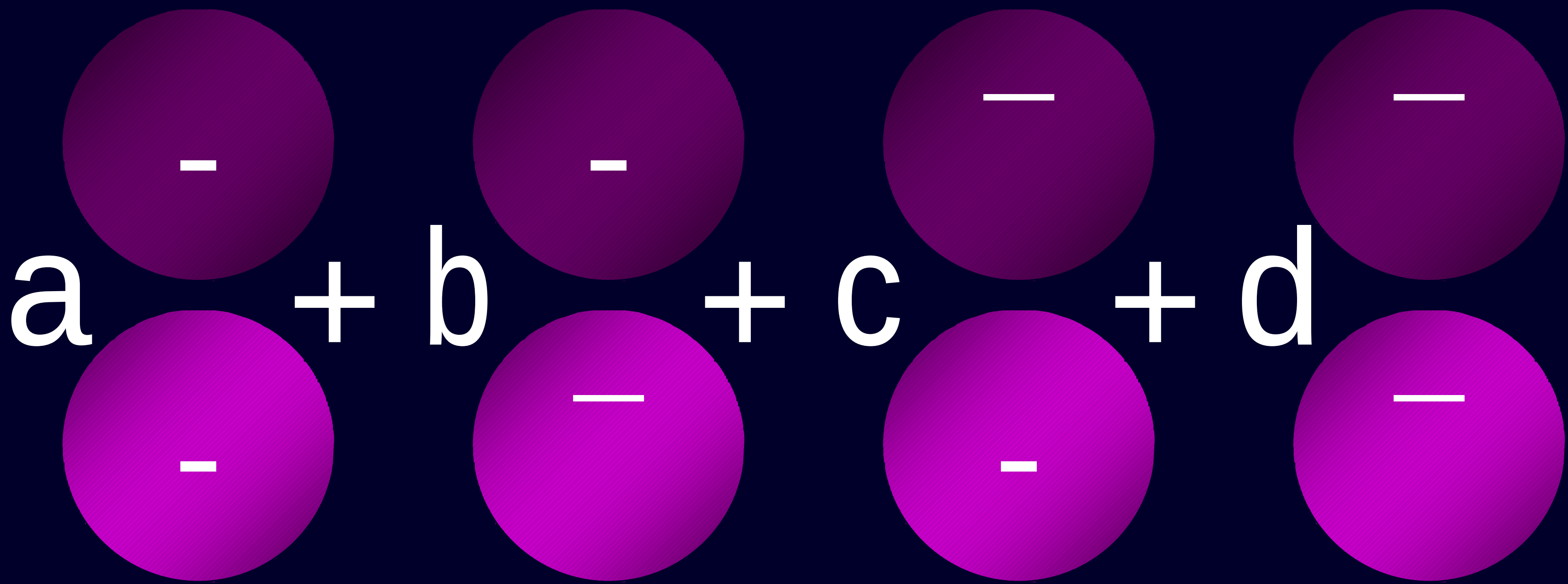
Two Spin-½ Particles



Four Distinguishable States



Continuous State Space



$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1 \quad a, b, c, d \in C$$

Continuous State Space

$$a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1 \quad a, b, c, d \in C$$

Continuous State Space

$$|\psi\rangle = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1 \quad a, b, c, d \in C$$

State Evolution

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle$$ (Continuous form)

$$|\psi'\rangle = U|\psi\rangle$$ (Discrete form)

- H is Hermitian, U is Unitary
- Linear, deterministic, reversible

Measurement

$$|\psi'_m\rangle = \frac{1}{\sqrt{p(m)}} M_m |\psi\rangle$$

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$$

- Outcome *m* occurs with probability *p(m)*
- Operators M_m non-unitary
- Probabilistic, irreversible

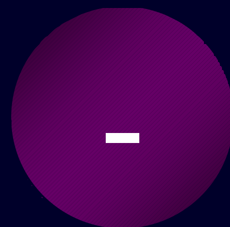
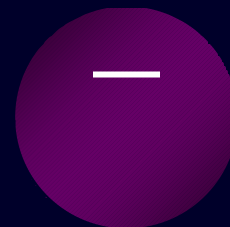
Deriving Measurement

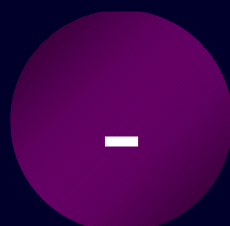
"Like a snake trying to swallow itself by the tail"

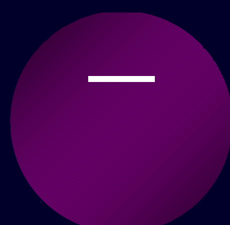
"It can be done up to a point… But it becomes embarrassing to the spectators even before it becomes uncomfortable for the snake"

– Bell

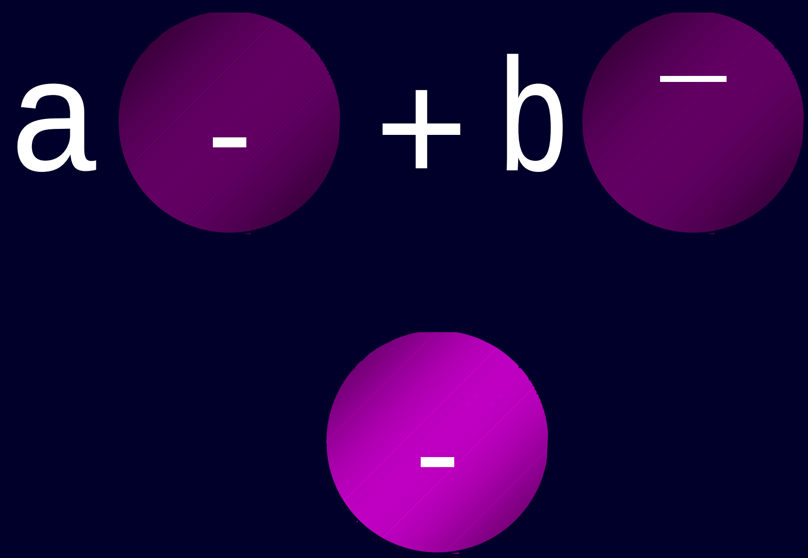
A Simple Measurement

a  + b 

Outcome  with probability $|a|^2$

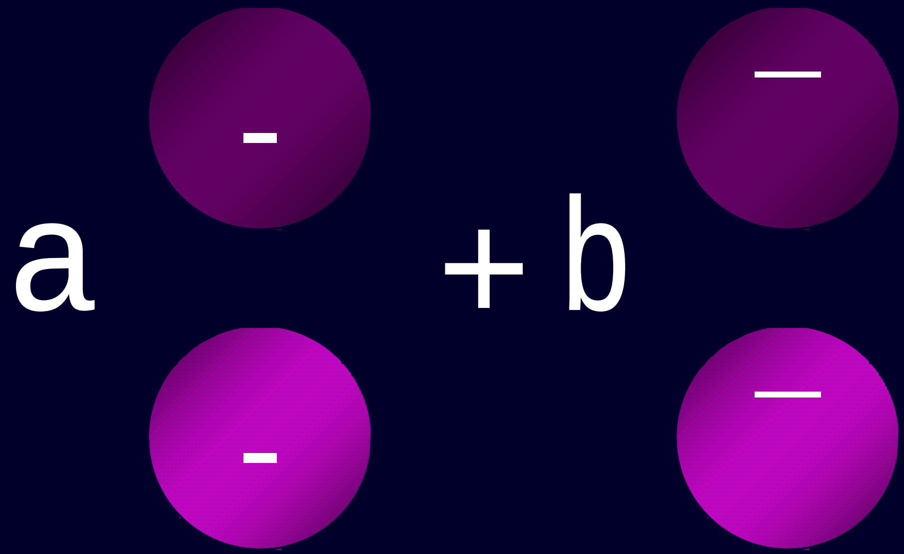
Outcome  with probability $|b|^2$

A Simulated Measurement



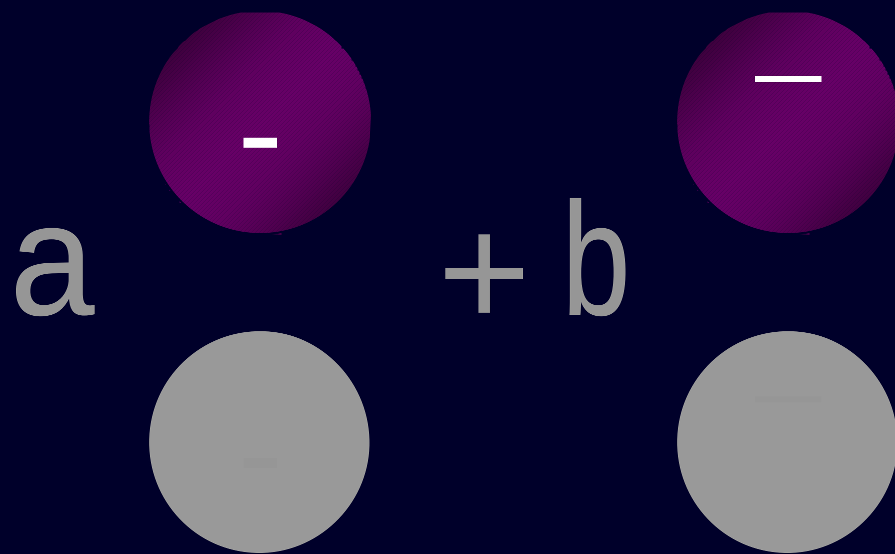
$$(a|0\rangle + b|1\rangle) \otimes |0\rangle$$

A Simulated Measurement

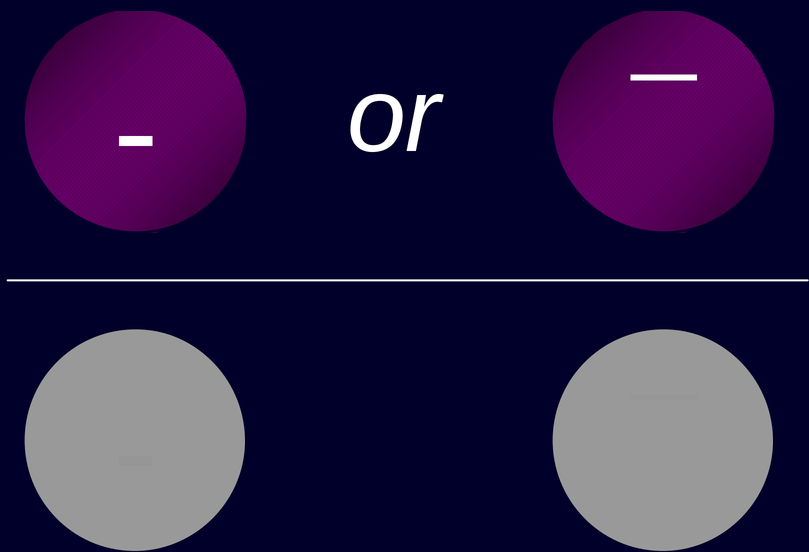


$$a|00\rangle + b|11\rangle$$

A Simulated Measurement



A Simulated Measurement



Terms remain orthogonal –
evolve independently, no interference

Density Operator Representation

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \begin{pmatrix} a^* & b^* & c^* & d^* \end{pmatrix}$$

Mixed States

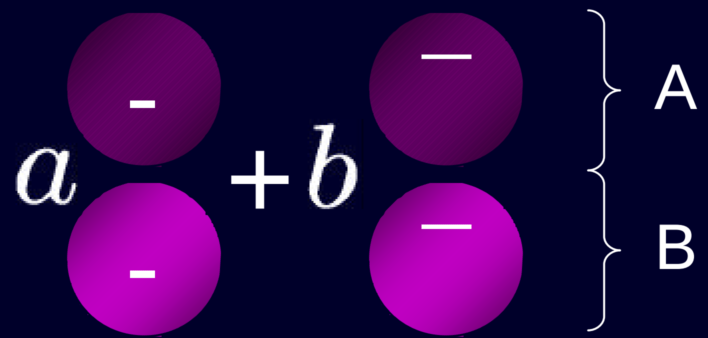
p_1 p_2

$|\psi_1\rangle$ $|\psi_2\rangle$

$$\rho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2|$$

Partial Trace

$$\mathrm{tr}_B(|a_0\rangle\langle a_1| \otimes |b_0\rangle\langle b_1|) = |a_0\rangle\langle a_1|\mathrm{tr}(|b_0\rangle\langle b_1|)$$



$$|\psi\rangle = a|00\rangle + b|11\rangle$$

$$\begin{aligned} \rho_A &= tr_B(|\psi\rangle\langle\psi|) \\ &= |a|^2\,|0\rangle\langle 0| + |b|^2\,|1\rangle\langle 1| \end{aligned}$$

Discarding a Qubit

$$|\psi\rangle = \sum_{x\in\{0,1\}^m} a(x)|x\rangle|g(x)\rangle$$

$$\rho_{reduced} = \frac{1}{2}|\psi_{good}\rangle\langle\psi_{good}| + \frac{1}{2}|\psi_{bad}\rangle\langle\psi_{bad}|$$

$$|\psi_{good}\rangle = \sum_x a(x)|x\rangle$$

$$|\psi_{bad}\rangle = \sum_x (-1)^{g(x)} a(x)|x\rangle$$

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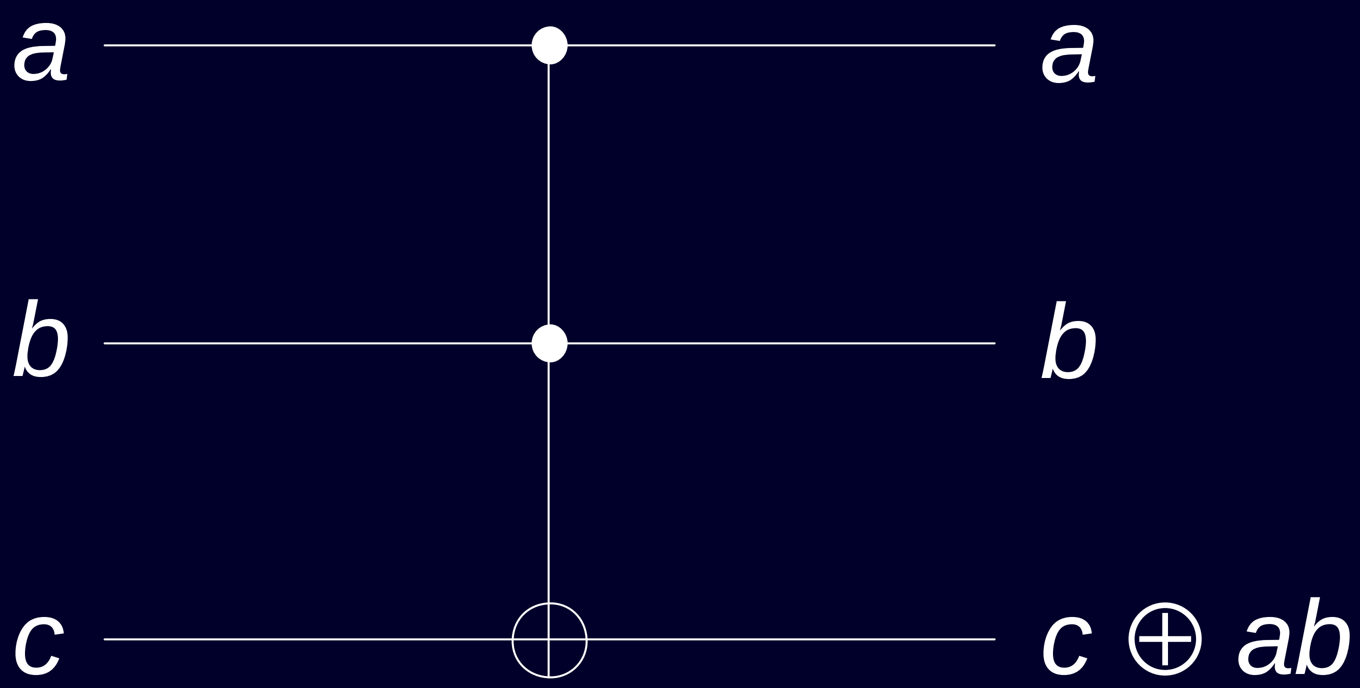
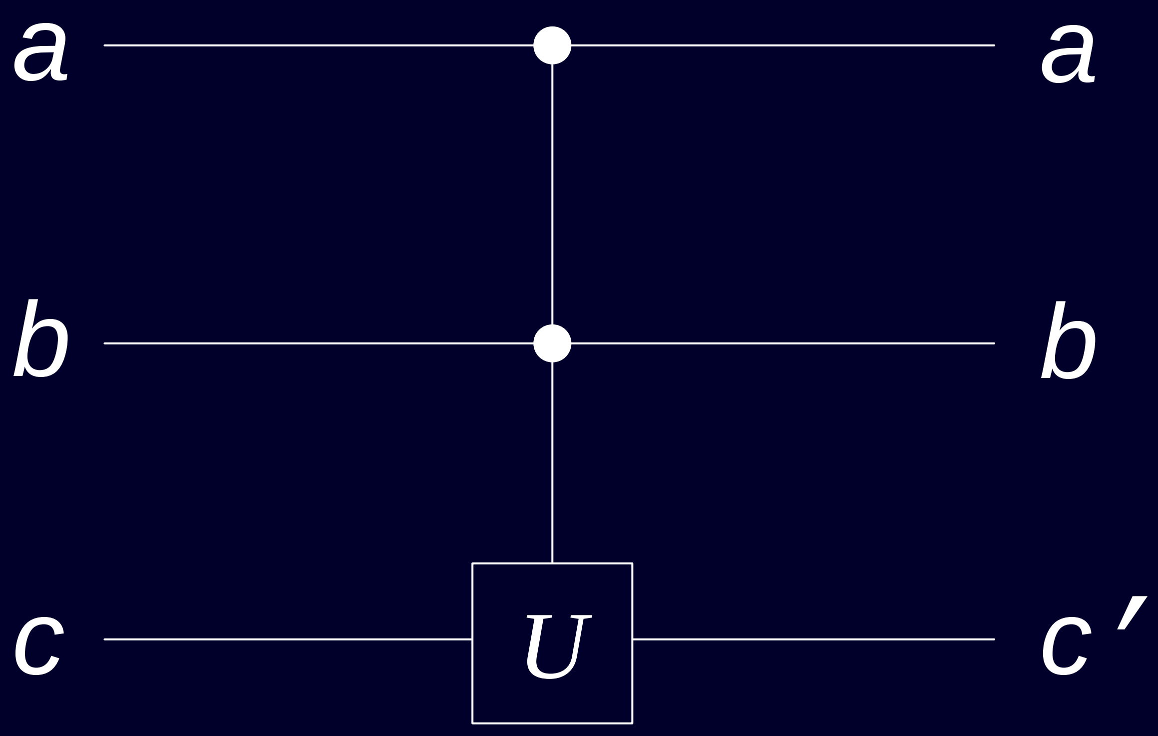
Toffoli Gate

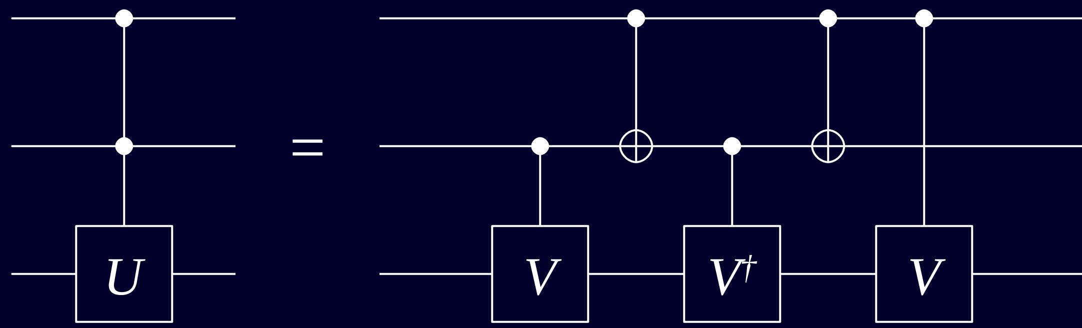


Deutsch's Controlled-U Gate



$$U = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ for Toffoli gate

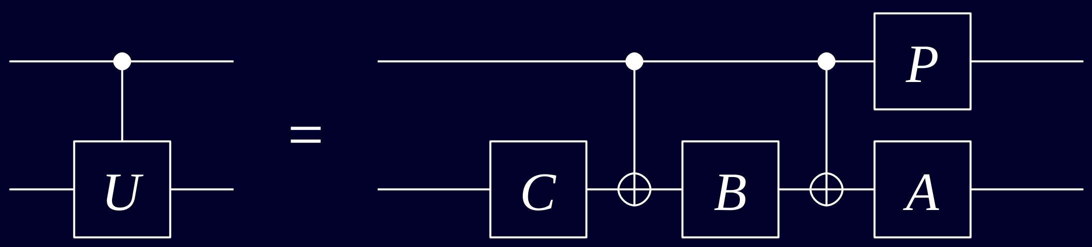
Equivalent Gate Array



$$V^2 = U$$

$$V = \frac{(1-i)(I+iX)}{2}$$ for Toffoli gate

Equivalent Gate Array



$$\begin{aligned} ABC &= I \\ e^{i\alpha} AXBXC &= U \end{aligned}$$

$$U = e^{i\alpha} \begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix} \begin{pmatrix} \cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix} \begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{pmatrix}$$

$$A = \begin{pmatrix} e^{-i\frac{\beta}{2}} & 0 \\ 0 & e^{i\frac{\beta}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\gamma}{4} & -\sin\frac{\gamma}{4} \\ \sin\frac{\gamma}{4} & \cos\frac{\gamma}{4} \end{pmatrix}$$

$$B = \begin{pmatrix} \cos\frac{\gamma}{4} & \sin\frac{\gamma}{4} \\ -\sin\frac{\gamma}{4} & \cos\frac{\gamma}{4} \end{pmatrix} \begin{pmatrix} e^{i\frac{\beta+\delta}{4}} & 0 \\ 0 & e^{-i\frac{\beta+\delta}{4}} \end{pmatrix}$$

$$C = \begin{pmatrix} e^{i\frac{\beta-\delta}{4}} & 0 \\ 0 & e^{-i\frac{\beta-\delta}{4}} \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$$

Almost Any Gate is Universal

$$U = e^{-iHt}$$

$$e^{i(A+B)\Delta t} = e^{iA\Delta t/2} e^{iB\Delta t} e^{iA\Delta t/2} + O(\Delta t^3)$$

$$e^{(A+B)\Delta t} = e^{A\Delta t} e^{B\Delta t} e^{-\frac{1}{2}[A,B]\Delta t^2} + O(\Delta t^3)$$

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Protecting against a Bit-Flip (X)

$$|0\rangle \longmapsto |000\rangle$$
$$|1\rangle \longmapsto |111\rangle$$

Input	*Even*	*Odd*	*Odd*	Output
$\|001\rangle$	$\|001\rangle$	$\|001\rangle$	$\|001\rangle$	$\|000\rangle$
$\|110\rangle$	$\|110\rangle$	$\|110\rangle$	$\|110\rangle$	$\|111\rangle$

Syndrome → third qubit flipped
(reveals nothing about state)

Protecting against a Phase-Flip (Z)

Phase flip (Z)

$$\begin{cases} |0\rangle \xmapsto{z} |0\rangle \\ |1\rangle \xmapsto{z} -|1\rangle \end{cases}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{aligned} |0\rangle &\longmapsto \frac{1}{2\sqrt{2}}\left(|0\rangle + |1\rangle\right)\left(|0\rangle + |1\rangle\right)\left(|0\rangle + |1\rangle\right) \\ |1\rangle &\longmapsto \frac{1}{2\sqrt{2}}\left(|0\rangle - |1\rangle\right)\left(|0\rangle - |1\rangle\right)\left(|0\rangle - |1\rangle\right) \end{aligned}$$

General Errors

- Pauli matrices form basis for 1-qubit operators:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- I is identity, X is bit-flip, Z is phase-flip
- Y is bit-flip and phase-flip combined (Y = i X Z)

9-Qubit Shor Code

$$|0\rangle \longmapsto \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$|1\rangle \longmapsto \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

- Protects against all one-qubit errors
- Error measurements must be erased
- Implies heat generation

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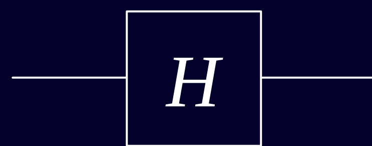
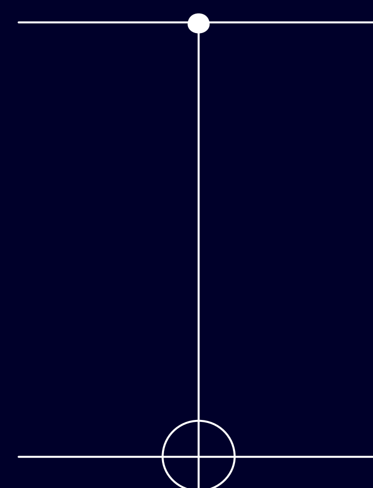
Fault Tolerant Gates



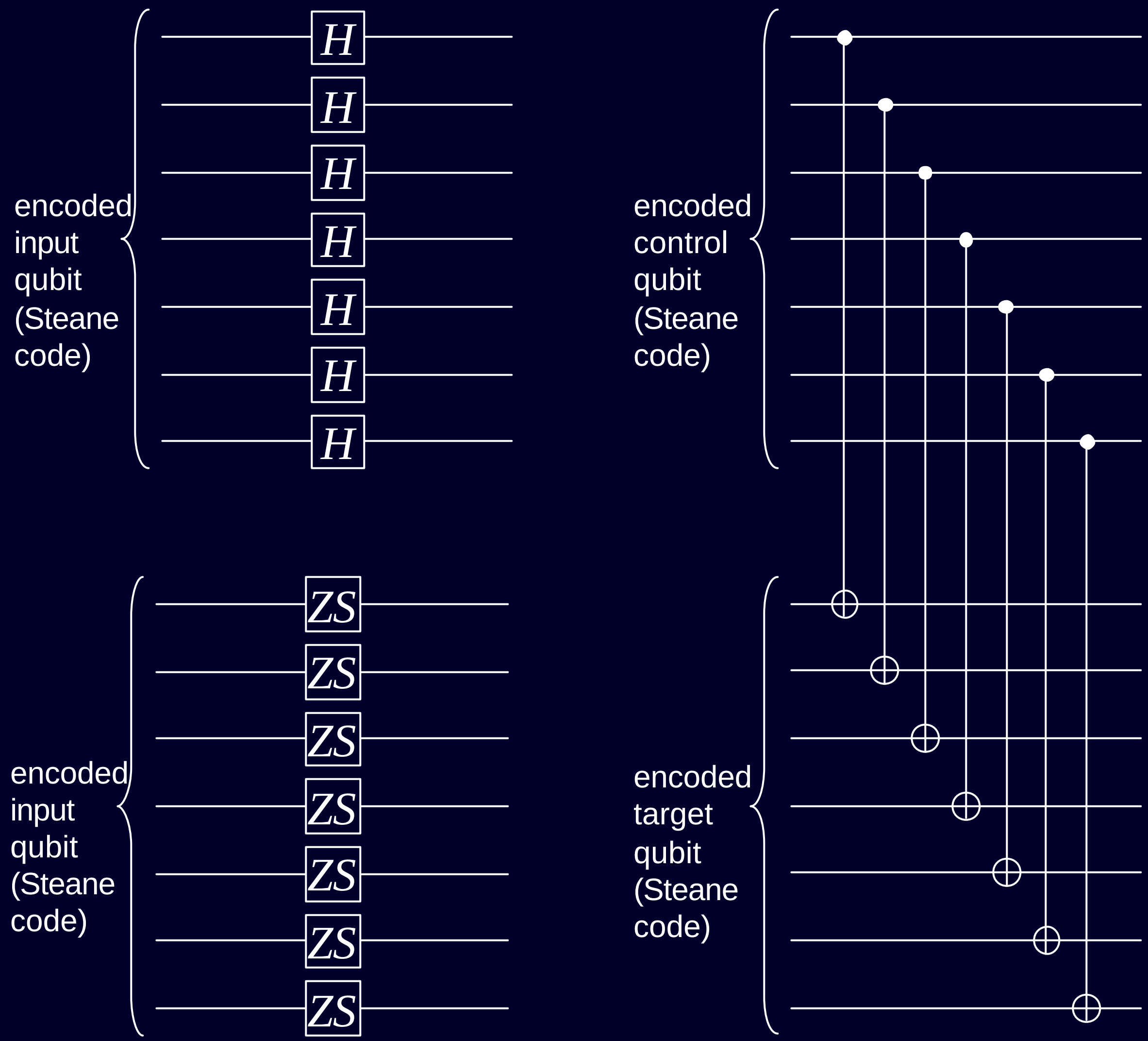
$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$



$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

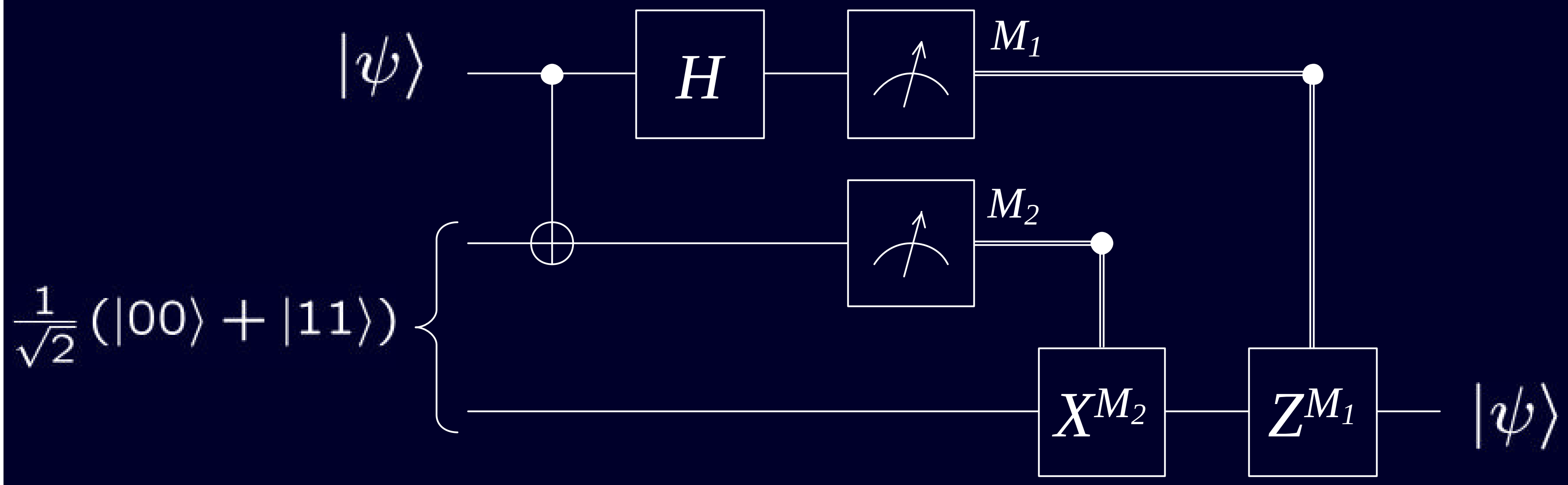
Fault Tolerant Gates



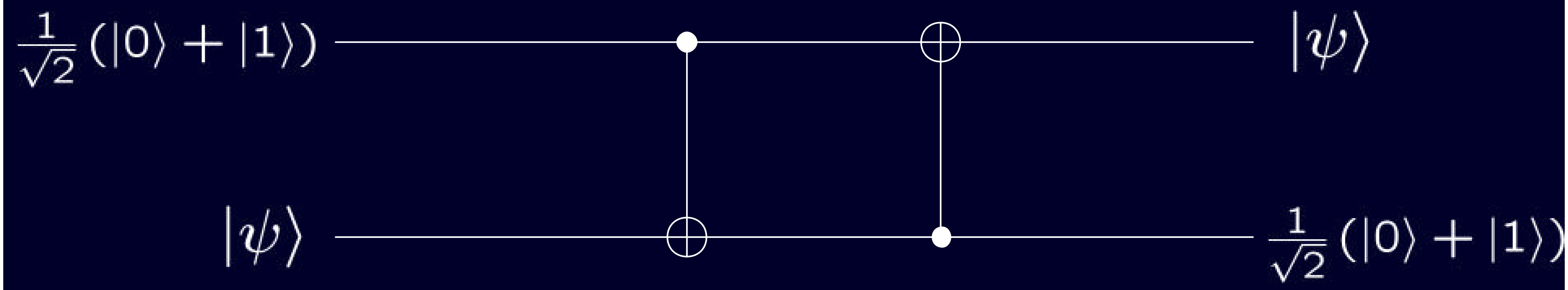
Clifford Group

- Encoded operators are tricky to design
- Manageable for operators in *Clifford group* using stabilizer codes, Heisenberg representation
- Map Pauli operators to Pauli operators
- Not universal

Teleportation Circuit

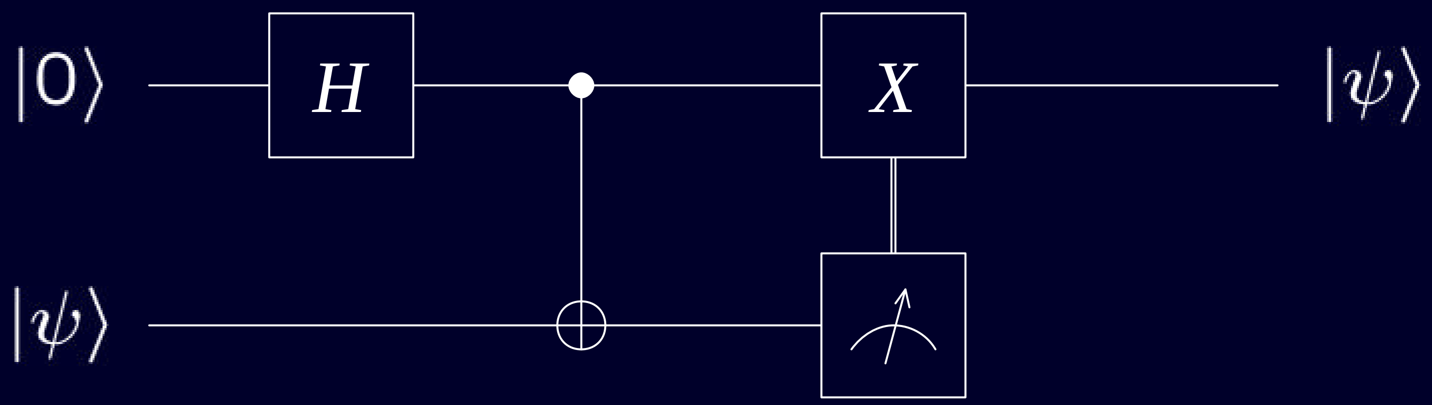


Simplified Circuit

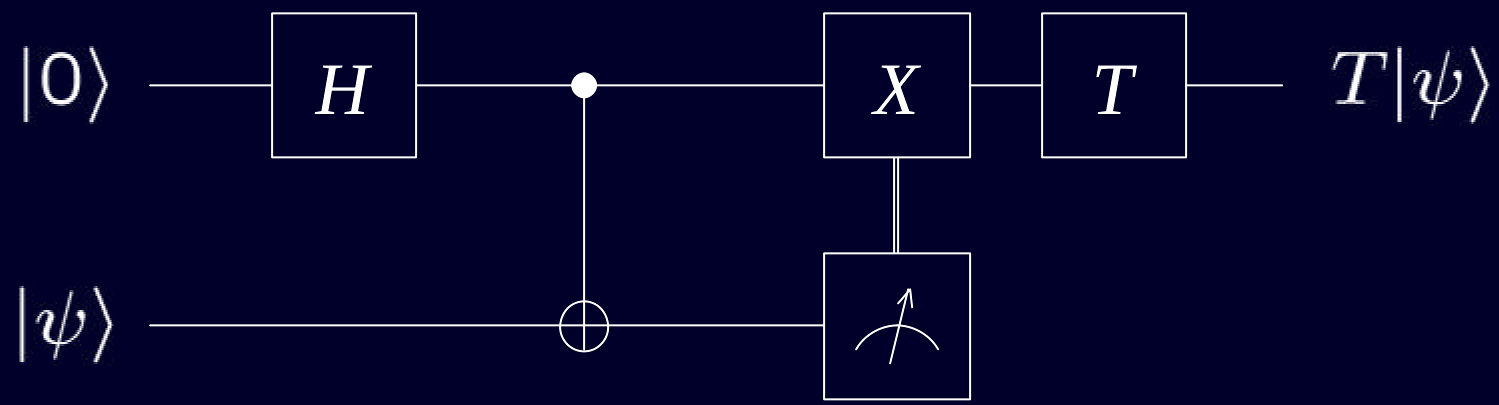


$$\begin{aligned}
\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(a|0\rangle + b|1\rangle) &= (a|00\rangle + b|01\rangle + a|10\rangle + b|11\rangle)/\sqrt{2} \\
&\mapsto (a|00\rangle + b|01\rangle + a|11\rangle + b|10\rangle)/\sqrt{2} \\
&\mapsto (a|00\rangle + b|11\rangle + a|01\rangle + b|10\rangle)/\sqrt{2} \\
&= (a|0\rangle + b|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)
\end{aligned}$$

Equivalent Circuit

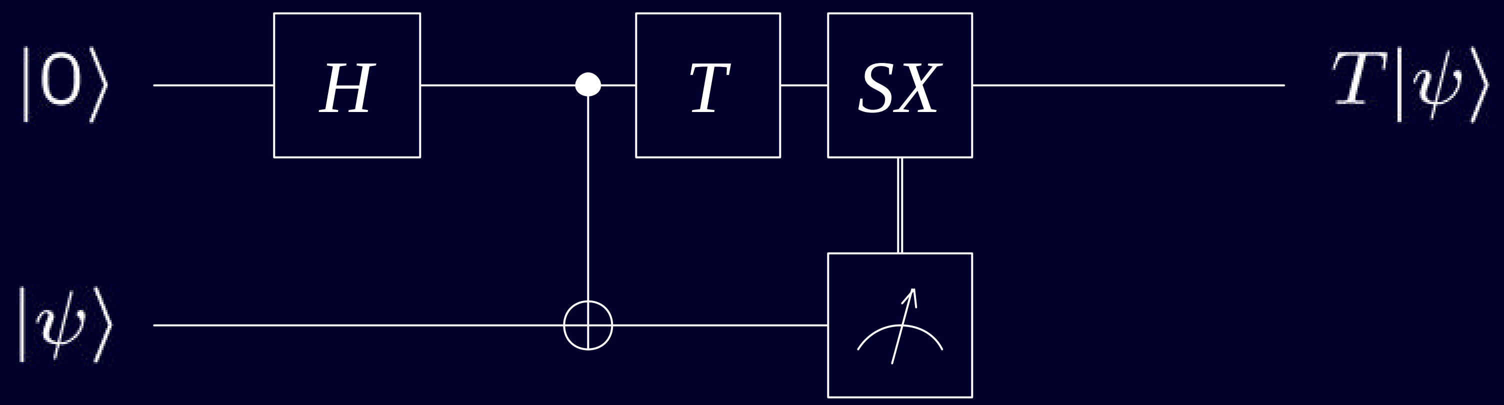


Implementing a Gate



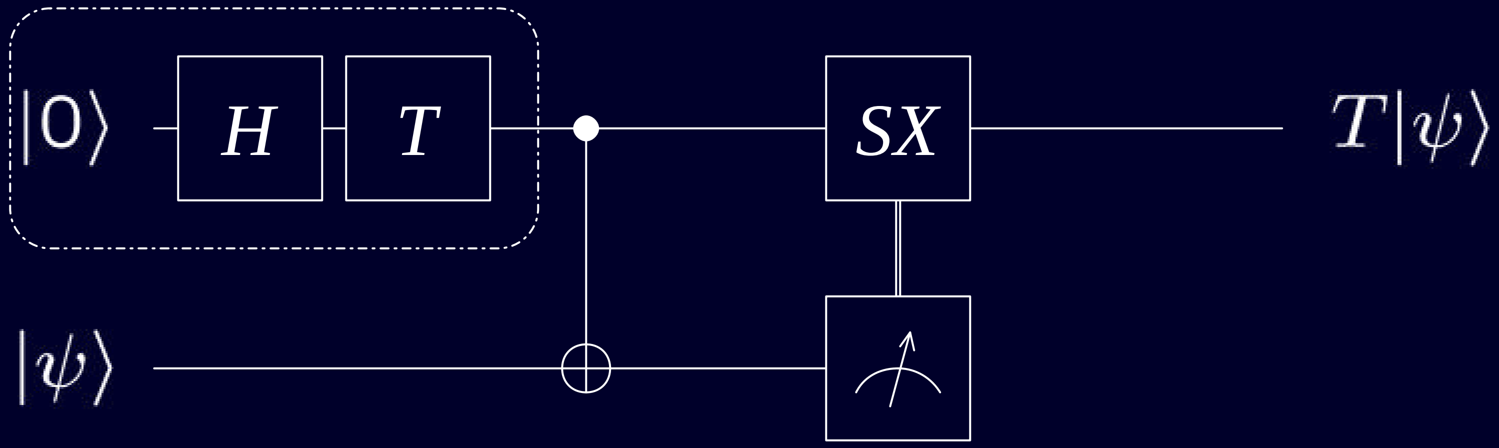
$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

Implementing a Gate



$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad SX = TXT^\dagger$$

Implementing a Gate



Works for U if $\forall P$, $UPU^\dagger$ is in the Clifford group

Conclusions

- Quantum computing requires logical reversibility
  - Entangled qubits cannot be erased by dispersion
- Does not require thermodynamic reversibility
  - Ancilla preparation, error measurement = refrigerator