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ON THE TIME REQUIRED TO RECOGNIZE PROPERTIES OF GRAPHS FROM THEIR ADJACENCY MATRICES†

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Abstract

Let P *be any non-trivial monotone property which applies to the class of* v*-vertex graphs. We show that, if graphs are represented by adjacency matrices, any algorithm for deciding if* P *holds or not of a given graph must, in the worst case, take time proportional to* v^2. *This provides a positive answer to the question raised by Aanderaa and Rosenberg in* [5].

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I. Introduction

Trying to relate the computational complexity of graph properties to the data-structure chosen for representing graphs is a natural and important question. Despite its many mathematical advantages, the adjacency matrix representation of graphs does not appear to be a good choice, if one is expecting to produce graph algorithms whose running time is faster than $\Omega(v^2)$,† v being the number of vertices (nodes) in the graph.

It has been conjectured by Aanderaa and Rosenberg in [5] that recognizing if a v-vertex graph has any particular non-trivial monotone property from its adjacency matrix requires, in the worst case, on the order of v^2 operations. A graph property P is monotone if adding edges to a graph where P holds does not make P false; it is non-trivial if P holds of the complete graph K_v and does not hold of its complement $E_v = \bar{K}_v$, the empty graph.

In this paper, we provide a proof of the validity of Aanderaa-Rosenberg's conjecture.

II. Notations for Graphs and Groups

Before attempting to establish any result, we need to set up some notations and definitions. We shall usually conform to traditional usage, as defined by Biggs [2] and Harary [3] for example, although this has not always been possible.

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†The notation $f(v) = \Omega(g(v))$ means $g(v) = O(f(v))$, i.e., there exists $K > 0$, for all v, $f(v) \geq Kg(v)$; it is the natural inverse of the "big-oh" notation.

2.1. Graphs

A v-graph or graph G (finite undirected labelled graph without self-loops or multiple edges) is a pair $(V(G),E(G))$ where $V(G)$ is a finite set of vertices, labelled 1 through $v = |V(G)|$, and $E(G) \subseteq V(G)^{|2|}$ is a subset of $V(G)^{|2|} = \{\{i,j\} | 1 \leq i,j \leq v, i \neq j\}$ of the symmetric cartesian product $V(G) \times V(G)$. Elements of $E(G)$ are edges and, if $e = \{u,v\} \in E(G)$, we say that e joins u and v. For example, the complete v-graph K_v has $v = |V(K_v)|$ and $E(K_v) = V(K_v)^{|2|}$; it is composed of v vertices and $\frac{1}{2}v(v-1)$ edges. Its complement, the empty v-graph $E_v = \bar{K}_v$ has $E(E_v) = \emptyset$; the complement $\bar{G}$ of a graph G is the graph $(V(G),V(G)^{|2|}-E(G))$.

Two v-graphs G_1 and G_2 are isomorphic if there exists a permutation σ of $\{1,\ldots,v\}$ such that $\{\sigma(u),\sigma(v)\} \in E(G_2)$ if and only if $\{u,v\} \in E(G_1)$. Graph isomorphism, denoted $G_1 \simeq G_2$, is an equivalence relation over the class of v-graphs. An unlabelled graph is an equivalence class of graphs under isomorphism.

Graph G_1 is a subgraph of G_2, denoted $G_1 \leq G_2$, if there exists $G_1' \simeq G_1$ such that $V(G_1') = V(G_2)$ and $E(G_1') \subseteq E(G_2)$. Relation $\leq$ is a partial ordering of v-graphs; it has a minimal element E_v and a maximal element K_v.

The adjacency matrix $M(G) = [m_{i,j}]$ of a v-graph G is a symmetric $v \times v$ boolean matrix such that $m_{i,j} = 1$ if and only if $\{i,j\} \in E(G)$. Two v-graphs G_1 and G_2 are isomorphic $G_1 \simeq G_2$ if and only if there exists a permutation matrix P such that $M(G_1) = P^{-1}M(G_2)P$.

Consider G_1 a v_1-graph and G_2 a v_2-graph. Their sum $G_1 + G_2$ is the $(v_1 + v_2)$-graph G formed by placing a v_1-graph G_1 and a v_2-graph G_2 side by side, i.e., $\{i,j\} \in E(G)$ if and only if

($1 \leq i,j \leq v_1$ and $\{i,j\} \in E(G_1)$) or ($v_1 < i,j \leq v_1 + v_2$ and $\{i-v_1, j-v_1\} \in E(G_2)$). The product $G_1 \times G_2$ is obtained from the sum $G_1 + G_2$ by joining every vertex in G_1 to every vertex in G_2, i.e., $E(G_1 \times G_2) = E(G_1 + G_2) \cup \{\{i,j\} \mid 1 \leq i \leq v_1 < j \leq v_1 + v_2\}$. Clearly, $G_1 + G_2 \leq G_1 \times G_2$ and $\overline{G_1 + G_2} = \bar{G}_1 \times \bar{G}_2$; also, $E_n + E_m = E_{n+m}$ while $K_n \times K_m = K_{n+m}$. We denote by $K_{n,m} = E_n \times E_m$ the complete (n,m)-bipartite graph. A graph G is bipartite if and only if $G \leq K_{n,m}$ for some $n, m \geq 1$.

2.2. Groups

In order to minimize confusion, we use Greek letters for groups and permutations. If Γ is a permutation group on $\{1,\ldots,d\}$, we say that d is the degree of Γ and we denote by $|\Gamma|$ the order of Γ. If Γ_1 and Γ_2 are two permutation groups of degree d, $\Gamma_1 \leq \Gamma_2$ means that Γ_1 is a subgroup of Γ_2. We use $<$ for proper inclusion, and denote by Σ_d the symmetric group of degree d and order $|\Sigma_d| = d!$.

Let Γ_1 and Γ_2 be two permutation groups of degrees d_1 and d_2 respectively. The sum $\Gamma_1 + \Gamma_2$ is the group of degree $d_1 + d_2$ and order $|\Gamma_1 + \Gamma_2| = |\Gamma_1| \cdot |\Gamma_2|$ resulting from the action

$$(\sigma_1 + \sigma_2)(i) = \begin{cases} \sigma_1(i) & \text{if } 1 \leq i \leq d_1 \\ \sigma_2(i-d_1) + d_1 & \text{if } d_1 < i \leq d_1 + d_2 \end{cases} \quad \text{with} \begin{cases} \sigma_1 \in \Gamma_1 \\ \sigma_2 \in \Gamma_2 \end{cases}$$

of Γ_1 and Γ_2 on $\{1,\ldots,d_1+d_2\}$. The product $\Gamma_1 \times \Gamma_2$ is the group of degree $d_1 \times d_2$ and order $|\Gamma_1| \cdot |\Gamma_2|$ resulting from the action

$$(\sigma_1 \times \sigma_2)\langle i,j\rangle = \langle \sigma_1(i), \sigma_2(j)\rangle \quad \text{with } 1 \leq i \leq d_1,\ 1 \leq j \leq d_2,\ \sigma_1 \in \Gamma_1 \text{ and } \sigma_2 \in \Gamma_2,$$

of Γ_1 and Γ_2 on $\{1,\ldots,d_1\} \times \{1,\ldots,d_2\}$.

If Γ is a permutation group on $\{1,\ldots,d\}$, the pseudo-square $\Gamma^{|2|}$ is the permutation group of degree $\frac{1}{2}d(d-1)$ and order $|\Gamma^{|2|}| = |\Gamma|$ resulting from the action $\sigma(\{i,j\}) = \{\sigma(i),\sigma(j)\}$ for $1 \le i,j \le d$ and $\sigma \in \Gamma$ of Γ over $\{1,\ldots,d\}^{|2|}$. If $|\Gamma| > 1$, then $\Gamma^{|2|} < \Gamma \times \Gamma$.

A permutation group Γ on $\{1,\ldots,d\}$ is transitive if the orbit $i.\Gamma = \{j|\ 1 \le j \le d,\ \exists\sigma \in \Gamma: j = \sigma(i)\}$ of any $i \in \{1,\ldots,d\}$ in Γ has size $|i.\Gamma| = d$, i.e., $i.\Gamma = \{1,\ldots,d\}$. For example, Σ_d and $\Sigma_d^{|2|}$ are both transitive. If Γ, Γ_1 and Γ_2 are transitive, $\Gamma_1 \times \Gamma_2$ is also transitive but $\Gamma^{|2|}$ is not transitive in general.

An automorphism of a graph G is an isomorphism of G with itself. The set of automorphisms of a v-graph G is a permutation group $\Gamma(G) = \{\sigma \in \Sigma_v|\ \{i,j\} \in E(G) \text{ iff } \{\sigma(i),\sigma(j)\} \in E(G)\}$ called the automorphism group or the point group of G. The automorphisms of G also induce a permutation group $\Gamma(G)^{|2|}$ on the edges (lines) of G, called the line group of G. For example

$$\Gamma(K_v) = \Gamma(E_v) = \Sigma_v \quad \text{and} \quad \Gamma(K_v)^{|2|} = \Gamma(E_v)^{|2|} = \Sigma_v^{|2|};$$

$$\Gamma(K_{m,n}) = \Sigma_m + \Sigma_n \quad \text{and} \quad \Gamma(K_{m,n})^{|2|} = \Sigma_m \times \Sigma_n \quad \text{if} \quad n \neq m .$$

In general $\Gamma(G) = \Gamma(\bar{G})$.

2.3. Symmetric Graphs

Graph G is point-symmetric (respectively line-symmetric) if $\Gamma(G)$ (respectively $\Gamma(G)^{|2|}$) is transitive. If G is both line and point symmetric, we say that graph G is symmetric. For example, E_v, K_v and $K_{v,v}$ are symmetric. If $n \neq m$, $K_{n,m}$ is line symmetric but not point symmetric, while $(K_n + K_n) \times (K_n + K_n)$ is point symmetric but not line symmetric for $n > 1$. If G is symmetric, $G + G$ is also symmetric;

if G is point symmetric, so are $\bar{G}$, $G+G$ and $G \times G$.

We now define a family of symmetric graphs which will be useful later on. Let $v = 2^p$, where p is a non-negative integer.

<u>Definition D1</u>: For each $0 \leq i \leq p$, the <u>graphs</u> B_p^i are defined by:

(i) $B_p^p = K_v$ *with* $v = 2^p$;

(ii) $B_p^i = B_{p-1}^i + B_{p-1}^i$ *for* $0 \leq i < p$.

For example, $B_0^0 = \cdot$, $B_2^1 = ||$, $B_3^2 = \square\square$, etc. In general, B_p^i consists of 2^{p-i} copies of K_{2^i}. It is easy to establish that these graphs have the following properties:

<u>Lemma 1</u>: The family $\{B_p^i \mid 0 \leq i \leq p\}$ *of graphs defined by D1 has the properties:*

(a) $E_v = B_p^0$ *and* $K_v = B_p^p$ *with* $v = 2^p$;

(b) $B_p^i < B_p^{i+1}$ *for* $0 \leq i < p$;

(c) B_p^i *is symmetric;*

(d) $B_p^{i+1} \leq B_{p-1}^i \times B_{p-1}^i$ *for* $0 \leq i < p$.

III. <u>The Argument Complexity of Boolean Functions</u>

3.1. <u>Monotone Non-trivial Properties</u>

Let $\{0,1\}^d$ represent the set of all (boolean) d-tuples over $\{0,1\}$. For any two elements $\bar{x} = \langle x_1,\dots,x_d\rangle$ and $\bar{y} = \langle y_1,\dots,y_d\rangle$ of $\{0,1\}^d$, we write $\bar{x} \leq \bar{y}$ whenever $x_i \leq y_i$ for all $1 \leq i \leq d$. For example, a v-graph G can be represented by a boolean vector $g \in \{0,1\}^d$ with $d = \frac{1}{2}v(v-1)$, where g is the upper non-diagonal part of the adjacency matrix $M(G)$ of G. If another v-graph G' is represented in a similar fashion g', then $G \simeq G'$ if and only if $g = \sigma g'$ for some $\sigma \in \Sigma_v^{|2|}$; similarly, $G \leq G'$ if and only if $g \leq \sigma g'$ for some $\sigma \in \Sigma_v^{|2|}$.

Consider a boolean function (property) $P: \{0,1\}^d \to \{0,1\}$ mapping the set of boolean d-tuples into $\{0,1\}$. If $\bar{x} \le \bar{y}$ implies $P(\bar{x}) \le P(\bar{y})$ for all $\bar{x}, \bar{y} \in \{0,1\}^d$, we say that P is monotone. We denote by $M_d = \{P: \{0,1\}^d \to \{0,1\} \mid P \text{ monotone}, P(\bar{0}) \equiv 0, P(\bar{1}) \equiv 1\}$ the class of monotone non-trivial properties. "Property" will now mean "monotone non-trivial boolean property".

We say that property $P \in M_d$ with $d = \frac{1}{2}v(v-1)$ is "invariant under graph isomorphism", or simply that "P is a v-graph property" if, for any $g \in \{0,1\}^d$ and $\sigma \in \Sigma_v^{|2|}$, $P(g) \equiv P(\sigma(g))$. This boolean vector $g \in \{0,1\}^d$ can be regarded as the upper non-diagonal part of the adjacency matrix M(G) of some v-graph G. We write P(G) rather than P(g) or P(M(G)); this notation however means that graph G is represented as a boolean vector of $\frac{1}{2}v(v-1)$ entries. The class of v-graph properties is denoted by $P_v = \{P \in M_d \mid d = \frac{1}{2}v(v-1), P \text{ is a v-graph property}\}$.

To any property $P \in M_d$, we can associate a permutation group $\Gamma(P) = \{\sigma \in \Sigma_d \mid \forall \bar{x} \in \{0,1\}^d: P(\bar{x}) \equiv P(\sigma(\bar{x}))\}$ which is the maximal group of permutation of the argument positions leaving P invariant. For example, P is a v-graph property if it is invariant under graph-isomorphism, i.e., $\Sigma_v^{|2|} \le \Gamma(P)$.

Similarly, we say that P is an (m,n)-bipartite property if $\Sigma_m \times \Sigma_n \le \Gamma(P)$; the class of (m,n)-bipartite properties is denoted by $P_{m,n} = \{P \in M_{m\times n} \mid \Sigma_m \times \Sigma_n \le \Gamma(P)\}$.

3.2. Algorithms and Complexity of Properties

An algorithm for evaluating $P(x_1,\ldots,x_d)$ with $P \in M_d$ must examine some of the individual arguments x_i, since P is non-constant.

On any reasonable model of machine, the number of arguments which need to be examined determines a lower bound on the execution time of the algorithm. In order to formalize this idea, we define a decision-tree T for property P to be a binary tree whose internal nodes specify arguments to be tested and external nodes are marked according to the appropriate value of P.

For example, if P is the 3-graph property, $P(G) \equiv$ "3-graph G is connected", the following is a decision tree for P, where $\{i,j\}$ in an internal node means the algorithm is to test the entry $m_{i,j}$ of $M(G)$.

T =

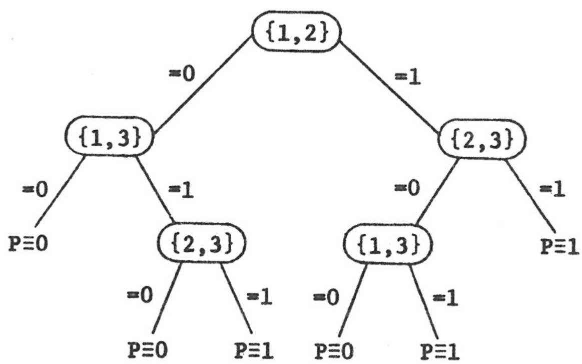

Figure 1

In general, we denote by $c(T,\bar{x})$ the number of tests made in determining $P(\bar{x})$ according to the decision tree T. For example, if graphs G_1 and G_2 are given respectively by the adjacency matrices

$$M(G_1) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } M(G_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ then } c(T,G_1) = 2 \text{ and } c(T,G_2) = 3.$$

The maximum number of tests made, $\max_{\bar{x} \in \{0,1\}^d} c(T,\bar{x})$, or, equivalently the maximum depth of the tree representation of T will be our measure of the cost of a particular decision tree T. The **argument complexity** $C(P)$ **of property** P will be the cost of the cheapest decision tree T for P:

Definition D2: The argument complexity $C(P)$ of property P is defined by:

$$C(P) = \min_{T \text{ a decision-tree for } P} \; \max_{\bar{x} \in \{0,1\}^d} \{c(T,\bar{x})\} \; .$$

As mentioned earlier, the argument complexity of property P is a lower bound on the time complexity of P. If $E \subseteq M_d$ is a class of properties, the complexity $C(E) = \min_{P \in E} \{C(P)\}$ is the minimum complexity of properties in the class. We are interested in graphs and bipartite properties:

Definition D3: We denote by $F(v)$ and $F(n,m)$ respectively the complexity of the classes of v-graph and (n,m)-bipartite properties, i.e., $F(v) = \min_{P \in P_v} \{C(P)\}$ and $F(n,m) = \min_{P \in P_{n,m}} \{C(P)\}$.

In general, if a class of functions is defined by an invariance permutation group,

Definition D4: The complexity $C(\Gamma)$ of a permutation group is the least complexity

$$C(\Gamma) = \min_{\{P \in M_d \mid \Gamma \leq \Gamma(P)\}} \{C(P)\} \text{ of properties } P \text{ left invariant by } \Gamma \; .$$

Using this notation gives $F(v) = C(\Sigma_v^{|2|})$ and $F(n,m) = C(\Sigma_m \times \Sigma_n)$.

It follows directly from (D4) that $\Gamma_1 \leq \Gamma_2$ and $\deg(\Gamma_1) = \deg(\Gamma_2)$ implies $C(\Gamma_1) \leq C(\Gamma_2)$. It is an easy exercise to show for example that $C(\Sigma_d) = d$.

In [4], Rivest and Vuillemin have shown that:

Theorem 1: *If the permutation group* Γ *is transitive and has degree* $d = q^\alpha$ *a prime power, then* $C(\Gamma) = d$.

This result has no direct implication as to the complexity of graph properties since the degree $\frac{1}{2}v(v-1)$ of $\Sigma_v^{|2|}$ is never a prime power unless $v = 2$ or 3. For bipartite properties however, we obtain $F(q^\alpha, q^\beta) = q^{\alpha+\beta}$ for any prime q and $\alpha, \beta \in \mathbb{N}$ as a corollary. The rest of the paper describes a way to embed some forms of bipartite properties into graph properties, so as to show $F(v) \geq Kv^2$ for some constant K.

IV. Proof of the Main Theorem

4.1. Embedding Technique

The general idea is to extract a subset of the entries in the adjacency matrix, and "give away" the other entries. We must keep enough symmetry into the problem so that $\Sigma_v^{|2|}$ acts transitively on the chosen subset and we can apply Theorem 1 in order to get $F(v) \geq Kv^2$. More precisely, we use:

Lemma 2: *Let* $P \in P_v$ *be a v-graph property,* G_1 *and* G_2 *a* v_1*- and* v_2*-graph respectively, with* $v_1 + v_2 = v$. *If* $P(G_1 + G_2) = 0$ *and* $P(G_1 \times G_2) = 1$, *then* $C(P) \geq C(\Gamma(G_1) \times \Gamma(G_2))$.

Proof: Let E_0 denote those edges in $G_1 \times G_2$ but not in $G_1 + G_2$, i.e. E is the set of edges joining vertices in G_1 with vertices in G_2, and is a subset of $E(K_{v_1,v_2})$:

$$E_0 = \{\{i,j\} \mid 1 \le i \le v_1 < j \le v_1+v_2 \text{ where } v_1 = |V(G_1)|,\ v_2 = |V(G_2)|\}\ .$$

Consider the function P', a restriction of P, mapping subsets S of E_0 into $\{0,1\}$ defined by:

$$P'(S) = P(G)\ , \quad \text{with}\quad G = (V(G_1+G_2), E(G_1+G_2) \cup S)\ .$$

By hypothesis, P' is a nontrivial, monotone function of S, since $E(G_1+G_2) \cup E_0 = E(G_1 \times G_2)$. By the definition of P' it follows that $C(P') \le C(P)$, since any decision tree for P can also be used for P' (P' is just P on a restricted domain).

It remains to show that $C(P') \ge C(\Gamma(G_1) \times \Gamma(G_2))$ by showing $\Gamma(P') \ge \Gamma(G_1) \times \Gamma(G_2)$. Now P is left invariant by $\Gamma(P) \ge \Sigma_v^{|2|}$, thus also by the subgroup Γ' of $\Sigma_v^{|2|}$ which fixes $E(G_1+G_2)$. But $\Gamma' \ge (\Gamma(G_1) + \Gamma(G_2))^{|2|}$ (acting on $(V(G_1) \cup V(G_2))^{|2|}$), which contains the subgroup $\Gamma(G_1) \times \Gamma(G_2)$ acting on E_0. □

In order to apply Theorem 1, we need that $\Gamma(G_1) \times \Gamma(G_2)$ be transitive and $v_1 \times v_2$ be a prime power. As noticed earlier, it is sufficient, in order for $\Gamma(G_1) \times \Gamma(G_2)$ to be transitive, that $\Gamma(G_1)$ and $\Gamma(G_2)$ be both transitive, i.e., that G_1 and G_2 be point symmetric. For the requirement $v_1 \times v_2$ is a prime power, we first consider v-graphs where v is a power of 2.

4.2. Graphs of Size 2^p

Using Lemma 2, it is now easy to prove

Lemma 3: *If* $v = 2^p$, $p \ge 1$, *then* $F(v) \ge \frac{1}{4}v^2$.

Proof: Consider the graphs B_p^i for $0 \leq i \leq p$ defined by (D1). Any graph property $P \in \mathcal{P}_v$ must be such that $0 \leq i \leq j$ implies $P(B_p^i) \equiv 0$ and $j < i \leq p$ implies $P(B_p^i) \equiv 1$ for some j such that $0 \leq j < p$ (this follows from monotonicity of P and Lemma 1, (a) and (b)). In particular, $P(B_p^j) \equiv P(B_{p-1}^j + B_{p-1}^j) \equiv 0$ and $P(B_p^{j+1}) \equiv 1$. Since we proved in Lemma 1, (d) that $B_p^{j+1} \leq B_{p-1}^j \times B_{p-1}^j$, and P is monotone, $P(B_{p-1}^j \times B_{p-1}^j) \equiv 1$. Applying Lemma 2 then yields $C(P) \geq C(\Gamma(B_{p-1}^j) \times \Gamma(B_{p-1}^j))$. As noticed in Lemma 1, (c), graph B_{p-1}^j is symmetric, therefore $\Gamma(B_{p-1}^j) \times \Gamma(B_{p-1}^j)$ is transitive. Since the degree of this group is $2^{p-1} \times 2^{p-1} = \frac{1}{4}v^2$ which is a prime power, Theorem 1 gives us $C(P) \geq \frac{1}{4}v^2$. This bound is valid for any $P \in \mathcal{P}_v$, thus $F(v) \geq \frac{1}{4}v^2$. □

This proves $F(v) \geq Kv^2$ for $v = 2^p$ a power of two. The construction can be adapted (at some cost) to powers of 3, and prime powers in general. What to do with numbers v which are not prime powers is not clear however. Instead of following this approach, we shall prove that $F(v)$ is more or less increasing with v, thus obtaining $F(v) \geq K'v^2$ for all v, the constant K' being lower than the one $(K = \frac{1}{4})$ which applies for $v = 2^p$ a power of two.

4.3. General Case

Proving directly that $F(v) \geq F(v-1)$ is not easy,† no matter how intuitively obvious this appears to be. We prove the following weaker result, which will be sufficient for our purposes:

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†As a matter of fact, this question is unresolved as far as the authors are concerned. This might not be much simpler than proving $F(v) = \frac{1}{2}v(v-1)$.

<u>*Lemma 4:*</u> *For all* $v \in \mathbb{N}$,

$$F(v) \geq \min(F(v-1), 2^{2K-2})$$

where $2^K \leq v < 2^{K+1}$.

<u>Proof</u>: For an arbitrary property $P \in \mathcal{P}_v$, one of three cases holds:

(i) $P(K_1 + K_{v-1}) = 1$,

(ii) $P(K_1 \times E_{v-1}) = 0$, or

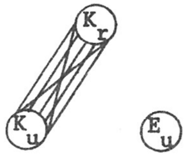
(iii) neither of the above.

Cases (i) and (ii) imply that $F(v) \geq F(v-1)$ directly, since we may induce a function $P' \in \mathcal{P}_{v-1}$ from P by suitably restricting the domain: $P'(G) = P(K_1 + G)$ in case (i) and $P'(G) = P(K_1 \times G)$ in case (ii). In either case P' is a monotone nontrivial graph property.

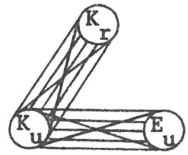
In case (iii), using u to denote 2^{K-1} and r to denote $v - 2u$, we have

$$P((K_u \times K_r) + E_u) = 0 \quad \text{since} \quad P(K_1 + K_{v-1}) = 0$$
$$\text{and} \quad ((K_u \times K_r) + E_u) \leq K_1 + K_{v-1} \;; \quad \text{and}$$
$$P((K_r + E_u) \times K_u) = 0 \quad \text{since} \quad P(K_1 \times E_{v-1}) = 1$$
$$\text{and} \quad ((K_r + E_u) \times K_u) \geq K_1 \times E_{v-1} \,.$$

The function P' defined as P restricted to those edges between K_u and E_u satisfies all the requirements of Theorem 1: We have just shown that it is nontrivial, it is monotone since it is a restriction of the monotone function P, and it is invariant under the action of $\Sigma_u \times \Sigma_u$, acting on the vertices of K_u and E_u, a transitive group. Since $C(P) \geq C(P')$ and P' is exhaustive, this proves the lemma. □



$(K_u \times K_r) + E_u$



$(K_r + E_u) \times K_u$

Figure 2

Combining lemmas 3 and 4 yields directly:

<u>Theorem 4</u>: *If P is a nontrivial monotone graph property of v-graphs, then*

$$C(P) \geq v^2/16 .$$

<u>Proof</u>: If $v = 2^K + r$ with $0 \leq r < 2^K$, then lemmas 3 and 4 give $C(P) \geq 2^{2K-2} \geq v^2/16$. □

Of course, this result also applies to other classes of graphs, directed, or multi-edges. It can be used directly as a lower bound, or the construction can be adapted so as to improve the constant.

V. Conclusion

The tantalizing remaining question is the exact value of $F(v)$. It is widely conjectured that $F(v) = \frac{1}{2}v(v-1)$ and this has been proved in [4] for $v = 1,2,4,5,7,11,13$. This is part of a more general problem discussed in [4]: <u>Is it true that any transitive permutation group</u> Γ <u>of degree</u> d <u>has complexity</u> $C(\Gamma) = d$? The results of the paper indicate that it might be easier to prove the existence of a constant K such that any transitive Γ of degree d has $C(\Gamma) \geq Kd$.

The monotonicity requirement is also discussed in [4] and, in fact, there is nothing to stop us from believing that $C(P) \geq Kv^2$ for any (monotone or non-monotone) v-graph property, provided $P(E_v) \neq P(K_v)$.

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† A complete bibliography on the problem can be found in [4].