Lecture 19

Local Computation Algorithms:

Maximal Independent Set

Maximal Independent Set:

\[
\text{def } U \subseteq V \text{ is a } \text{"Maximal Independent Set" (MIS)} \text{ if}
\]

1. \( \forall u, u' \in U, (u, u') \notin E \)

2. \( \exists w \in V \setminus U \text{ st. } U \cup \{w\} \text{ is independent } \text{"maximal"} \)

Today's assumption:

\( G \) has max degree \( d \)

Note: MIS can be solved via greedy (not NP-complete)
Distributed Algorithm for MIS:
“Luby’s Algorithm” (actually a variant)

- MIS = ∅
- all nodes set to “live”

repeat K times in parallel:
- A nodes V, color self “red” with prob = \( \frac{1}{2d} \), else “blue”. Send color to all nbrs.
- If v colors self “red” & no other nbr of v colors self red then red "volunteer to be in MIS"
  - add v to MIS
  - remove v & all nbrs from graph (set to “dead”)

(for purposes of analyses, continue to select selves after die, but don’t do anything else)

\[ \Pr[\text{# rounds til graph empty} \geq 8d \log n] \leq \frac{1}{n} \]

\[ E[\text{# rounds}] \text{ is } O(d \log n) \iff \text{can improve!} \]
Main Lemma

\[ \Pr[v \text{ live } \& \text{ added to MIS in round}] \leq \frac{1}{4d} \]

\[ \implies \]

Then \( v \) "dies".

Proof

\[ \Pr[v \text{ colors self red}] = \frac{1}{2d} \]

\[ \Pr[\text{any } w \in N(v) \text{ colors self red}] \leq \sum_{w \in N(v)} \frac{1}{2d} \]

\[ \leq \frac{1}{2} \]

(union bound)

(bound on degree)

\[ \therefore \Pr[v \text{ colors self red } \& \text{ no other nbr colors self red}] \leq \frac{1}{2d} \left( 1 - \frac{1}{2} \right) = \frac{1}{4d} \]

\[ \Rightarrow \]

Corrected \( \Pr[v \text{ live after } 4Kd \text{ rounds}] \leq \left( 1 - \frac{1}{4d} \right)^{4Kd} = e^{-K'} \)

Setting \( K' = \log n \)

if \( K = O(d \log n) \), \( \Pr[v \text{ live at end}] \leq e^{-O(\log n)} = \frac{1}{n} \)

(can do better)
See slides for Local Computation Algorithm (LCA) model.

Problem when sequentially simulate k-round algorithm:

get \( d^k \) complexity

\( K = O(\log n) \Rightarrow d^k \) not sublinear

What to do? run fewer rounds

many nodes will not be decided yet \( \Rightarrow \) is it ok?
Local Computation Algorithm to compute Luby's answer: "Luby Status"

- Run Luby with $K = O(d \log d)$ rounds
  
  at end each node $v$ is one of:
  
  \[
  \begin{cases}
  \text{live} & \text{in MIS} \\
  \text{not in MIS} & \text{taken out by nbr who is in MIS}
  \end{cases}
  \]

- Use "Parnas-Ron" reduction:
  
  simulate $v$'s view of computation in sequential manner:
  
  \[
  d^K = O(d \log d)
  \]

  queries

  - if $v$ is in/not in then done

  else $v$ is alive \(< \) what do we do now?

Luby:

- set $K = O(d \log d)$ "Luby Status"
- MIS $\leftarrow \emptyset$
- all nodes set to "live"
- repeat $K$ times in parallel:
  
  - A nodes $v$, color self "red" with prob $\geq \frac{1}{2d}$, else "blue". Send color to all nbrs.
  
  - if $v$ colors self "red" & no other nbr of $v$ colors self red then
    - add $v$ to MIS
    - remove $v$ + all nbrs from graph (set to "dead")

  \[
  v
  \]

  Old Log d

  that determines result of "Luby status"

  use "Parnas- Ron" reduction:
Questions:

What is $\Pr[v \text{ is alive}]$?

How are live nodes distributed after $O(d \log d)$ rounds?

lots of live, few dead?

$\Pr[v \text{ survives } O(d \log d) \text{ rounds}] \leq e^{-c} \leq \frac{1}{d^c}$

Most "die"
few live but clumped together.

No!

Surviving nodes will be in small connected components.
Surviving nodes will be in small connected components "shattered".

This relies heavily on degree bound of graph

- # conn comp subgraphs small
- survival of components ~ independent
Luby status: Luby with $k = O(d \log d)$:

given $v$, is it:

- live
- in MIS $\leftarrow$ set self to red + no nbrs did not in MIS $\leftarrow$ taken out by nbr

Luby:

- $MIS \leftarrow \emptyset$
- all nodes set to "live"

repeat $K$ times in parallel:

- A nodes $v$, color self "red" with prob $\geq \frac{1}{2d}$ else "blue". Send color to all nbrs.
- If $v$ colors self "red" & no other nbr of $v$ colors self red then:
  - add $v$ to MIS
  - remove $v$ + all nbrs from graph (set to "dead")

LCA for $MIS(v)$:

- Run sequential version of Luby status ($v$)
- if it is in/out output answer + halt
- else, (1) do BFS to find $v$'s connected component
  - of live nodes
  - (2) Compute lexicographically 1st MIS $M'$ for that connected component (consistent with nbrs that are decided)
- (3) Output whether $v$ in/out of $M'$

Runtime

$O(d \log d)$

$O(d \log d) \times \text{size of component}$

what is this?
Bounding size of connected components:

Claim: After $O(d \log d)$ rounds, connected components of survivors of size $\leq 0(\text{poly}(d) \log n)$

$\Rightarrow$ runtime of above procedure is $\sim d^{o(d \log d)} \times \text{polylog}(d) \cdot \log n$

Main difficulty: survival of $v$ & neighbors not independent
Bounding survivors:

\[ A_v = \begin{cases} 1 & \text{if } v \text{ survives all rounds} \\ 0 & \text{o.w.} \end{cases} \]

\[ B_v = \begin{cases} 1 & \text{if } \text{no } w(\text{nbr of } v) \text{ colors self} \\ 0 & \text{o.w.} \end{cases} \]

Note: \( A_v = 1 \implies B_v = 1 \)

\[
\Pr [ B_v = 1 ] \leq \left( 1 - \frac{1}{\sqrt[4]{d^3}} \right)^{cd \log d} \leq \frac{1}{8d^3} \quad \text{for } c \geq 20
\]
Bounding size of connected components:

\[ A_v = \begin{cases} 1 & \text{if } v \text{ survives all rounds} \\ 0 & \text{otherwise} \end{cases} \]

\[ B_v = \begin{cases} 1 & \text{if there exists a set } s_t \text{ s.t. } v \text{ colors self } + \text{ no } w \text{ in } N(v) \text{ colors self} \\ 0 & \text{otherwise} \end{cases} \]

Note: \( A_v = 1 \Rightarrow B_v = 1 \)

eg. \( v \) survives \( \Rightarrow \) there exists a set \( s_t \) s.t. \( v \) colors self + no \( w \) in \( N(v) \) colors self

we care about \( A_v \)'s, but \( B_v \)'s have nice independence properties
Bounding size of connected components.

Claim: After $O(d \log d)$ rounds, connected components of survivors of size $\leq O(p(\text{poly}(d) \cdot \log n))$ can find whole component via BFS.

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Proof idea:
- any large component has lots of nodes that are independent (distance $\geq 3$)
- these independent nodes unlikely to simultaneously survive

do we need union but over all sets of size $w$? NO
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Claim: After \( O(d \log d) \) rounds, connected components of survivors of size \( \leq O(\text{poly}(d) \log n) \)

**Proof**

Let \( H \leftarrow \text{graph st. nodes } \sim \text{Bv} \)

edges \( \sim \text{Bv} + \text{Bw} \) distance 3 in \( G \)

represent independent events

\[
\deg(H) \leq d^3
\]

Observe: # components in \( H \) of size \( w \)

\[
\leq \# \text{ size w subtrees in } H
\]

Why? map each component \( C \) in \( H \) to arbitrary spanning tree of \( C \)

mapping is 1-1

but could have many spanning trees per component

\( \text{great!} \)

we are good at counting subtrees

\( \text{not even connected!} \)
How many subtrees in a degree bounded graph?

\[ \text{Known Thm: } \text{# non isomorphic trees on } w \text{ nodes} \leq 4^w \]

\[ \cdots \]

\[ \text{Corr: } \text{# size } w \text{ subtrees in } N \text{-node graph of degree } D \leq 0 \]
\[ \text{is } \leq N \cdot 4^w \cdot D^w = N (4D)^w \]

Why?

- choose root in \( H \)
- choose size \( w \) tree shape from known thm
- choose placement in \( H \)

\[ \text{Total # choices: } N \cdot 4^w \cdot D^w \]
Claim: After \( O(d \log d) \) rounds, connected components of survivors of size \( \leq O(\text{poly}(d) \cdot \log n) \)

Proof:

Let \( H \leftarrow \text{graph s.t. nodes } \sim B_v \)

- \( \deg(H) \leq d^3 \)
- \( \# \text{ components in } H \leq \# \text{ size } w \text{ subtrees of size } w \text{ in } H \leq n \cdot (4d^3)^w \) 

- \( \Pr[ \text{ node } u \text{ survives }] \leq \frac{1}{8d^3} \)

\[ \Pr[ \text{ component of size } w \text{ in } H \text{ survives }] \leq \left( \frac{1}{8d^3} \right)^w \]

\[ \Pr[ \text{ any component of size } w \text{ survives in } H] \leq n \cdot (4d^3)^w \cdot \left( \frac{1}{8d^3} \right)^w = \frac{n}{2^w} \]

\( \Rightarrow \) for \( w = \Omega(\log n) \),

\[ \Pr[ \exists \text{ surviving component of size } w \text{ in } H] \leq \frac{1}{n} \]

Component of size \( w \) in \( H \Rightarrow \)

Component of size \( \leq w \cdot d^2 \) in \( G \)

So unlikely to have any surviving component of size \( \Omega(d^2 \log n) \)