Lecture 19

Local Computation Algorithms:

Maximal Independent Set

Maximal Independent Set:

\[ \text{def } u \subseteq V \text{ is a "Maximal Independent Set" (MIS) if} \]

1. \( \forall u_1, u_2 \in U, \ (u_1, u_2) \notin E \)
2. \( \exists w \in V \setminus U \text{ s.t. } U \cup \{w\} \text{ is independent} \)

Today's assumption:

\( G \) has max degree \( d \)

Note: MIS can be solved via greedy (not NP-complete)
Distributed Algorithm for MIS:

"Luby's Algorithm" (actually a variant)

- MIS $\leftarrow \emptyset$
- all nodes set to "live"

repeat $K$ times in parallel:

- A nodes $V$, color self "red" with prob $\geq \frac{1}{2d}$, else "blue". Send color to all nbrs.

- If $v$ colors self "red" & no other nbr of $v$ colors self red then
  - add $v$ to MIS
  - remove $v$ & all nbrs from graph (set to "dead")

(for purposes of analyses, continue to select selves after die, but don't send color to nbors)

Thm \[ Pr \left[ \text{# phases till graph empty} \geq 8d\log n \right] \leq \frac{1}{n} \]

Corr \[ E[\text{# phases}] \text{ is } O(d \log n) \quad \Leftarrow \text{can improve!} \]
Main Lemma

\[
\Pr[v \text{ live } \text{ and added to MIS in round}] = \frac{1}{yd}
\]

Proof

\[
\Pr[v \text{ colors self red}] = \frac{1}{2d}
\]

\[
\Pr[\text{any } w \in N(v) \text{ colors self red}] \leq \sum_{w \in N(v)} \frac{1}{2d} \leq \frac{1}{2}
\]

\[
\therefore \Pr[v \text{ colors self red and no other nbr colors self red}] = \frac{1}{2d} \left( 1 - \frac{1}{2} \right) = \frac{1}{4d}
\]

\[
\Rightarrow \quad \Pr[v \text{ live after } 4kd \text{ rounds}] \leq \left( 1 - \frac{1}{4d} \right)^{4kd} = e^{-k}
\]

Setting K:

if \( K = O(d \cdot \log n) \), \( \Pr[v \text{ live at end}] = e^{-O(\log n)} = \frac{1}{n^c} \)

(can do better)
See slides for Local Computation Algorithm (LCA) model.

Problem when sequentially simulate k-round algorithm get $d^k$ complexity.

If $k = O(\log n)$ then not sublinear.

What to do? Run fewer rounds, many nodes will not be decided yet. Is it ok?
Local Computation Algorithm to compute Luby's answer: "Luby status"

- Run Luby with $K = O(d \log d)$ rounds.

At the end, each node $v$ is one of:

- Live in MIS
- Not in MIS
- Taken out by nbr

Use "Parnas-Ron" reduction:

Simulate $v$'s view of computation in sequential manner:

- If $v$ is in/not in then done

else $v$ is alive $\leftarrow$ What do we do?
Questions:

What is \( \Pr[v \text{ is alive}] \)?

How are live nodes distributed after \( O(d \log d) \) rounds?

lots of live, few dead?

NO!

\[ \Pr[v \text{ survives } O(d \log d) \text{ rounds}] \leq e^{-O(d \log d)} \leq \frac{1}{d^2} \]

Most will "die"

\[ \text{don't worry it won't be painful!} \]
surviving nodes will be in small connected components

few live but clumped together?

NO!
surviving nodes will be in small connected components.

This relies heavily on degree bound of graph

- # conn subgraphs small
- survival of components independent
"Luby status" Luby with $k = O(d \log d)$:

- Given $v$, is it:
  - Live in MIS $\leftarrow$ set self to red + no nbrs did
  - Not in MIS $\leftarrow$ taken out by nbr

Luby:
- MIS $\leftarrow \emptyset$
- All nodes set to "live"
- Repeat $K$ times in parallel:
  - A nodes $v$, color self "red" with prob $\geq \frac{1}{2d}$; else "blue". Send color to all nbrs.
  - If $v$ colors self "red" & no other nbr of $v$ colors self red then
    - Add $v$ to MIS
    - Remove $v$ & all nbrs from graph (set to "dead")

LCA for MIS($v$):
- Run sequential version of Luby status ($v$)
- If it is input/output: output answer & halt
- Else, (1) do BFS to find $v$'s connected component of live nodes
  (2) Compute lexicographically 1st MIS $M'$ for that connected component
  (3) Output whether $v$ in/out of $M'$

Runtime:
- $O(d \log d)$
- $O(d \log d) \times$ size of component

What is size of component?
Bounding size of connected components:

Claim: After $O(d \log d)$ rounds, connected components of survivors of size $\leq O(p \log d \cdot \log n)$ can find whole component via BFS "brute force"

Main difficulty: survival of $v$ and neighbors are not independent
Bounding survivors:

\[ A_v = \begin{cases} 1 & \text{if } v \text{ survives all rounds} \\ 0 & \text{otherwise} \end{cases} \]

\[ B_v = \begin{cases} 1 & \text{if } \text{no } w \in N(v) \text{ colors self} \\ 0 & \text{otherwise} \end{cases} \]

Note: \( A_v = 1 \Rightarrow B_v = 1 \)

\[ \Pr[B_v=1] \leq \left(1 - \frac{1}{4d}\right)^c \cdot \log d \leq \frac{1}{8d^3} \text{ for } c \geq 20 \]

Luby:

- \( M_S \leftarrow \text{all nodes set to } \text{"live"} \)
- Repeat \( k \) times in parallel:
  - If nodes \( v \), color self \("red"\) with prob \( \geq \frac{1}{2d} \), else \("blue"\). Send color to all nbrs.
  - If \( v \) colors self \("red"\) & no other nbr of \( v \) colors self \("red"\), then:
    - Add \( v \) to \( M_S \)
    - Remove \( v \) + all nbrs from graph (set to \("dead"\)).

These events are independent.
Bounding size of connected components:

\[ A_v = \begin{cases} 1 & \text{if } v \text{ survives all rounds} \\ 0 & \text{otherwise} \end{cases} \]

\[ B_v = \begin{cases} 1 & \text{if } \exists \text{ round s.t. } v \text{ colors self} + \text{ no } w \in (N(v)) \text{ colors self} \\ 0 & \text{otherwise} \end{cases} \]

Note: \( A_v = 1 \Rightarrow B_v = 1 \)

e.g., \( v \) survives \( \Rightarrow \exists \text{ round s.t. } v \text{ colors self} + \text{ no } w \in (N(v)) \text{ colors self} \)

we care about \( A_v \)'s, but \( B_v \)'s have nice independence properties

\[ Luby: \]
\[ \text{MIS } \iff \]
\[ \text{all nodes set to "live"} \]
\[ \text{repeat } K \text{ times in parallel:} \]
\[ \text{if nodes } v, \text{ color self "red" with prob } \frac{1}{2d}, \text{ else "blue". Send color to all nbrs} \]
\[ \text{if } v \text{ colors self "red" + no other nbr of } v \text{ colors self red} \]
\[ \text{add } v \text{ to MIS} \]
\[ \text{remove } v + \text{ all nbrs from graph (set to "dead")} \]

might not mean that \( v \in \text{MIS} \)

\( e.g. \) if \( v \) died due to nbr being put in MIS

distance 3:
\[ B_v \text{ depends on } B_w \]
\[ B_w \text{ depends on } B_z \]
\[ \text{but } B_w + B_v \text{ independent} \]

\[ \text{deg} \leq d \Rightarrow \text{ each } B_w \text{ depends on } \leq d^2 \]
\[ \text{other } B_w \text{'s} \]
Bounding size of connected components:

Claim: After $O(d \log d)$ rounds, connected components of survivors of size $\leq O(p \pi d \log \log n)$ can find whole component via BFS "brute force".

Proof idea:

- Any large connected component has lots of nodes that are independent (distance $\geq 3$).
- These independent nodes are unlikely to simultaneously survive.

\[ \exists \text{ do we need to union bound over all sets of size } \frac{p \pi d \log \log n}{n} \]
Claim: After $O(d \log d)$ rounds, connected components of survivors of size $O(p^{\text{poly} \log d} \log n)$

Proof:

Let $H \leftarrow$ graph st. nodes $\sim B_v$
edges $\sim B_v \cup B_w$ distance 3 in $G$
represent independent events!!

$\deg(H) \leq d^3$

Observe: # components in $H$
of size $w$
$\leq$
# size $w$ subtrees in $H$

Why? map each component $C$
to arbitrary spanning tree
of $C$

not even connected!

Great! we are good at
country trees!!

mapping is 1-1
but could have many spanning trees per component
How many subtrees in a degree bounded graph?

Known Thm: # non-isomorphic trees on w nodes ≤ 4^w

Corr: # size w subtrees in N-node graph of degree ≤ D is ≤ N \cdot 4^w \cdot D^w = N (4D)^w

Why?
- Choose Root in H
- Choose size w tree (shape) from known thm
- Choose placement in H

Total # choices: N \cdot 4^w \cdot D^w
Claim: After $O(d \log d)$ rounds, connected components of survivors of size $\leq O(poly \log d \cdot \log n)$

Proof:
1. Let $H \leftarrow$ graph s.t. nodes $\sim B_v$
   - $\deg(H) \leq d^3$
   - $\deg(H) \leq d^3$

2. $\#$ components in $H \leq \#$ size $w$ subtrees in $H$ of size $w$ $\leq \#$ size $w$ subtrees in $H$ $\leq n \cdot (4d^3)^w$

3. $\Pr[\text{node } u \text{ survives}] \leq \frac{1}{8d^3}$

4. $\Pr[\text{component of size } w \text{ in } H \text{ survives}] \leq \left(\frac{1}{8d^3}\right)^w \leftarrow \text{since independent!}$

5. $\Pr[\text{any component of size } w \text{ survives in } H] \leq n \cdot (4d^3)^w = \frac{n}{(8d^3)^w}$

6. $w = \Omega(\log n)$,

   \[ \Pr[\exists \text{ surviving component of size } w \text{ in } H] \leq \frac{1}{n} \]

7. Component of size $\leq w$ in $H$ $\Rightarrow$ Component of size $\leq w \cdot d^3$ in $G$

   So unlikely to have any surviving component of size $O(d^3 \log n)$