# Lecture 19

## Local Computation Algorithms:

### Maximal Independent Set

**Maximal Independent Set:**

not maximum

NP Complete

def $U \subseteq V$ is a "Maximal Independent Set" (MIS) if

(1) $\forall u_1, u_2 \in U$, $(u_1, u_2) \notin E$ "independent"

(2) $\nexists\, w \in V \setminus U$ s.t. $U \cup \{w\}$ is independent "maximal"

Today's assumption:

$G$ has max degree $d$

Note: MIS can be solved via greedy (not NP Complete)

# Distributed Algorithm for MIS:
## "Luby's Algorithm" (actually a variant)

**Maximal Independent Set:**

<u>def</u> $U \subseteq V$ is a "Maximal Independent Set" (MIS) if
(1) $\forall u_1, u_2 \in U$, $(u_1, u_2) \notin E$
(2) $\nexists w \in V \setminus U$ s.t. $U \cup \{w\}$ is independent

- MIS $\leftarrow \varphi$
- all nodes set to "live"
- repeat K times in parallel:
  - $\forall$ nodes v, color self "red" with prob $\geq \frac{1}{2d}$, else "blue". Send color to all nbrs.
  - If v colors self "red" + no other nbr of v colors self red then
    - add v to MIS
    - remove v + all nbrs from graph (set to "dead")

K rounds

(for purposes of analyses, continue to select selves after die, but don't send color to nbrs)

<u>Thm</u> $\Pr[\#\text{ phases til graph empty} \geq 8d\log n] \leq \frac{1}{n}$

<u>Corr</u> $E[\#\text{ phases}]$ is $O(d \log n)$ $\Leftarrow$ can improve!

- $MIS \leftarrow \varphi$
- all nodes set to "live"
- repeat $K$ times in parallel:
  - $\forall$ nodes $v$, color self "red" with prob $\geq \frac{1}{2d}$, else "blue". Send color to all nbrs.
  - If $v$ colors self "red" & no other nbr of $v$ colors self red then
    - add $v$ to MIS
    - remove $v$ & all nbrs from graph (set to "dead")

# Main Lemma

$$\Pr[v \text{ live } \& \text{ added to MIS in round}] \geq \frac{1}{4d}$$

## Proof

$$\Pr[v \text{ colors self red}] = \frac{1}{2d}$$

$$\Pr[\text{any } w \in N(v) \text{ colors self red}] \leq \sum_{w \in N(v)} \frac{1}{2d}$$ (union bnd)

$$\leq \frac{1}{2}$$ (bound on degree)

$$\therefore \Pr[v \text{ colors self red } \& \text{ no other nbr colors self red}] \geq \frac{1}{2d}\left(1 - \frac{1}{2}\right) = \frac{1}{4d} \qquad \blacksquare$$

$\Rightarrow$ **Corr** $\Pr[v \text{ live after } 4Kd \text{ rounds}] \leq \left(1 - \frac{1}{4d}\right)^{4Kd} = e^{-K}$

**Setting K:**

if $K = O(d \log n)$, $\Pr[v \text{ live at end}] \leq e^{-O(\log n)} = \frac{1}{n^c}$

(can do better)

See slides for

Local Computation Algorithm (LCA)

model

Problem when sequentially simulate $k$-round algorithm

get $d^k$ complexity

$k = O(\log n)$ $\Rightarrow$ not sublinear

What to do? run fewer rounds

many nodes will not be decided yet ← is it ok?

# Local Computation Algorithm to compute Luby's answer:

Luby:

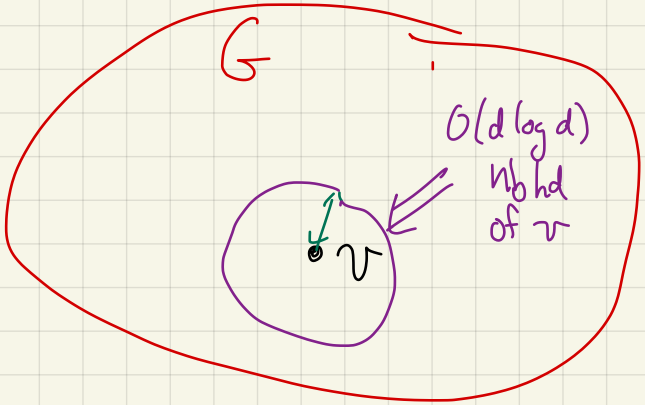
- MIS $\leftarrow \varphi$
- all nodes set to "live"
- repeat $K$ times in parallel:
  - $\forall$ nodes $v$, color self "red" with prob $\geq \frac{1}{2d}$, else "blue". Send color to all nbrs.
  - If $v$ colors self "red" + no other nbr of $v$ colors self red then
    - add $v$ to MIS
    - remove $v$ + all nbrs from graph (set to "dead")

"Luby status"

- Run Luby with $K = O(d \log d)$ rounds

at end, each node $v$ is one of:

- live
- in MIS $\leftarrow$ set self to red + no nbrs did
- not in MIS $\leftarrow$ taken out by nbr




- Use "Parnas - Ron" reduction:

simulate $v$'s view of computation in sequential manner: $d^K = d^{O(d \log d)}$ queries

+ determine whether $v$ is live/in/not in

degree

# rounds

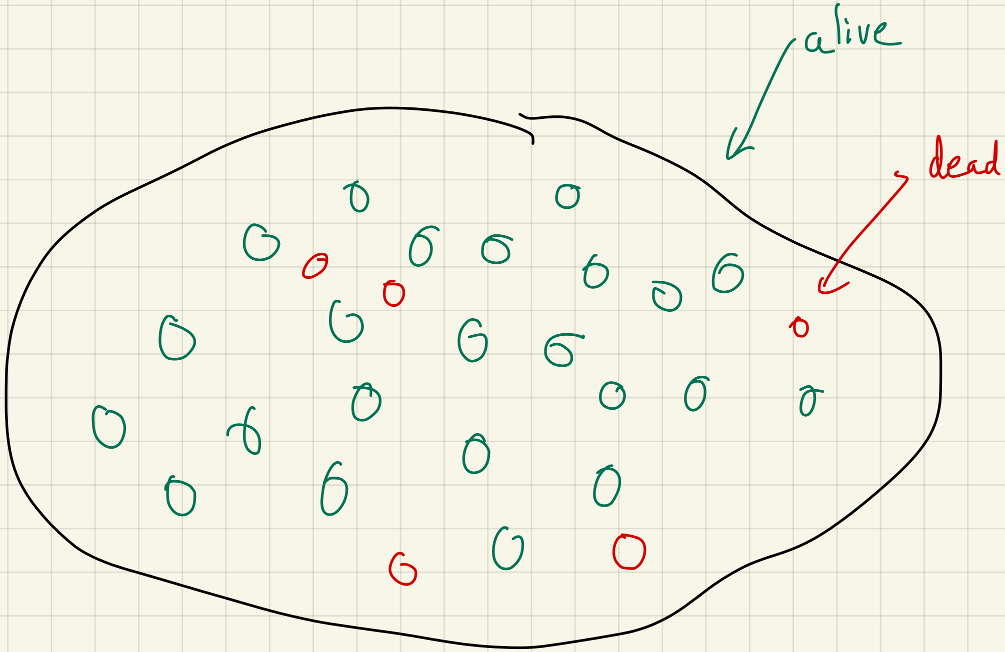
- if $v$ is in/not in then done

else $v$ is alive $\leftarrow$ What do we do?

Questions:

What is prob $v$ is alive?

How are live nodes distributed after $O(d \log d)$ rounds?

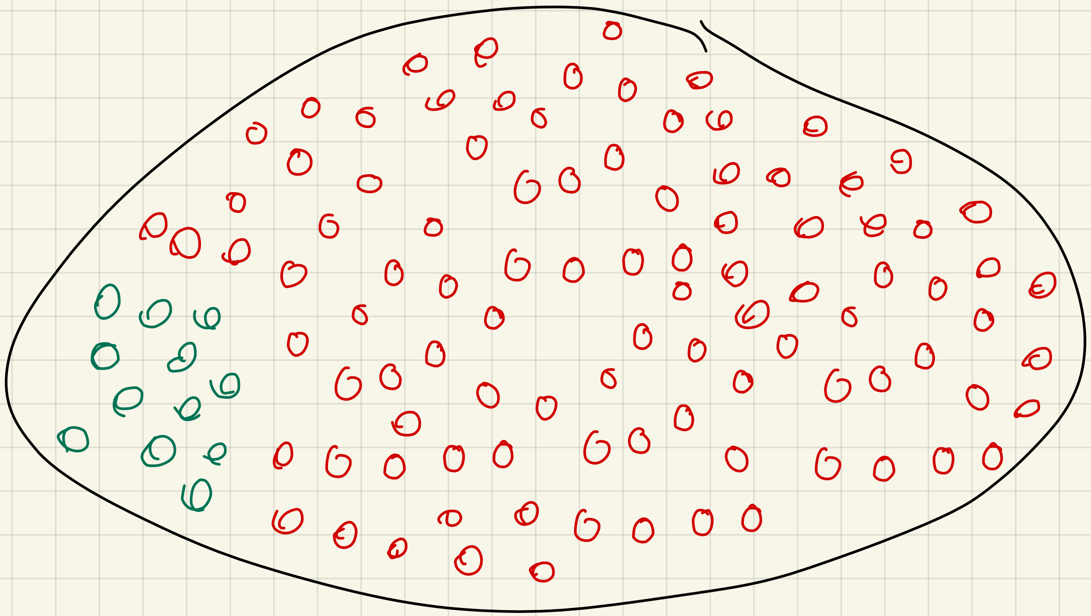



lots of live, few dead?

NO!

$$\Pr[v \text{ survives } O(d \log d) \text{ rounds}] \le e^{-O(\log d)} \le \frac{1}{d^c}$$

Most will "die" ← don't worry it won't be painful!

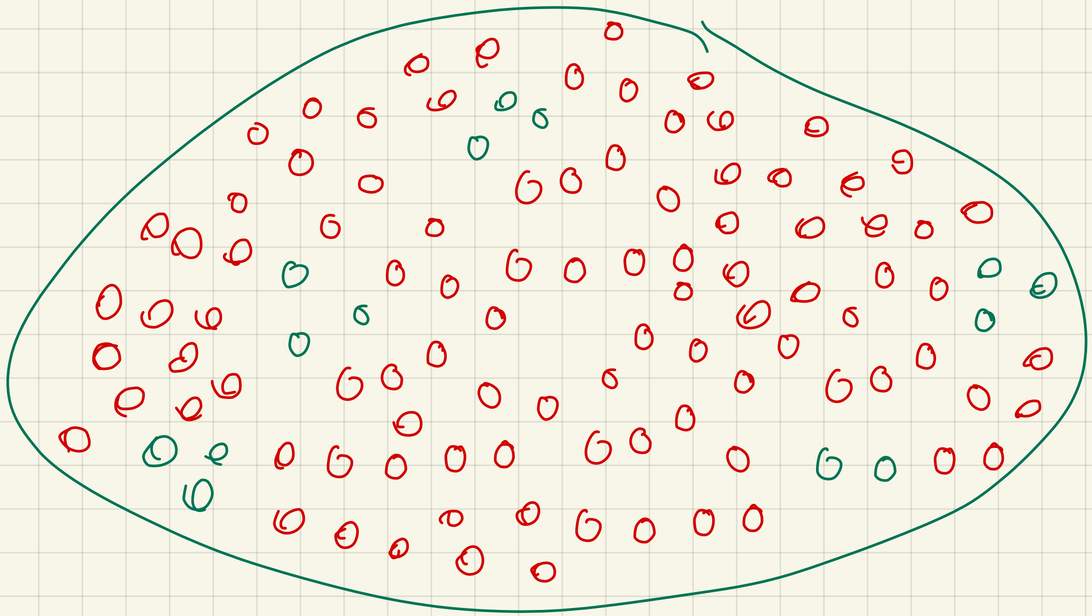


few live but clumped together?

NO!

surviving nodes will be in small connected components

Surviving nodes will be in small connected components



This relies heavily on degree bound of graph

- # conn subgraphs small
- survival of components ≈ independent

"Luby status" Luby with $k = O(d \log d)$:

given $v$, is it:

- live
- in MIS ← set self to red + no nbrs did
- not in MIS ← taken out by nbr

Luby:

- MIS ← $\varphi$
- all nodes set to "live"
- repeat $k$ times in parallel:
  - $\forall$ nodes $v$, color self "red" with prob $\geq \frac{1}{2d}$, else "blue". Send color to all nbrs.
  - If $v$ colors self "red" + no other nbr of $v$ colors self red then
    - add $v$ to MIS
    - remove $v$ + all nbrs from graph (set to "dead")

# LCA for MIS($v$):

Runtime

- run sequential version of Luby status ($v$) — $d^{O(d \log d)}$
- if it is in/out output answer + halt
- else, (1) do BFS to find $v$'s connected component of live nodes — $d^{O(d \log d)} \times$ size of component

  (2) Compute lexicographically 1st MIS $M'$ for that connected component — size of component

  (3) Output whether $v$ in/out of $M'$

what is size of component?

Bounding size of connected components:

Claim After $O(d \log d)$ rounds, connected components of survivors of size $\leq O(\mathrm{poly}\log(d) \cdot \log n)$

$\Longrightarrow$ can find whole component via BFS

"brute force"

Main difficulty: survival of $v$ & neighbors are not independent

## Bounding survivors:

$$A_v = \begin{cases} 1 & \text{if } v \text{ survives all rounds} \\ 0 & \text{o.w.} \end{cases}$$

$$B_v = \begin{cases} 1 & \text{if } \nexists \text{ round s.t. } v \text{ colors self \& no } w \in N(v) \text{ colors self} \\ 0 & \text{o.w.} \end{cases}$$

might not mean that $v \in MIS$

e.g. if $v$ died due to nbr being put in MIS

**Luby:**

- $MIS \leftarrow \varphi$
- all nodes set to "live"
- repeat $K$ times in parallel:
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    - add $v$ to MIS
    - remove $v$ & all nbrs from graph (set to "dead")

Note: $A_v = 1 \Rightarrow B_v = 1$

e.g. $v$ survives $\Rightarrow \nexists$ round s.t. $v$ colors self & no $w \in N(v)$ colors self

these events are independent

$$\Pr[B_v = 1] \leq \underbrace{\left(1 - \frac{1}{4d}\right)}_{\text{prob survive one round}}^{c \cdot d \log d} \leq \frac{1}{8d^3} \quad \text{for} \quad c \geq 20$$

Bounding size of connected components:

$$A_v = \begin{cases} 1 & \text{if } v \text{ survives all rounds} \\ 0 & \text{o.w.} \end{cases}$$

$$B_v = \begin{cases} 1 & \text{if } \nexists \text{ round s.t. } v \text{ colors self + no } w \in N(v) \text{ colors self} \\ 0 & \text{o.w.} \end{cases}$$

might not mean that $v \in MIS$

e.g. if $v$ died due to nbr being put in MIS

Luby:

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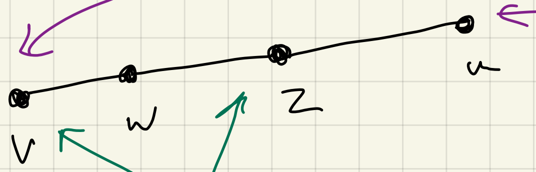
Note: $A_v = 1 \Rightarrow B_v = 1$

eg. $v$ survives $\Rightarrow \nexists$ round s.t. $v$ colors self + no $w \in N(v)$ colors self

we care about $A_v$'s, but $B_v$'s have nice independence properties




distance 3:

$B_v$ depends on $B_w$

$B_u$ depends on $B_z$

but $B_u + B_v$ indep!

distance 2:

$B_v + B_z$ depend on $w$'s colors so not independent

$\deg \leq d \Rightarrow$ each $B_u$ depends on $\leq d^2$ other $B_w$'s

Bounding size of connected components:

**Claim** After $O(d \log d)$ rounds, connected components of survivors of size $\le O(\text{poly} \log(d) \cdot \log n)$

$\Longrightarrow$ can find whole component via BFS "brute force"

**Proof idea:**

- any large conn component has lots of nodes that are independent (distance $\ge 3$)
- these indep nodes unlikely to simultaneously survive

do we need to union bnd over all sets of size w?

NO

Claim After $O(d \log d)$ rounds, connected components of survivors of size $\le O(\mathrm{poly}\log(d) \cdot \log n)$

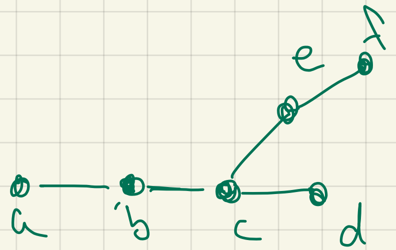
Proof

Let $H \leftarrow$ graph s.t. nodes $\sim B_v$

edges $\sim B_v$ & $B_w$ $\underbrace{\text{distance } 3}$ in $G$

represent independent events!!

$\deg(H) \le d^3$

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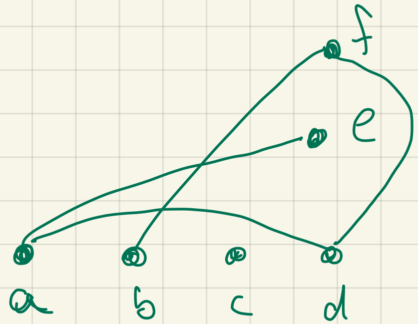


H

not even connected!

Great! we are good at counting trees!! $\Longrightarrow$

Observe: # components in $H$ of size $w$ $\le$ # size $w$ subtrees in $H$

why? map each component $C$ to arbitrary spanning tree of $C$

mapping is 1-1 but could have many spanning trees per component

How many subtrees in a degree bounded graph?

**Known Thm** # non isomorphic trees on $w$ nodes $\leq 4^w$ ← ignores names of nodes & root (just shape)

**Corr** # size $w$ subtrees in $N$-node graph of degree $\leq D$ is $\leq N \cdot 4^w \cdot D^w = N(4D)^w$ ← considers names of nodes & root

why?

- choose Root in $H$
- choose size $w$ tree (shape) from known thm
- choose placement in $H$

**choices**

$N$

$4^w$

$D$ choices for $1^{st}$ child

" " " $2^{nd}$ "

⋮

total # choices: $N \cdot 4^w \cdot D^w$

**Claim** After $O(d \log d)$ rounds, connected components of survivors of size $\le O(\mathrm{poly} \log(d) \cdot \log n)$

**Proof**

- Let $H \leftarrow$ graph s.t. nodes $\sim B_v$
  edges $\sim B_v$ & $B_w$ distance 3 in $G$
  represent independent events!!
- $\deg(H) \le d^3$
- # components in $H$ of size $w$ $\le$ # size $w$ subtrees in $H$ $\le n \cdot (4d^3)^w$
- $\Pr[\text{node } u \text{ survives}] \le \frac{1}{8d^3}$

$\Pr[\text{component of size } w \text{ in } H \text{ survives}] \le \left(\frac{1}{8d^3}\right)^w$ ← since independent!

$\Pr[\text{any component of size } w \text{ survives in } H] \le \frac{n(4d^3)^w}{(8d^3)^w} = \frac{n}{2^w}$

$\Rightarrow$ for $w = \Omega(\log n)$,

$\Pr[\exists \text{ surviving component of size } w \text{ in } H] \le \frac{1}{n}$

what about $G$?

Component of size $\le w$ in $H$ $\Rightarrow$
Component of size $\le w \cdot d^3$ in $G$

So unlikely to have any surviving component of size $\Omega(d^3 \log n)$ ☒