LCAs for spanners

Graph Spanners:

Given $G=(V,E)$

• same nodes
• $E' \subseteq E$

<u>def.</u> k-spanner is subgraph $H=(V,E')$ s.t.

$\forall u,v \ \ dist_H(u,v) \le k \cdot dist_G(u,v)$

<u>Known</u> $\forall\, G,\ \exists\ (2k-1)$-spanner with $O(n^{1+\frac{1}{k}})$ edges } e.g. $k=2$ $\exists$ 3-spanner with $O(n^{3/2})$ edges

Optimal? yes for $k=2,3,5$
Erdos girth conjecture $\Rightarrow$ yes for all k

Equivalent Characterization:

$\forall\ (u,v) \notin H$, $\exists$ path from u to v in H of length $\le k$ } So, whenever we omit an edge (u,v) we will make sure a path of length $\le k$ remains between u + v

<u>Question</u>: LCA which given graph G provides queries to spanner H?

<u>How is G given?</u> Assume following probes:

neighbor: given u, i output i^{th} nbr of u
adjacency: given (u,v) output whether $(u,v) \in G$
or
output j if $(u,v) \in G$ + "no" if $(u,v) \notin G$

j: index to edge (u,v)

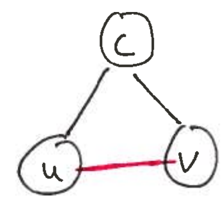
degree: given u output $deg(u)$

Today's goal:

LCA for 3-spanner with $\tilde{O}(n^{3/2})$ edges + $O(n^{3/4})$ time/query:

First, a thought:

Pick centers randomly
if u,v both connected
to same center, can
delete edge (u,v)



$\text{dist}_G(u,v)=1$ but $\text{dist}_H(u,v)=2$ (ok, since k=3)

but: (1) will we delete enough edges this way?
2 Can we figure out that u,v connected to same center
in sublinear time?

Today: will assume max degree is $n^{3/4}$

- still nontrivial
- general case builds on ideas today

Global Construction of 3-spanners with $\tilde{O}(n^{3/2})$ edges [Baswana Sen 07]

$\Uparrow$ note: ave degree $\approx n^{1/2}$

Construction of H: (not sublinear time)

- Pick $S \subseteq V$ s.t. $|S| = \Theta(\sqrt{n} \cdot \log n)$ ← each node tosses coin with prob $\Theta\left(\frac{\log n}{\sqrt{n}}\right)$

  "cluster centers" ⇐ each one defines a "cluster"

- main insight ⇒ $\boxed{\text{whp, } \forall\, u \in V \text{ s.t. } u \text{ has degree } \geq \sqrt{n}, \text{ then } u \text{ adjacent to at least one center } v \in S}$ useful observation *

  u chooses one $v \in S$ (arbitrarily?) ← or lowest id #? to be its "cluster center"

- Constructing H:

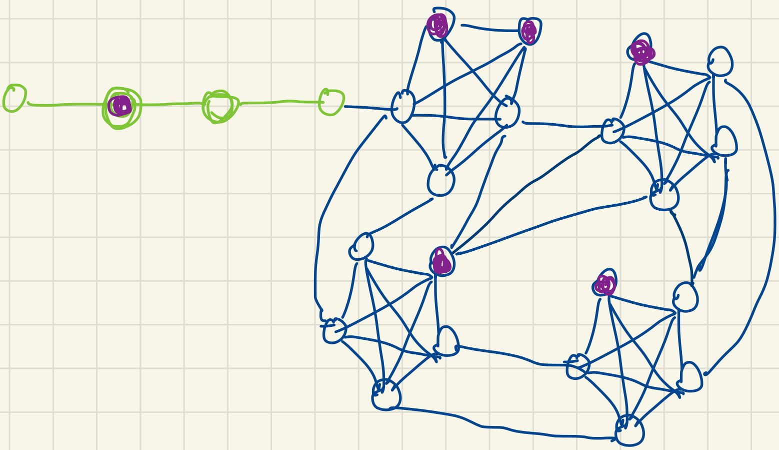
#edges	
$< n \cdot \sqrt{n}$	(1) if u low degree ($< \sqrt{n}$), add all edges (u,v)
$< n \cdot 1$	(2) if u high degree ($\geq \sqrt{n}$), add edge to its cluster center
$< n \cdot \underbrace{\sqrt{n} \cdot \log n}_{\#\text{ clusters}}$	(3) if u high degree ($\geq \sqrt{n}$), add <u>one</u> edge to every adjacent cluster
$\tilde{O}(n^{3/2})$ total	

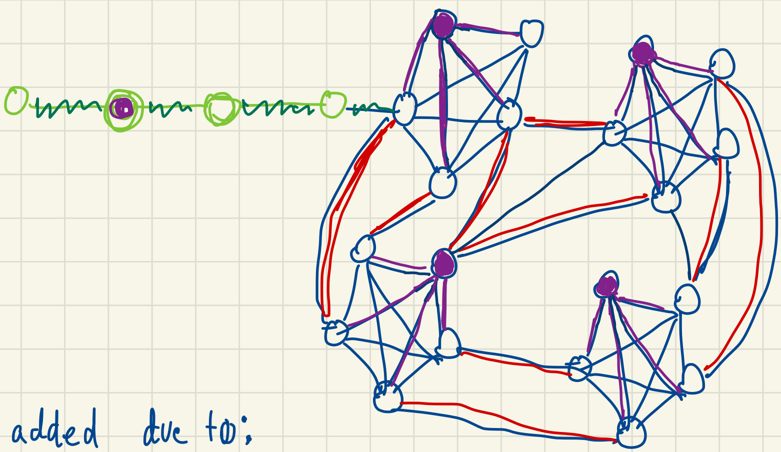
Example:

low degree

high degree

cluster center





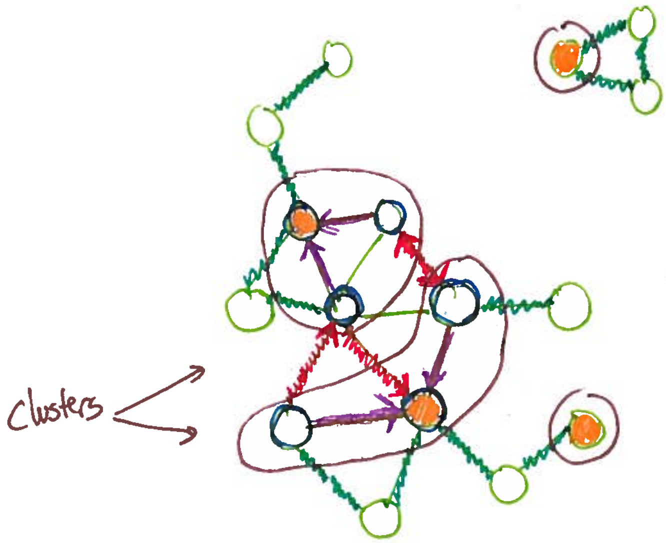
edges added due to:

rule 1: low degree

rule 2: cluster center

rule 3: adjacent cluster

Example:



low degree
high degree
cluster center

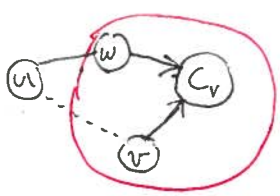
edges added due to

rule 1: low degree
rule 2: cluster center
rule 3: adjacent cluster

(directed edges only indicate who made the choice, actual edges are all undirected)

Stretch?

- for u, v in same cluster, both $u + v$ keep edge to center c $\Rightarrow \text{dist}_H(u,v) = 2$
- for u, v in different clusters:



if $(u,v) \notin H$ then must have kept some other edge (u,w) s.t. w in v's cluster.

So either $w = c_v$ or $(w, c_v) \in H$

$\Rightarrow (u,w)\,(w, c_v)\,(c_v, v) \in H$

$\Rightarrow \text{dist}_H(u, v) = 3$

Local Algorithm for constructing H:

given $(u,v) \in G$, is $(u,v) \in H$?

Rule (1): if u or v low degree, yes! 2 degree probes ✓

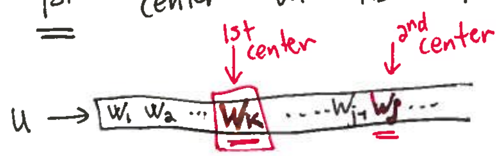
Rule (2): if v is u's center (or if u is v's center)

Rule (3): if (u,v) is "chosen" edge from u to v's cluster (or v to u's cluster)

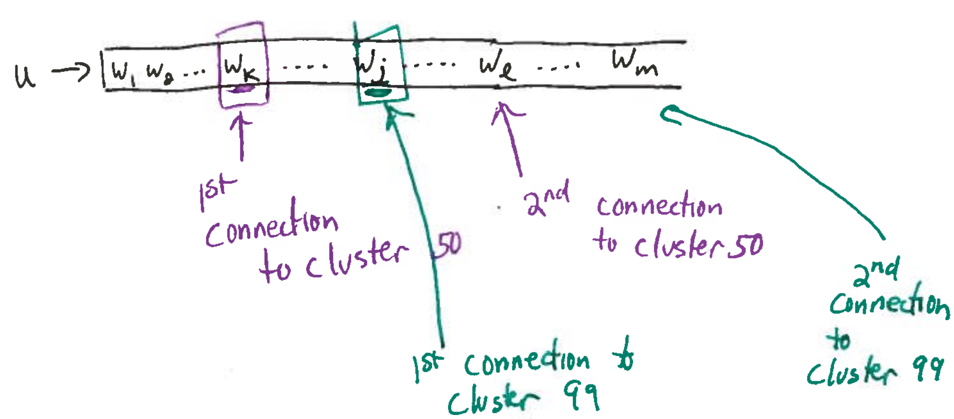
How do we know?

Naive idea: "First center Attempt"

u chooses <u>1st</u> center on its incidence list



u chooses <u>1st</u> connection to each cluster in ~~in~~cidence list



Implementing rule 2:

on query (u,v): if v the chosen center of u?

need to also check if u is chosen center of v

- check if v is a center
  (check v's coin toss)
- check if any node preceding v on u's incidence list is a center

runtime: $O(\text{max degree})$

better runtime: $O(\sqrt{n})$ by observation *'

Implementing rule 3:

on query (u,v): does v introduce u to a new cluster?

- find v's cluster center c_v $O(\sqrt{n})$
- check all nbrs of c_v + check if come earlier in u's incidence list?

need to make sure c_v is w's center

$O(\Delta \cdot (1+\sqrt{n}))$

$\forall w \in N(c_v)$

adjacency probe (u,w) returns locn in u's list

not sublinear for $\Delta = \Omega(\sqrt{n})$ (regime of interest)

Improved Plan: "Multiple Centers"

Rule 2: u chooses <u>all</u> centers in first $\sqrt{n}$ locns of incidence list $C_u = \{ v \mid v$ is in 1st $\sqrt{n}$ locns of u's incidence list & v is a center$\}$

<u>Observation</u> WHP, $\forall$ u s.t. $deg(u) \geq \sqrt{n}$, $1 \leq |C_u| \leq \log n$ $\boxed{\text{check this!}}$

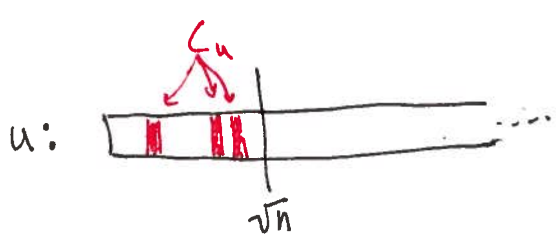
How does this change things?

- degree from rule 2 "keep all edges between u & C_u"
  is $O(\log n)$ per node
  $\Rightarrow O(n \log n)$ total [before it was $O(n)$ total]

- Verifying if $v \in C_u$:
  - adjacency probe (u,v) returns v's locn in u's list in one step
  - check if v is a center by looking at random ($)'s

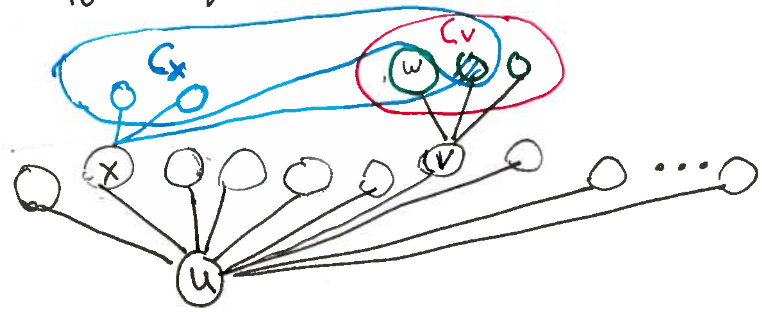
SAVINGS!! ⟹ 1 adjacency probe

- computing C_u:
  - check 1st $\sqrt{n}$ locns in u's list to see which are centers



$\sqrt{n}$ neighbor probes

Rule 3: u chooses first edge v which introduces u to v's cluster



How to determine?

- compute C_v — $\sqrt{n}$ neighbor probes
- for each $w \in C_v$, test if v "introduces" w to u:

1st attempt

For each nbr x of u until reach v: — $\deg(u)$

Find C_x — $\sqrt{n}$ nbr probes

cross off $C_x \cap C_v$ — $O(\log n)$

If any $w \in C_v$ not crossed off

then keep (u,v) in H

else discard (u,v)

Total: $\underbrace{O(\Delta \cdot \sqrt{n} \cdot \log n)}_{\text{bad!!}}$

Smarter method to determine if v "introduces" cluster to u:

- Compute C_v — $\sqrt{n}$ nbr probes
- For each nbr x of u up to v: — $\deg(u)$

  For each $w \in C_v$, — $O(\log n)$

  if w is center of x — 1 adjacency probe

  cross w off

If any $w \in C_v$ not crossed off

Keep (u,v) in H

else discard.

Total:

$\sqrt{n} + \deg(u) \times \log n \times 1$

$= O(\deg(u) \cdot \log n)$

If $\max_u \deg(u) \leq n^{3/4}$, we are done!!