Testing $\triangle$-freeness in dense graphs: a lower bound

superpoly dependence on ε is required!

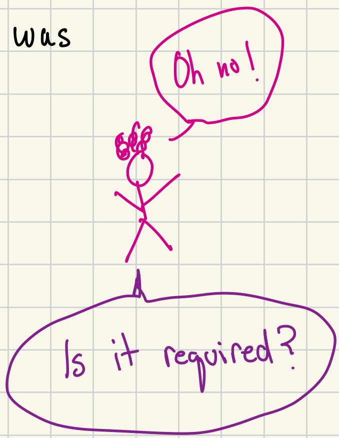
Last time:

Algorithm for testing Δ-freeness

with <u>no</u> dependence on n.

<u>but</u> dependence on ε was

$$\left.2^{2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}}\right\} \log 1/\varepsilon$$



Intriguing characterization of bipartite graphs:

Thm [Alon] In adjacency matrix model, complexity of 1-sided testing H-freeness property is

- if H bipartite: poly$(1/\varepsilon)$
- if H not bipartite: no poly$(1/\varepsilon)$ suffices

today will show for Δ's

More specifically:

Thm $\exists$ const c s.t. any 1-sided tester for whether graph G is Δ-free needs

$$\geq \left(\frac{c}{\varepsilon}\right)^{c \log 1/\varepsilon} \text{ queries}$$

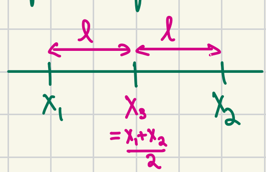
A first main tool:

Additive number theory lemma

<u>Lemma</u> $\forall m$, $\exists X \subset M = \{1,2,\dots m\}$

of size $\geq \dfrac{m}{e^{10\sqrt{\log m}}}$ with no nontrivial

solution to $\underbrace{x_1 + x_2 = 2x_3}_{\text{no 3 evenly spaced points}}$



will use X to construct graphs which are

(1) far from Δ-free

(2) any algorithm needs <u>lots</u> of queries to find Δ (in terms of ε)

examples:

Bad X: $\{1,2,3\}$

$\{5,9,13\}$

Good X:

how big? $\rightarrow$ $\{1,2,\cancel{3},4,5,\cancel{6},\cancel{7},\cancel{8},\cancel{9},10,\ldots\}$

only size $O(\log m)$ $\rightarrow$ $\{1,2,4,8,16,32,\ldots\}$

Proof of lemma:

Let d be integer (will set to $e^{10\sqrt{\log m}}$)

$$k \leftarrow \left\lfloor \frac{\log m}{\log d} \right\rfloor - 1 \quad \left(\text{so } k \approx \frac{\log m}{10\sqrt{\log m}} \approx \frac{\sqrt{\log m}}{10}\right)$$

$$\text{define } X_B := \Big\{ \underbrace{\sum_{i=0}^{k} x_i d^i}_{\substack{\text{view } x \in M \\ \text{in base } d \\ \text{representation} \\ x = (x_0 \cdots x_k)}} \;\Big|\; \underbrace{x_i < \frac{d}{2} \text{ for } 0 \le i \le k}_{\textcircled{1} \text{ no carries on sums!}} \;\&\; \underbrace{\sum_{i=0}^{k} \overbrace{x_i^2}^{\text{can be any convex fctn of } x_i\text{'s}} = B}_{\textcircled{2}} \Big\}$$

partitions the x's into groups

why the constraints?

① x_i's $< \frac{d}{2}$ $\Rightarrow$ summing pairs of elements in X_B doesn't generate a carry in any location

(we'll see why useful soon...)

② use (along with ①) to show "sum-free"

Claim $X_B \subseteq M$ (so X_B's partition M)

why? largest int in $X_B \le d^{k+1} \le d^{\lfloor \frac{\log m}{\log d} \rfloor - 1 + 1} \le d^{\log_d m}$

$= m^{\log_d d} = m$ ■

Pick B s.t. $|X_B|$ maximized: ave size of $X_B = \frac{|\bigcup_B X_B|}{\# B}$

$\underbrace{\text{how big can } B \text{ be?}}_{\substack{\text{equivalently,} \\ \text{how many } B\text{'s} \\ \text{are there?}}}$ $\underbrace{B \le (k+1)\left(\frac{d}{2}\right)^2 < k \cdot d^2}_{\text{using that } \sum x_i^2 = B}$

how small can $\sum |X_B|$ be?

$$\left|\bigcup_B X_B\right| \underset{\text{disjoint}}{=} \sum_B |X_B| \geq \left(\frac{d}{2}\right)^{k+1} > \left(\frac{d}{2}\right)^k$$

so $\exists$ B s.t. $|X_B| \geq \dfrac{\left(\frac{d}{2}\right)^k}{k \cdot d^2}$

using settings of $d, k,$ get

$$|X_B| \geq \frac{m}{e^{10\sqrt{\log m}}}$$

So if this B is also sum-free, we have the lemma!

(actually all X_B's are sum-free, so we will be good!)

<u>Claim</u> $\forall B$, X_B is sum-free

i.e. $\nexists\ x, y, z \in X_B$ st. $x + y = 2z$

Proof of Claim:

for $x, y, z \in X_B$:

$$x + y = 2 \cdot z \iff \sum_{i=0}^{k} x_i \cdot d^i + \sum_{i=0}^{k} y_i \cdot d^i = 2 \cdot \sum_{i=0}^{k} z_i \cdot d^i$$

$$\iff \left.\begin{array}{l} x_0 + y_0 = 2z_0 \\ x_1 + y_1 = 2z_1 \\ \vdots \\ x_k + y_k = 2z_k \end{array}\right\} \text{ since no carries}$$

this is where we use constraint 1

<u>note</u> $\forall i \quad x_i + y_i = 2z_i$

$\implies \forall i \quad x_i^2 + y_i^2 \geq 2z_i^2$ with equality only if $x_i = y_i = z_i$

this is where we use constraint 2

why? $f(a) = a^2$ is convex

using Jensen's ≠:

$\frac{1}{2}(f(a_1) + f(a_2)) \geq f\left(\frac{a_1 + a_2}{2}\right)$ with equality only if $a_1 = a_2 = \frac{a_1 + a_2}{2}$

$\implies \frac{1}{2}(x_i^2 + y_i^2) \geq \left(\frac{2z_i}{2}\right)^2$ " " " " $x_i = y_i = 2z_i$

■ (proof of note)

So if $x, y, z \in X_B$ s.t. not $(x = y = z)$

then $\exists\, i$ s.t. not $(x_i = y_i = z_i)$

For this i: $\quad x_i^2 + y_i^2 > 2z_i^2$

For all other j: $\quad x_j^2 + y_j^2 \geq 2z_j^2$

$$\text{So} \quad \underbrace{\sum_i x_i^2}_{=B} + \underbrace{\sum_i y_i^2}_{=B} > \underbrace{2 \cdot \sum z_i^2}_{=2B}$$

CONTRADICTION!

$\blacksquare$

So we have sum-free X_B s.t. $|X_B| \geq \frac{m}{e^{10\sqrt{\log m}}}$

for l.b. on Δ-testing, not enough, will need another idea,

(but won't do it here)

A second main tool:

Goldreich - Trevisan Thm: (home work)

Adj matrix model

Property P

Tester T with $q(n, \varepsilon)$ queries

possibly adaptive

$\Rightarrow$ Tester T': "Natural tester"

pick $q(n, \varepsilon)$ nodes

query submatrix

decide

$O(q^2)$ queries

nonadaptive

Consequences:

- l.b. for natural tester of $\Omega(q')$ queries
  $\Longrightarrow$ l.b. for <u>any</u> tester of
  $\Omega(\sqrt{q'})$ queries
- reduction preserves 1-sidedness, so l.b. implication does too.

In our case:

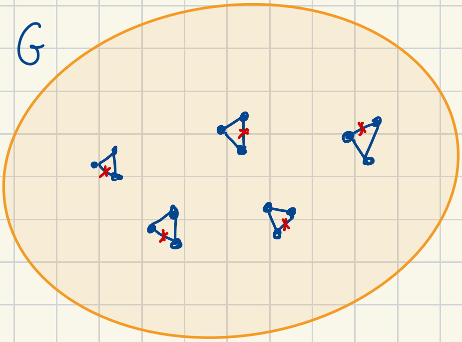
$$\text{prob}[\text{fail}] \approx \frac{\#\Delta\text{'s in } G}{\#\text{ node triples}}$$

we will calculate $\#\Delta$'s for our lower bound family of graphs

need to find a class of graphs on which natural tester doesn't find Δ (in $q \approx \left(\frac{1}{\varepsilon}\right)^{\log \frac{1}{\varepsilon}}$ queries) + distance is big ($> \varepsilon$)

A difficulty:

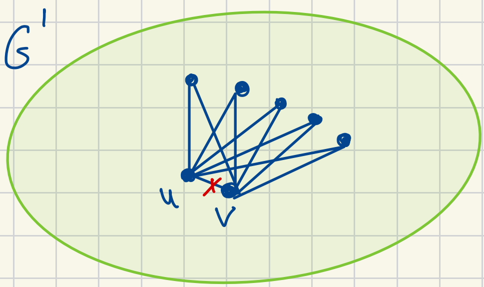
distance to Δ-free $\not\approx$ # Δ's



distance to Δ-free:
need to delete ≥ 1 edge in each Δ

$\Longrightarrow$ dist $\geq$ # disjoint Δ's

5 Δ's s.t. all Δ's disjoint

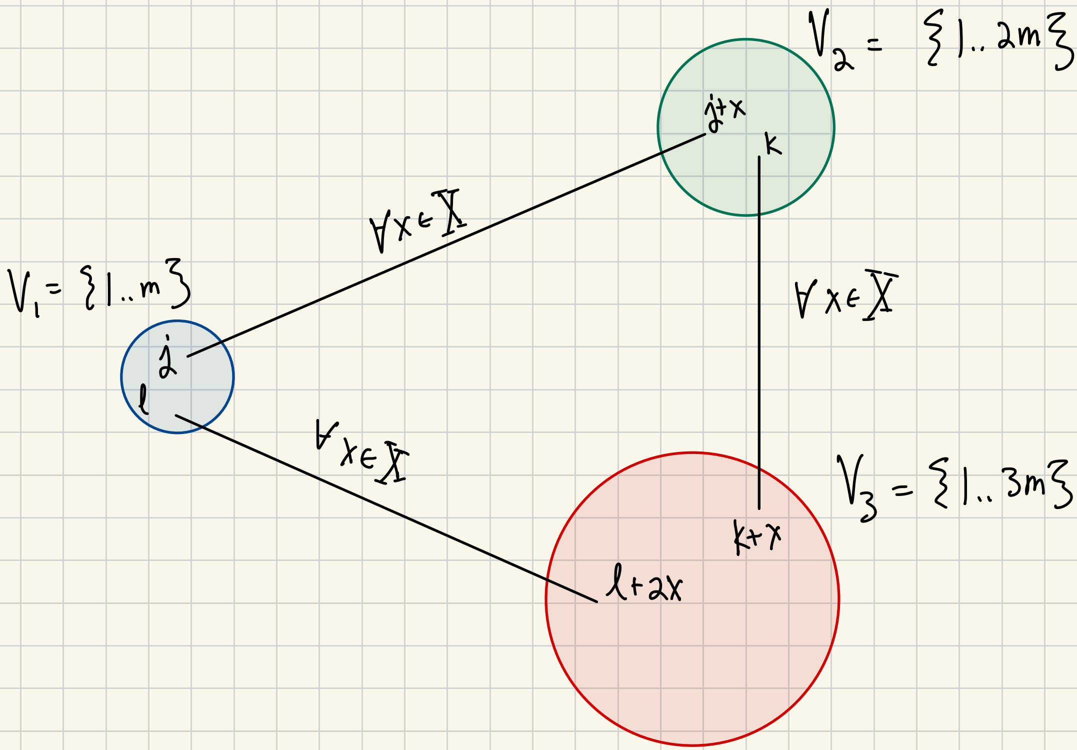


dist to Δ-free $= 1$

5 Δ's s.t. all Δ's share edge (u,v)

Graphs on which natural tester needs lots of queries:

given sum-free $X \subseteq \{1..m\}$



- will abuse notation:

  node should be (i, j)

  i in $\{1,2,3\}$

  $j \in \{1..m\}$, $\{1..2m\}$, $\{1..3m\}$

  will drop i if easy to see from context

- # nodes $= 6m$
- # edges $= \Theta(m \cdot |X|) = \Theta(n^2 / e^{10\sqrt{\lg n}})$ ← not exactly dense

- # Δ's : [important for determining detection probability of algorithm]

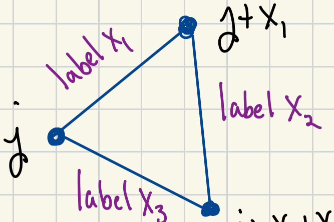
"intended": of form $j,\ j+x,\ j+2x$

$$m \cdot |X| = \Theta\left(n^2 / e^{10\sqrt{\lg n}}\right)$$

"unintended":

$V_1\, V_2\, V_3$ have no internal edges $\Rightarrow$

any Δ has $v_1 \in V_1$ $v_2 \in V_2$ $v_3 \in V_3$



$\left.\begin{array}{l} j+x_1+x_2 \\ = j+2x_3 \end{array}\right\} \Rightarrow x_1+x_2 = 2x_3$

$\Rightarrow \underbrace{x_1 = x_2 = x_3}_{\text{but these are intended!}}$ since X is sumfree

Conclusion:

no unintended Δ's

$\Pr[\text{pick } \Delta] \approx \left(n^2 / e^{10\sqrt{\lg n}}\right) / \binom{n}{3}$ ← small, but so is distance

- \# <u>disjoint</u> Δ's

[important for determining distance to Δ-free:

given set of disjoint Δ's, must remove ≥ 1 edge in <u>each</u>

$\Downarrow$

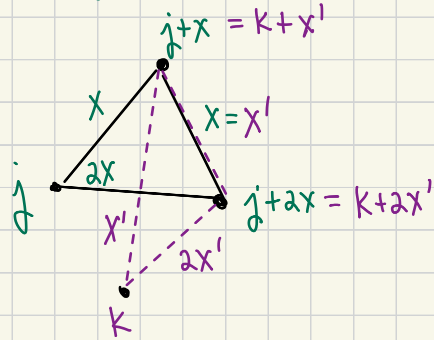
"absolute" distance from Δ-free $= \Omega(\#\text{ disjoint } \Delta\text{'s})$]

<u>Claim</u>: <u>All</u> intended Δ's are disjoint

(share no edges at all!)

suppose not:

(argument for why they can't share 2nd edge, similar argument holds for other edges)



since $x = x'$, $j = k$ $\rightarrow\leftarrow$

so distance to Δ-free $= \Theta(\#\Delta\text{'s}) = \Theta\left(\frac{n^2}{e^{10\sqrt{\log n}}}\right) = \Theta(m|X|)$

Problem need ε distance,

we only have $\Theta\left(\frac{1}{e^{10\sqrt{\log n}}}\right)$ distance

Idea for fix: "s - blow up"

$$G \Rightarrow G^{(s)}$$

$\sim m \cdot s$ node u in G $\Rightarrow$ size s independent set in $G^{(s)}$

$\sim m \cdot |X| \cdot s^2$ edge (u,v) in G $\Rightarrow$ complete bipartite graph in $G^{(s)}$

$\sim m \cdot |X| \cdot s^3$ Δ in G $\Rightarrow$ $\underbrace{s^3}_{\text{need to prove a lemma to show this}}$ Δ's in $G^{(s)}$ $\Leftarrow$ much more but no longer disjoint

how many edge disjoint Δ's?

Lemma absolute dist of $G^{(s)}$ from Δ-free $\geq$ # edge disjoint Δ's $\geq m \cdot |X| \cdot s^2$

Proof show Δ in $G \Rightarrow s^2$ disjoint Δ's in $G^{(s)}$

use $s \approx \frac{n}{6m} \approx n \cdot \left(\frac{\varepsilon}{c}\right)^{c \log \varepsilon/c}$

pick m largest int $\varepsilon \leq \frac{1}{e^{10\sqrt{\log m}}}$ so $m \geq \left(\frac{c}{\varepsilon}\right)^{c \log 1/\varepsilon}$

So <u>relative</u> distance of $G^{(s)}$ $\approx \frac{m|X|s^2}{(ms)^2}$ ← size of adj matrix

$$= \frac{|X|}{m} \geq \frac{1}{e^{10\sqrt{\log m}}} \geq \varepsilon$$

choice of m

$$\# \Delta\text{'s} \approx m \cdot |X| \cdot s^3$$

$$\approx \left(\frac{\varepsilon}{c'}\right)^{c' \log(c'/\varepsilon)} \cdot n^3$$

If take sample of size q + run natural algorithm

$$E[\# \Delta\text{'s in sample}] < \binom{q}{3} \left(\frac{\varepsilon}{c'}\right)^{c' \log(c'/\varepsilon)}$$

$$\ll 1 \quad \text{unless } q > \left(\frac{c''}{\varepsilon}\right)^{c'' \log(c''/\varepsilon)}$$

so by Markov's ≠, $\Pr[\Delta \text{ in sample}] \ll 1$

since 1-sided error, must see Δ in sample to fail

$\blacksquare$