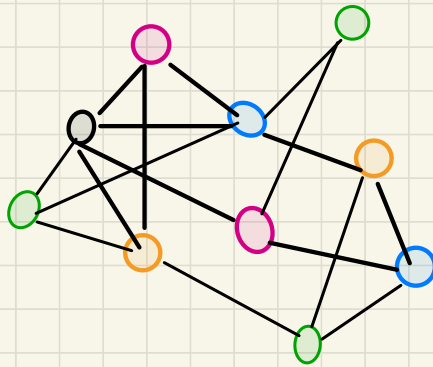
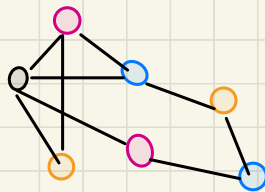


Lecture 3

Sublinear time algorithms
for coloring graphs



Graph Coloring



def a proper c -coloring of G
assigns color c_v from $\{1 \dots c\}$
to each $v \in V$ st, $\forall (u, v) \in E \quad c_u \neq c_v$

- in general, NP-complete
- important case solvable in linear time " $\Delta+1$ -coloring"

$$\begin{aligned} \text{max degree } \Delta \\ C = \Delta + 1 \end{aligned}$$

runtime:
 $O(n\Delta)$
 $= O(m)$

Greedy algorithm:

for each $v \in V$
assign c_v different from
all c_u for $u \in N(v)$

always exists since
 $\leq \Delta$ nbrs + can
choose from $\Delta+1$
colors

Can we do better?

Yes!

Thm [Assadi Chen Khanna] can $\Delta+1$ color in
time $\tilde{O}(n\sqrt{n})$ time.

This lecture: simplification of [ACK] by [Assadi Yazdanyar]
(slightly less efficient in log factors)

Note: if Δ is $\tilde{O}(\sqrt{n})$, greedy algorithm achieves thm.

Question: what to do if Δ is $\Omega(\sqrt{n})$?

new idea:

v picks random color $c \in 1 \dots (\Delta+1)$

$\rightarrow$ goes over all vtes colored c to check if
one is a nbr.

$\nwarrow$ how many vtes?

Thought experiment: if vtes randomly
colored (which they aren't) expect
 $n/\Delta = \tilde{O}(n/\sqrt{n})$

$\nwarrow$ since assumed $\Delta = \Omega(\sqrt{n})$

Greedy algorithm:

$\forall v$, check all nbrs + see which colors free

New algorithm:

$\forall v$, check all vts colored c + see if nbr

Greedy is good when "few" nbrs

New algorithm is good when "few" vts of
each color (equivalently, lots
of colors)

Algorithm:

1. Let $C_1 \dots C_{\Delta+1} = \emptyset$ Initialize.
invariant: $C_c = \text{vtes colored } c$

2. Let $v_1 \dots v_n$ be vtes in **RANDOM** order

For $i \leftarrow 1$ to n :

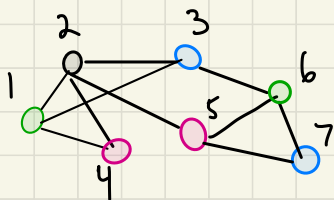
a) sample $c \in_u [\Delta+1]$

b) $\forall u \in C_c$ check if a nbr of v_i

c) if no nbrs in C_c ,
color v_i with c

update $C_c \leftarrow C_c \cup \{v_i\}$ + halt

else go to 2a + try again



When color node 6:

$C_{\text{green}} \{1\}$

$C_{\text{pink}} \{4, 5\}$

$C_{\text{blue}} \{3\}$

$C_{\text{black}} \{2\}$

Input representation

adjacency matrix

(need $O(1)$ query time in 2(b) - is $(u,v) \in E$?)

Behavior

terminates $\Rightarrow$ legal $\Delta+1$ -coloring

Runtime will show $O\left(\frac{\sqrt{n}}{\Delta} \log \Delta\right)$

let $X_{v_i} :=$ queries to color v_i

if v_i tries colors $c_1 \dots c_k$ before success,

$$X_{v_i} = \sum_{j=1}^k |C_{c_j}^{(i)}|$$

← at time v_i is colored

↑ can be ∞ , but unlikely

$$\leq \sum_{j=1}^k |C_{c_j}|$$

← at end

$$E[\text{Runtime of algorithm}] = E\left[O(1) \cdot \sum_i \chi_{v_i}\right]$$

← sum over all vertices

$$= O(1) \sum_i E[\chi_{v_i}]$$

linearity of expectation

bounding $E[\chi_{v_i}]$: First Fix ordering of v_i 's

Let $\deg^<(v_i) = \#$ of nbrs of v_i appearing before v_i in ordering

Claim 1 $E[\chi_{v_i}] \leq \frac{n}{\Delta+1 - \deg^<(v_i)}$

← this value depends on ordering of v_i 's

Warmups:

(i) $\Pr[\text{color} \in \text{"good" for } v_i]$

= prob not used by previously colored nbr.

$\leq \deg^<(v_i)$ of these

$$\geq 1 - \frac{\deg^<(v_i)}{\Delta+1}$$

(2) Expected # queries for single color in part (b):

randomly chosen
↓

fix v_i
+ only count one loop

$$E_{c \in [\Delta+1]} [\text{\# queries for } c] = \frac{1}{\Delta+1} \sum_{c=1}^{\Delta+1} |C_c|$$

at end,
 C_c 's partition V
so sum $\leq n$

$$\leq \frac{n}{\Delta+1}$$

Proof of Claim 1

$$E[X_v] = \sum_{c=1}^{\Delta+1} \Pr[\text{1st sampled color for } v \text{ was } c] \cdot E[X_v \mid \text{1st sampled color for } v \text{ was } c]$$

what is this?

X_v involves $|C_c|$ queries

with what probability are we in each case?

→ if c good for v then $X_v = |C_c| + \text{done!}$

→ else c bad:

$$E[X_v \mid \text{1st sampled color for } v \text{ is } c] = E[|C_c| + X_v] = |C_c| + E[X_v]$$

$B(v) \equiv$ bad colors for v

$$|B(v)| \leq \deg^-(v)$$

↖ back to beginning

$$E[X_v] = \sum_{c=1}^{\Delta+1} P[\text{1st sampled color for } v \text{ was } c] \cdot E[X_v \mid \text{1st sampled color for } v \text{ was } c]$$

$$= \sum_{c=1}^{\Delta+1} \frac{1}{\Delta+1} |C_c| + \sum_{c \in B(v)} \frac{1}{\Delta+1} E[X_v]$$

$$\leq \frac{n}{\Delta+1} + \frac{\deg^<(v)}{\Delta+1} E[X_v]$$

a worry:
not so great
if $\deg^<(v) = \Delta$

$$\Rightarrow E[X_v] \leq \frac{\frac{n}{\Delta+1}}{\frac{\Delta+1 - \deg^<(v)}{\Delta+1}} = \frac{n}{\Delta+1 - \deg^<(v)}$$

□

$$\text{So, } E[\text{total runtime}] = O(1) \cdot \sum_v E[X_v]$$

$$= O(1) \sum_v \frac{n}{\Delta+1 - \deg^<(v)}$$

depends on
random ordering

need to bound

This was analysis for fixed ordering Π

Observe :

if $\deg^<(v) \leq \frac{\Delta}{2}$ then random color c
is free with $\text{prob} \geq \frac{1}{2}$

if $\frac{\Delta}{2} \leq \deg^<(v) \leq \frac{3\Delta}{4}$ then c free
with $\text{prob} \geq \frac{1}{4}$

if $\frac{3\Delta}{4} \leq \deg^<(v) \leq \frac{7\Delta}{8}$ then c free with
 $\text{prob} \geq \frac{1}{8}$

still not so bad...

but what if $\deg^<(v) = \Delta$? that is pretty bad...
& it could happen at the end.

Idea it would be great if $\deg^<(v)$ random!

the ordering on vertices is randomly chosen

$\Rightarrow$ Claim rank of v among $v \cup N(v)$
is uniform in $[1, \dots, \deg(v)+1]$
 $\Rightarrow \deg^<(v)$ unif in $[0, \dots, \deg(v)]$

Now do analysis for random ordering π

$$E[\text{runtime}] = \sum_{\pi} E[\text{runtime} | \pi]$$

$$= O(1) \cdot \sum_v E_{\pi} \left[\underbrace{\frac{n}{\Delta+1 - \deg^{\leftarrow}(v)}}_{*} \right]$$

$$* = \sum_{d=0}^{\deg(v)} \frac{1}{\deg(v)+1} \cdot \frac{n}{\Delta+1-d} \leq \sum_{d=0}^{\Delta} \frac{1}{\Delta+1} \cdot \frac{n}{\Delta+1-d}$$

average over rank(v) in $v \cup N(v)$
 by claim
 why?

$$\text{Let } A = \underbrace{\left\{ \frac{n}{\Delta+1}, \frac{n}{\Delta+1-1}, \frac{n}{\Delta+1-2}, \dots, \frac{n}{\Delta+1-\Delta} \right\}}_{\text{RHS}}$$

RHS = ave value of A

LHS = ave value of smallest $\deg(v)$ values in A

$$\leq \frac{n}{\Delta+1} \sum_{d=0}^{\Delta} \frac{1}{\Delta+1-d} = \frac{n}{\Delta+1} \sum_{i=1}^{\Delta+1} \frac{1}{i}$$

$$\leq \frac{n}{\Delta+1} (\ln(\Delta+1) + 1)$$

more complicated analysis removes $\log \Delta$

$$\text{So } E[\text{runtime}] \leq O(1) \sum_v \frac{n}{\Delta+1} (\ln(\Delta+1) + 1) = O\left(n^2 \frac{\log \Delta}{\Delta}\right)$$