Testing "Triangle Freeness" for Dense Graphs

def. \( G \) is \( \Delta \)-free if \( \# \{ x, y, z \mid A(x, y) = A(y, z) = A(x, z) = 1 \} \)

Claim (will prove in homework)
If there is a properly testing algorithm for \( \Delta \)-free-ness, then there is an algorithm that works as follows:
- pick random \( x, y, z \)
- test if \( A(x, y) = A(y, z) = A(x, z) = 1 \)

But the question remains... how many times do you need to repeat the test?

Let's take a detour:
- How many triangles in a random tripartite graph?

\[ \forall u \in A, v \in B, w \in C: \]
\[ P_r [u \sim v \sim w] = \eta^3 \]
\[ E [6_{v, v, w}] = \eta^3 \]
\[ E [\# \text{triangles}] = E \sum_{u \in A, v \in B, w \in C} 6_{u, v, w} = \eta^3 \cdot |A| |B| |C| \]
One possibility:

Density & Regularity of set pairs:

def. for $A, B \subseteq V$ s.t.

1. $A \cap B = \emptyset$
2. $|A|, |B| > 1$

Let $e(A, B) = \#$ edges between $A \cup B$

+ density $d(A, B) = \frac{e(A, B)}{|A| |B|}$

Say $A, B$ is $\gamma$-regular if $\forall A', B' \subseteq B$

st. $|A'| \geq \gamma |A|$

$|B'| \geq \gamma |B|$

$|d(A', B') - d(A, B)| \leq \delta$

Lemma [Komlos Simonovits]

For $\gamma > 0$ there exists $\delta$ (regularity parameter, depends only on $\gamma$) = $\frac{1}{2} \gamma \approx \delta^2(\gamma)$

$\delta$ (no triangles, depends only on $\gamma$) = $(1 - \gamma) \frac{\gamma^3}{8} \leq \frac{\gamma^3}{16} = \delta^3(\gamma)$

st. if $A, B, C$ disjoint subsets of $V$, each pair

is $\gamma$-regular with density $\geq \gamma$

then $G$ contains $\geq \delta |A| |B| |C|$ distinct $A's$ with vertex

from each of $A, B, C$
\textbf{Proof} (simplification of \cite{Alon Fischer Krivelevich Szegedy})

\[ A^* \leftarrow \text{nodes in } A \text{ with } \geq (\eta - \gamma) |B| \text{ nbrs in } B \]
\[ \eta \geq (\eta - \gamma) |C| \text{ nbrs in } C \]

**Claim** \[ |A^*| \geq (1 - 2\gamma)|A| \]

**Proof of Claim**

\[ A' \leftarrow \text{"bad" nodes of } A \text{ w.r.t. } B \text{ (i.e., } \leq (\eta - \gamma) |B| \text{ nbrs in } B) \]
\[ A'' \leftarrow \text{"..." nbrs of } C \text{ ("..." nbrs in } C) \]

Then \[ |A'| \leq \gamma |A| \]
\[ \eta |A''| \leq \gamma |A| \]

**Why?** otherwise consider pair \( A', B \)

\[ d(A', B) \leq \frac{|A'| (\eta - \gamma) |B|}{\eta |A'| |B|} = \eta - \gamma \]

by assumption for contradiction

\[ \text{size } \geq \gamma |A| \]

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\[ \text{trivially } \geq \gamma |B| \text{ since } \gamma < 1 \]

\[ d(A', B) \geq \eta \]

So \[ |d(A', B) - d(A, B)| > \gamma \]

Contradicts \( \gamma \)-regularity !

But \[ A^* = A \setminus (A' \cup A'') \]

So \[ |A^*| \geq |A| - |A'| - |A''| \]
\[ \geq |A| - 2\gamma |A| \]
\[ = (1 - 2\gamma)|A| \]  \( \square \)  \( \text{End of proof of claim} \)
Finishing proof of lemma:

For each \( u \in A^* \):

\[
\begin{align*}
\text{def. } \quad B_u &= \text{nbrs of } u \text{ in } B \\
C_u &= \quad \text{in } C
\end{align*}
\]

\( |B_u| \leq (\eta - \gamma) |B| \leq \gamma |B| \)

\( |C_u| \leq (\eta - \gamma) |B| \leq \gamma |B| \)

Since \( \gamma \) chosen s.t. \( \gamma < \frac{\eta}{2} \), \( \eta - \gamma > \gamma \)

Note: \# edges between \( B_u + C_u \Rightarrow \) lower bound on \# distinct triangles with \( u \) as a vertex

\[
\begin{align*}
d(B, C) &= \eta \\
\Rightarrow d(B_u, C_u) &\geq \eta - \gamma \quad (\text{since } |B_u|, |C_u| \text{ big enough } + B, C \text{ \( \gamma \)-regular}) \\
\Rightarrow e(B_u, C_u) &\geq (\eta - \gamma) |B_u| |C_u| \\
&\geq (\eta - \gamma)^3 |B| |C| \quad \text{gives lb on \# triangles with } u
\end{align*}
\]

\[
\begin{align*}
\Rightarrow \text{total } \pm A's &\geq (1 - 2\gamma) |A|, (\eta - \gamma)^3 |B| |C| \\
&\geq (1 - \eta) (\eta / 2)^3 |A| |B| |C| = (1 - \eta) \frac{\eta^3}{8} |A| |B| |C| \\
\text{choosing } \gamma = \frac{\eta}{2}
\end{align*}
\]
Do any interesting graphs have regularity properties? In some sense, all graphs do! I.e. every graph (in some sense) can be approximated by random graphs.

**Szemerédi's Regularity Lemma**

Would like it to say:

"One can equipartition the nodes V into V₁...Vₖ (for some constant K) s.t. all pairs (Vᵢ,Vⱼ) are ε-regular".

More useful version:

**Lemma**

∀ m, ε > 0. ∃ T = T(m, ε) St.

given G = (V,E) St. |V| > T

there is an equipartition of V into sets

then exist equipartition B into k sets which

refine A + s.t. m ∈ k ≤ T

ε = ε(1/k) set pairs not ε-regular.
"Picture":

Why is this good?

- partition big graph into "constant" # partitions
- st. each pair behaves like random bipartite graph
- random bipartite graphs have nice properties

Why was SRL first studied?

to prove conjecture of Erdős & Turán:
sequences of integers must always contain long arithmetic progressions
An application of the SRL:

Property testing 1

Given \( G \), adjacency matrix format

**Desired Behavior**

- if \( G \) is \( \Delta \)-free, output PASS
- if \( G \) is \( \varepsilon \)-far from \( \Delta \)-free, \( \Pr[\text{output PASS}] \geq \frac{3}{4} \)
  - most deletion \( \leq \varepsilon n^2 \) edges
  - to make it \( \Delta \)-free

How much time does this require?

trivial \( O(n^3) \), \( O(n^2) \), \( \ldots \), \( O(1) \)?

**Algorithm**

\[
\text{do } O(\varepsilon^{-1}) \text{ times}
\]

- Pick \( V_1, V_2, V_3 \)
- if \( \Delta \) reject and halt

Accept
**Thm**  $A \in E(s)$ st. $A \in st. |V|=n$  
+ st. $G$ is $\varepsilon$-far from $A$-free 
then $G$ has $\geq \delta(n^3)$ distinct $A$'s

**Corollary.** Algorithm has desired behavior

ie. if $A$-free, accepts with prob $1$

if $\varepsilon$-Far, $\geq \delta(n^3)$ $A$'s

$Pr \left[ \text{don't find } A \text{ in } \frac{\varepsilon}{8} \text{ loops} \right] \leq (1-\delta)^{\frac{\varepsilon}{8}}$

$\leq \varepsilon^{-C} < \frac{\delta}{4}$

for big enough $C$

**Proof of Thm**

Use regularity to get equipartition $\xi \{ V_1 \ldots V_{k-3} \}$ 

st. $\frac{5}{2} \leq k \leq \Upsilon \left( 5\varepsilon^{-1}, \varepsilon^1 \right)$

equivalently: $\frac{\varepsilon n}{5} \geq \frac{n}{k} \geq \frac{n}{\Upsilon(5\varepsilon^{-1}, \varepsilon^1)}$

$\# \text{nodes per partition}$

(do this by starting with arbitrary equipartition into $\frac{5}{2}$ sets as $A$)

for $\varepsilon^1 = \min \left\{ \frac{\varepsilon}{5}, \Upsilon \left( \frac{3}{5} \right) \right\}$

st. $\leq \varepsilon^1(\frac{k}{2})$ pairs not $\varepsilon^1$-regular
Need: # of partitions fairly large st. # edges inside a partition not too big

\[ G' \equiv \text{take } G \text{ and } \]

1) delete edges of \( G \) internal to any \( V_i \)

\[ \text{how many? } \leq \frac{n}{k} \cdot n \leq \frac{\varepsilon n^2}{5} \]

\[ \text{deg w/in } V_i \text{ sum over all n nodes} \]

\[ \text{since } |V_i| \leq \frac{n}{k} \]

2) delete edges between \( \varepsilon' \)-non-regular pairs

\[ \text{how many? } \leq \varepsilon'(\frac{k}{5}) \left( \frac{n}{k} \right)^2 \leq \frac{\varepsilon}{5} \cdot \frac{k^2}{2} \cdot \frac{n^2}{k^2} \leq \frac{\varepsilon}{10} n^2 \]

\[ \text{max # edges per pair max non-regular pairs here we use equipartition } \Rightarrow |V_i| = \frac{n}{k} \]

3) delete edges between low density pairs

\[ \text{how many? } \leq \frac{\varepsilon}{5} \left( \frac{n}{k} \right)^2 \leq \frac{\varepsilon n^2}{10} \]

\[ \text{note } \varepsilon(\frac{n}{k})^2 \leq \binom{n}{2} \]

So total deleted edges from \( G \) \( \leq \varepsilon n^2 \)

\[ \Rightarrow \text{so cheater is not so bad} \]
But, $G$ was $\varepsilon$-far from $\Delta$-free, so $G'$ must still have a $\Delta$!!!

Furthermore, by the way we constructed $G'$, we knew a lot about the $\Delta$: $\forall \Delta' \exists a,b \in V_i, V_j, V_k$

1) it must be that $i,j,k$ distinct since removed all edges within partitions

2) $(i,j), (j,k), (j,k)$ are regular pairs since removed non-regular pairs

3) $(i,j), (j,k), (j,k)$ are high density pairs since removed low density pairs

$\therefore$ $\exists i,j,k$ distinct st. $a \in V_i, b \in V_j, c \in V_k$

$V_i, V_j, V_k$ all $\geq \frac{\varepsilon^2}{5}$ - density pairs

$+ \ U^{\Delta} \left( \frac{\varepsilon}{5} \right) - regular$

$\Rightarrow \frac{\eta}{2} \geq \frac{\varepsilon}{10}$

$\Delta$-counting Lemma $\Rightarrow$

$\geq \delta^a \left( \frac{\varepsilon^3}{5} \right)!_V \Delta^s$ triangles in $G'$

$\geq \delta^a \left( \frac{\varepsilon^3}{5} \right) \frac{\eta^3}{3} \left( \frac{T(\frac{5}{8}, \varepsilon')}{3} \right)^3 \Delta^s$

$\geq \frac{1}{2} \frac{\varepsilon^3}{8000} = \frac{\varepsilon^3}{16000}$

$\geq \delta'(\frac{\eta}{2}) \Delta^s$ in $G'$ thus in $G$

for $\delta' = 6 \delta^a \left( \frac{\varepsilon^3}{5} \right) \left( \frac{T(\frac{5}{8}, \varepsilon')}{3} \right)^3$
Extensions

- Komlos-Simonovits holds for all const sized subgraphs

- Almost "as is" can use method to test all 1st order graph properties

\[ \forall u_1, u_2, u_3 \ldots u_k \quad \forall v_1, v_2 \quad R(u_1, u_k, v_1, v_2) \]

defined by \( v, k, l \) neighbors

\[ \forall u_1, u_2, u_3 \quad R(u_1, u_2, u_3) \]

encodes

\[ \forall (u_1 \sim u_2, u_2 \sim u_3, u_1 \sim u_3) \]

H-freeness for const size H

Induced \( \square \) vs. \( \Box \) forbidden

not induced

* 1-sided const time \( \propto \) hereditary graph props [Alan Shapira]

closed under vertex removal (not necessarily edges)

includes monotone graph props

Chordal

perfect interval graph

difficulty: infinite set of forbidden subgraphs also forbidden as induced

* 2-sided const time \( \propto \) regular partition is hardest testing problem

properly testable iff can reduce to testing [Alan Fisher Neiman

properly testable iff satisfies one of finitely many Szemeredi partitions.

see also work by [Borgs Chayes Lovasz Sos Szegedy Vesztergombi]