6.896 Sublinear Time Algorithms February 28, 2017

Lecture 5

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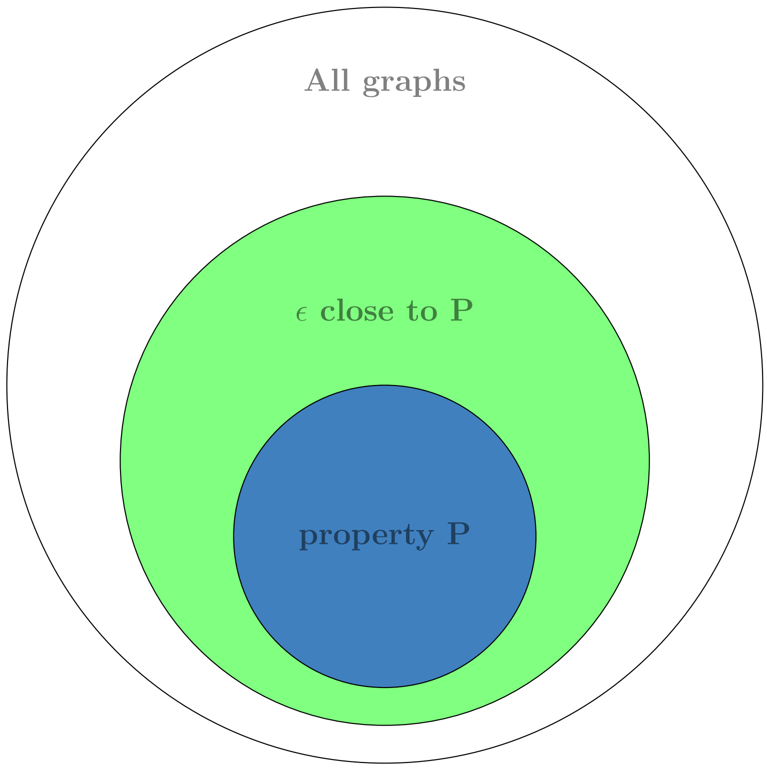
1 Property testing

It takes linear time to distinguish graphs that are connected and disconnected. However, it is much quicker to distinguish two graphs if they are close to each other. If there is a graph with property p and there is another one that is ϵ-close to P, then to distinguish these two may only take the sub-linear time.



1.1 Compromise

Can We distinguish the graphs with propoerty P and those that are far away from P?

i.e. G (degree $\leq d$) is "ϵ-far" from planar if we need to remove $\geq \epsilon d_{max} n$ edges to make it planar.

1.2 Property testing algorithm

$if\ G\ planar$
    $then\ output\ pass\ with\ probability\ \geq 1-\beta$
$if\ G\ is\ "\epsilon{-}far"\ from\ planar$
    $then\ output\ fail\ with\ probability\ \geq 1-\beta$

2 Testing Planarity

All graphs have max degree $\leq d$

2.1 Testing H-minor freeness

Definition 1 *H is minor of G if you could obtain H from G via either vertex removals, edge removals or edge contractions.*

Definition 2 *G is "H-minor-free" if H is not a minor of G.*

Definition 3 *G is "ϵ to H-minor-free" if we can remove $\leq \epsilon dn$ edges to make it H-minor-free.*

Definition 4 *G is has minor closed property p if all the minors of G have property P.*

Theorem 5 (Robertson & Seymour) *Every minor-closed property is expressible as a constant number of excluded minors.*

because the minor closed graph has this unique property: breaking them into pieces will only require remove very few edges.

Definition 6 *G is (k, ϵ)-hyperfinite if one can remove $\leq \epsilon n$ edges and remain with components of size $\leq k$.*

Definition 7 *G is p-hyperfinite if $\forall\ \epsilon > 0$, G is (ϵ, $p(\epsilon)$)-hyperfinite.*

Theorem 8 (Useful Theorem) *Given $H\ \exists\ C_H$ such that $\forall\ 0 < \epsilon < 1$, every H-minor free graph of deg $\leq d$ is $(\frac{C_H}{\epsilon^2}, \epsilon d)$-hyperfinite. (i.e. Remove $\leq \epsilon dn$ edges and components of size $O(\frac{1}{\epsilon^2})$)*

note:
whenever you have a minor close property, this graph has hyper-finite, they depends on ϵ.
Each of ϵ_k is still planar, which is still hyperfinite, can even be broke down to smaller planars.
sub-graphs of H-minor free graphs are also H-minor free and hyperfinite but only remove number of edges in porportion to number of nodes in the subgraphs

2.2 Why is hyperfinite useful?

Partition G into G'
$-$remove at most $\epsilon' dn$ edges
$-$ Constant size component remain
$-$if no way to do this, G is not a planar
If G' is close to planar, so is G
$-$ so let G' by picking random components & seeing if they have the property

3 Partition oracle

3.1 Partition Oracle

Assume we have "partition oracle" P with parameter k, $\frac{\epsilon d}{4}$, such that $\forall\ v \in V$,

$$|P[v]| \leq k$$

$$P[v] connected$$

If G is H-minor free with prob $\geq \frac{9}{10}$

Given partition oracle:

$estimate\ \hat{f} = number\ of\ edges\ (u,v)\ such\ that\ P[u] \neq P[v]\ to\ additive\ error \leq \frac{\epsilon dn}{8}.$

$if\ \hat{f} > \frac{3}{8}\epsilon dn,\ output\ "fail"\ and\ halt$

$else\ choose\ S = O(\frac{1}{\epsilon})\ nodes\ randomly$

$if\ for\ any\ s \in S,\ P[s]\ not\ planar,\ "fail"\ and\ halt$

$Accept$

If G planar, $E[\hat{f}] = \frac{\epsilon dn}{4}$

$Sampling\ bounds\ \ \hat{f} \leq \frac{\epsilon dn}{4} + \frac{\epsilon dn}{8} = \frac{3}{8}\epsilon nd$

$All\ partitions\ planar,\ then\ pass$

If G is "ϵ-far" from H-minor free,

$Case\ 1$

$p's\ output\ is\ such\ that\ |\epsilon(u,v) \in E | p(u) \neq p(v)| \leq \frac{\epsilon dn}{2}$
$sampling\ bounds\ \hat{f} \leq \frac{\epsilon dn}{4} + \frac{\epsilon dn}{8} = \frac{3}{8}\epsilon nd$
$output\ fail\ with\ prob \geq \frac{9}{10}$

$Case\ 2$

$p's\ output\ satisfies\ |\epsilon(u,v) \in E | p(u) \neq p(v)| \leq \frac{\epsilon dn}{2}$
$G' = G\ with\ cross\ edges\ removed$
$if\ G'\ is\ \frac{\epsilon}{2}-far\ from\ having\ property,\ third\ step\ likely\ to\ fail$
$else\ G'\ is\ \frac{\epsilon}{2}\ close\ to\ property\ \&\ G\ is\ \frac{\epsilon}{2}\ close\ to\ G'$
$so\ G\ is\ \epsilon-close\ to\ having\ property$

$if\ G'\ is\ \frac{\epsilon}{2}-far\ from\ planar,\ need\ to\ remove \geq \frac{\epsilon}{2}dn\ edges\ to\ make\ planar$

3.2 Global partitioning algorithm

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let π_1,...,π_n be random labelling of nodes, π_i ≠ π_j, π_i ∈ [n]
p = φ
For i = 1...n do
    if π_i is still in the graph then
        if ∃(k,δ) − isolated neighborhood of π_i in remaining graph
            then s = this nbhd
        else s = {π_i}
        p = p ∪ s
        remove s from graph
```

For hyperfinite graphs, most nodes have (k, δ)-isolated nbhds

Lemma 9 *if G is hyperfinite, most nodes have (k, δ)-isolated nbhd*

To compute $p[v]$ locally, recursively compute $p[w]$ $\forall$ w of rank ¡ rank$[v]$ with distance k of v

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if ∃ w such that v ∈ p[w] and rk(w) ≤ rk(v), then p[v] = p[u]
else look for (k,δ) isolated nbrhd of v
    if find it p[v]
    else p[v]
```

3.3 Local simulation of oracle

assign random number $\in (0,1)$ *to v*

when first see it, use rank orders to define π

to compute $p[v]$

recursively compute $p[w]$ $\forall w$ *of rank* $< v$ *within distance* $\leq k$ *of v*

if $\exists w$ *such that* $v \in p[w]$ *then* $p[v] = p[w]$

else look for $(k, \delta)-$*isolated nbhd of v*

(ignoring any node which is in p[w] for any w with smaller rank)

if find it, $p[v] =$ *this nbhd*

else $p[v] = \{v\}$

3.4 Query complexity

d^k *nodes within distance k*

$2^{d^{O(k)}}$ *using [NO] analysis &* $k \approx p(\frac{\epsilon^3}{big\ constant})$