Reducing Randomness Via Random Walks on special graphs

Reducing Randomness

For decision problem L,

Let A be algorithm s.t. 1) $\forall x \in L \quad \Pr[A(x)=1] \geq 99/100$ almost always correct

2) $\forall x \notin L \quad \Pr[A(x)=0] = 1$ always correct

To get error $< 2^{-k}$:

Method:	# random bits used
1) run k times & output "$x \notin L$" if ever see "$x \notin L$" else output "$x \in L$"	$O(kr)$
2) use p.i. random bits	$O(k+r)$
3) today: use random walk on graph to choose random bits	$r + O(k)$

Plan:

- associate all (random) strings in $\{0,1\}^n$ with nodes of a graph G
- problem of picking a random string is now equivalent to problem of picking a random node ← easier?

  picking several random strings $\Rightarrow$ picking several nodes ← easier?

  picking several strings, one of which is "good" $\Rightarrow$ picking several nodes, one of which is "good" ← "easier"!

we get to pick G!

The graph G:

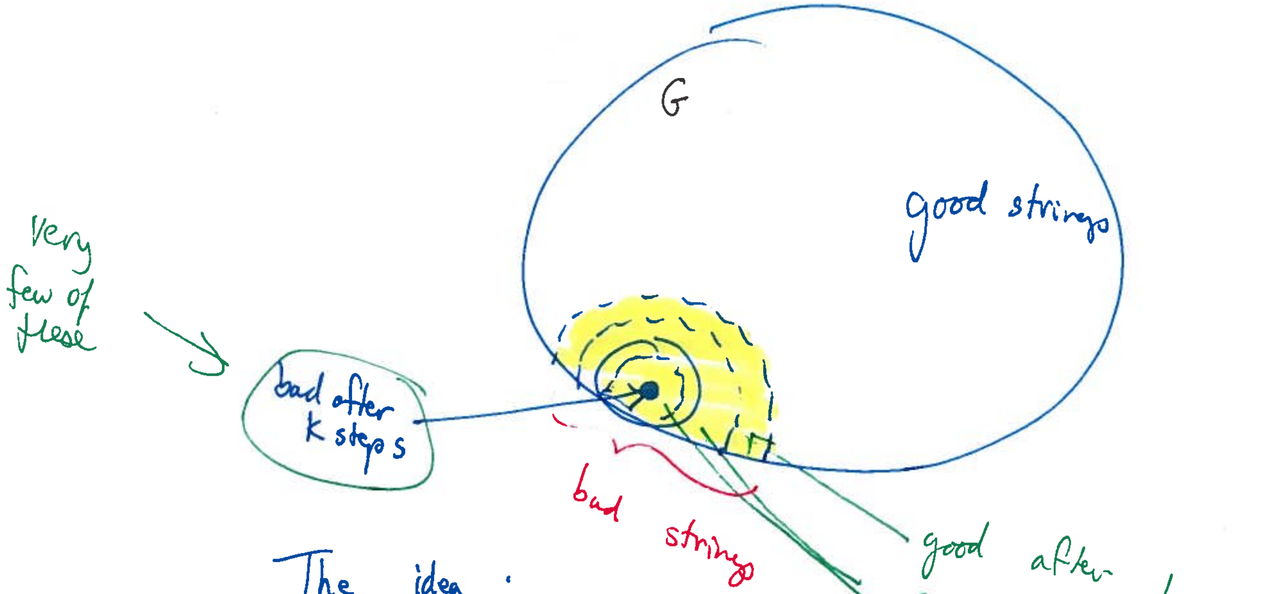
- Constant degree d-regular, connected, nonbipartite
- transition matrix P for r.w. on G has $|\lambda_2| \le \frac{1}{10}$
  stationary dist π uniform since d-reg
- $\#$nodes $= 2^r$ $\sim$ r random bits

The Algorithm:

- pick random start node $w \in \{0,1\}^r$ — r bits
- Repeat K times:
  - $w \leftarrow$ random neighbor of w — $O(1)$ bits $\times K$
  - run $A(x)$ with w as random bits
    - if A outputs "$x \in L$" then output "$x \in L$" + halt
    - else continue
- Output "$x \notin L$"

total: $r + O(K)$ random bits

Claim: error of new algorithm $\le \left(\frac{1}{5}\right)^K$ for $x \in L$
(still 0-error for $x \notin L$)



The idea:

very unlikely to pick a start location that is bad after k steps.

Behavior :

idea :

bad case - walk only on "bad" random strings + never get out to "good" random strings

why would this not work on arbitrary G?

e.g. G = line

if $x \notin L$: algorithm never errs (there are no bad strings)

if $x \in L$:

most random bits say $x \in L$: $\geq \frac{99}{100} \cdot 2^r$

define $B \leftarrow \{ w \mid \mathcal{A}(x)$ with random bits w is incorrect i.e. says $x \notin L \}$

"Bad w's"

$$|B| \leq \frac{2^r}{100}$$

Want linear algebraic way of describing walks that stay in badset :

define N diagonal matrix such that

$$N_w = \begin{cases} 1 & \text{if } w \in B \leftarrow \text{incorrect} \\ 0 & \text{o.w.} \leftarrow \text{correct} \end{cases}$$

Bad w's

$$N = \begin{pmatrix} 1,1 & & & & & \\ & 0\;0\;0\;0 & & & 0 & \\ & & 1,1 & & & \\ & 0 & & 0\;0\;0\;0 & & \\ & & & & 1,1 \end{pmatrix}$$

e.g. $q = (\tfrac{1}{4}\ \tfrac{3}{4})$

$N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$qN = (\tfrac{1}{4}\ \ 0)$

zeroed out

q any probability distribution, $q \cdot N$ is ??

$\|qN\|_1 = \Pr_{w \in q}[w \text{ is bad}]$ i.e. qN deletes weight that finds a witness to $x \in L$

Can compose:

$\|q \cdot PN\|_1 = \Pr_{w \in q}[\text{start at } q, \text{ take a step + land on "bad"}]$

$\vdots$

$\|q \cdot (PN)^i\|_1 = \Pr_{w \in q}[\text{start at } q, \text{ take } i \text{ steps + each is "bad"}]$

ignores whether start node is bad, this just hurts us so it is ok to ignore

Lemma $\forall \pi$ $\|\pi PN\|_2 \leq \frac{1}{5}\|\pi\|_2$

First: How do we use the lemma?

If always see bad w's, then answer incorrect

$$\Rightarrow \Pr[\text{incorrect}] \leq \|p_0 \cdot (PN)^k\|_1$$

$$\leq \sqrt{2^r}\,\|p_0 (PN)^k\|_2$$

since $\|p\|_1 \leq \sqrt{\text{domain size}} \cdot \|p\|_2$

$$\leq \cancel{\sqrt{2^r}} \cdot \underbrace{\|p_0\|_2}_{\cancel{\frac{1}{\sqrt{2^r}}}} \left(\tfrac{1}{5}\right)^k$$

apply lemma k times

since start at uniform + L_2 norm of uniform $= \sqrt{\sum \left(\frac{1}{2^r}\right)^2} = \sqrt{\frac{1}{2^r}}$

$$= \left(\tfrac{1}{5}\right)^k$$

Proof of lemma let $V_1 \dots V_{2^r}$ be e-vects of P, + V_1 is s.t. $\|V_1\|_2 = 1$

note, $V_1 = \left(\frac{1}{\sqrt{2^r}}, \dots, \frac{1}{\sqrt{2^r}}\right)$

then $\pi = \sum_{i=1}^{2^r} \alpha_i V_i$

Note: 1) $\|\pi\|_2 = \sqrt{\alpha_i^2}$ (from before)

2) $\forall\, w \quad \|wN\|_2 = \sqrt{\sum_{i \in B} w_i^2} \leq \sqrt{\sum_i w_i^2} = \|w\|_2$

So:

$$\|\pi P N\|_2 = \left\| \sum_{i=1}^{2^n} \alpha_i v_i P N \right\|_2$$

$$= \left\| \sum_{i=1}^{2^n} \alpha_i \lambda_i v_i N \right\|_2$$

$$\leq \underbrace{\|\alpha_1 \lambda_1 v_1 N\|_2}_{A} + \underbrace{\left\| \sum_{i=2}^{2^n} \alpha_i \lambda_i V_i N \right\|_2}_{B}$$

Cauchy-Schwarz

bounding A:

$$\|\alpha_1 \lambda_1 v_1 N\|_2 = \|\alpha_1 v_1 N\|_2 \quad \text{since } \lambda_1 = 1$$

$$= |\alpha_1| \sqrt{\sum_{i \in B} \left(\frac{1}{\sqrt{2^r}}\right)^2} \quad \text{since } V_1 = \left(\frac{1}{\sqrt{2^r}}, \dots, \frac{1}{\sqrt{2^n}}\right)$$

use that uniform is unlikely to be on bad string

$$= |\alpha_1| \sqrt{\frac{|B|}{2^n}}$$

$$\leq \frac{|\alpha_1|}{10} \quad \text{since } \frac{|B|}{2^r} \leq \frac{1}{100}$$

$$\leq \frac{\|\pi\|_2}{10} \quad \text{since } \|\pi\|_2 = \sqrt{\sum \alpha_i^2}$$

Bounding (B) : $\left\| \sum_{i=2}^{2^n} \alpha_i \lambda_i v_i N \right\|_2 \leq \left\| \sum_{i=2}^{2^n} \alpha_i \lambda_i v_i \right\|_2$ from note

$= \sqrt{\sum (\alpha_i \cdot \lambda_i)^2}$

use "mixing"

$\leq \sqrt{\sum \alpha_i^2 \left(\frac{1}{10}\right)^2}$ $\lambda_i \leq 1/10$

$\leq \frac{1}{10} \|\pi\|_2$

So: $\|\pi P N\|_2 \leq \frac{\|\pi\|_2}{5}$ ■