Lecture 2

The Lovász Local Lemma

Another way to argue that "nothing bad happens"

If $A_1 \ldots A_n$ are "bad" events

how do we know if there is positive probability that none occur? (or is prob that any occurs <1?)

**usual way:** Union bound

\[ \Pr \left[ \bigcup A_i \right] \leq \sum \Pr[A_i] \]

no assumptions on $A_i$'s w.r.t. independence

then need $p < \frac{1}{n}$ to get anything interesting (i.e. sum <1)

**if $A_i$'s independent + "nontrivial":** $\iff$ "nontrivial" $\iff \Pr(A_i) > 1$

\[ \Pr \left[ \bigcup A_i \right] \leq 1 - \Pr \left[ \bigcap A_i \right] \]

\[ = 1 - \prod \Pr (A_i) \]

\[ > 0 \]

\[ \leq 1 \]

**always!!**

**What if $A_i$'s have "some" independence?**

**def** A "independent" of $B_1 \ldots B_k$ if $\forall J \subseteq [k]$ $J \neq \emptyset$

\[ \Pr \left[ A \land \bigcap_{j \in J} B_j \right] = \Pr[A] \cdot \Pr \left[ \bigcap_{j \in J} B_j \right] \]
**Definition:** \( A_1 \ldots A_n \) events

\( D = (V, E) \) with \( V = \llbracket n \rrbracket \) is a "dependency digraph of \( A_1 \ldots A_n \)"

if each \( A_i \) independent of all \( A_j \) that don't neighbor it in \( D \) (i.e., all \( A_j \) s.t. \((ij) \notin E\))

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**Lovász Local Lemma (Symmetric Version):**

\( A_1 \ldots A_n \) events s.t. \( \Pr(A_i) \leq p \) \( \forall i \)

with dependency digraph \( D \) s.t. \( D \) is of degree \( \leq d \).

If \( e_{D}(d+1) \leq 1 \) then

\[ \Pr[\bigwedge_{i=1}^{n} \overline{A_i}] > 0 \]

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**Application:**

Then, \( S_1 \ldots S_m \subseteq X \), \( |S_i| = l \), each \( S_i \) intersects at most \( d \) other \( S_j \)'s

if \( e_{D}(d+1) \leq 2^{l-1} \)

then can 2-color \( X \) s.t. each \( S_i \) not mono-chromatic

i.e. \( H \) is a hypergraph with \( m \) edges, each containing \( l \) nodes + each intersecting \( \leq d \) other edges.
**Proof:**

Color each elt of $X$ red/blue with prob $\frac{1}{2}$ iid.

$A_i$: event that $S_i$ monochromatic

$p = \Pr[L \cap A_i] = 2^{-l(l-1)}$

$A_i$ ind of all $A_j$ st. $S_i \cap S_j = \emptyset$

depends on $\leq d$ other $A_j$

Since $e^{p(d+1)} = e^{\frac{1}{2d-1}(d+1)} \leq 1$

$\text{LLL } \Rightarrow \exists$ 2-coloring

**Comparison:**

- $\# $ edges $= m$
- Size of edge $= l$
- $m < 2^{l-1}$

- $\# $ edges $= m$
- Size of edge $\geq l$
- Each edge intersects $\leq d$ others
- $\sum d+1 \leq \frac{2^{l-1}}{e}$
- No dependence on $m$

**A second application:**

Given CNF formula s.t. $l$ vars in each clause

*Each* var in $\leq k$ clauses.

If $\frac{e^{l(k+1)}}{2^{l-1}} \leq 1$ there is a satisfying assignment
How do you find a solution?

Partial history:

Lovász 1975 non-constructive
(no fast algorithm to find soln)

Beck 1991 randomized algorithm
but for more restrictive conditions

... on parameters

Moser 2009 negligible restrictions for SAT

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Moser-Tardos

Then: given $S_1, \ldots, S_m \subseteq \mathbb{X}$

- each $S_i$ intersects $\leq d$ other $S_j$'s
- if $e(d+1) \cdot c \leq 2^{d-1}$
- then can find 2-coloring of $\mathbb{X}$ s.t.
  each $S_i$ not monochromatic
  in time poly in $m, d, \mathbb{X}$

Moser-Tardos Algorithm:

1. 2-color all els of $\mathbb{X}$ randomly
2. While there is a monochromatic set:
   - pick arbitrary monochromatic $S_i$
   - randomly reassign colors to elements of $S_i$
Special Case & Slower algorithm: (based on Beck &alon)

**Stronger Assumption:**

Let $D = d(d-1)^3$

\[ l = c_1 + c_2 + l_3 \]

16 $D(1+d) \leq 2^d$ \hspace{1cm} (1)

16 $D(1+d) \leq 2^{d/2}$ \hspace{1cm} (2)

$2c(1+d) \leq 2^{l_3}$

For today, assume $l$ is constant.

Algorithm: Given $S_1, \ldots, S_m \leq X$

**First Pass:**

For each $j \in X$

if $j$ is "saved" do nothing

else pick color $c \in \{\text{red, blue}\}$ via coin flip

Consider all $S_i$ containing $j$

if $S_i$ has $l_1$ pts all same color

and $\leq 4$ pts in other colors

then $S_i$ becomes dangerous

and all uncolored pts become "frozen"

(by now, all pts in $X$ are $\in \{\text{red, blue, frozen}\}$)

If $S_i$ not yet 2-colored then $S_i$ "survives"

**Second Pass:**

Find coloring of surviving $S_i$ via brute force
Big Questions:
(1) Does it work?
(2) Runtime?

Analysis:
Consider a single \( S_i \):
\[
\Pr[\text{\( S_i \) survives}] \geq \Pr[\text{\( S_i \) becomes dangerous}]
\]
\[
= \frac{2}{2^L} = 2^{-L},
\]

When is survival of \( S_i + S_j \) independent?
not \( \iff S_i \cap S_j \neq \emptyset \)

(1) \( S_i \cap S_j \neq \emptyset \)

(2) \( S_i \cap S_k \neq \emptyset \) and \( S_k \cap S_j \neq \emptyset \)

(3) \( \exists k, l \) s.t.
\( S_i \cap S_k \neq \emptyset \)
\( S_k \cap S_l \neq \emptyset \)
\( S_i \cap S_j \neq \emptyset \)

\( S_k \) freezes pts in \( S_i + S_j \Rightarrow S_i \) not 2-colored
Useful Graph $G'$

\[ G \leftarrow \text{nodes } V \subseteq [m], \quad (\text{node } i \sim S_i) \]
\[ \text{edges } (i, j) \in E \iff S_i \cap S_j \neq \emptyset \]

Note: $S_i$ only depends on other $S_j$ within distance $\leq 3$

Observe: solution exists via LLL by condition 1 after 1st pass, solution for survivors via condition 2

Claim: (with high prob)
all surviving nodes in $G$ form components of size $O(\log m)$

Correct: if $c$ constant:
and pass needs only $O(2^c \cdot \log m) = O(m^{2c})$

What if $c$ not constant?
re-recur on components

Proof of Claim:

What is prob that component $C$ survives?

if $C$ survives, consider

the biggest $T \subseteq C$ st.

(i) all nodes in $T$ are
\[ \text{dist} \geq 4 \text{ in } G \]

(ii) if connected nodes in $T$ of $\text{dist} = 4$ then $T$ connected
picking $T$ greedily $\Rightarrow \exists T$ of size $\leq \frac{|C|}{d^3}$

if $C$ survives, $T \subseteq C$ also survives

$\Rightarrow \Pr [T \text{ survives}] \geq \Pr [C \text{ survives}]$

What is $\Pr [T \text{ survives}]$?

$\forall S_i \in T$, $S_i$ survives if

1. dangerous
2. next to dangerous $S_i$

which froze its elements

Note: if $S_i \neq S_j$
then $S_i \cap S_j = \emptyset$ since $S_i$ and $S_j$ are dist 4

For each $S_i \in T$
pick $S_i'$ possible dangerous from $S_i$ i.e. $\exists \ell \in \mathbb{N}$ ways to make this choice

all $S_i'$ are disjoint

$\Pr [\text{all } k \text{ } S_i' \text{ become dangerous}] \leq 2^{(1-\ell) \cdot k}$

$\Pr [\text{all } S_i \text{ survive}] \leq (d+1)^k \cdot 2^{(1-\ell) \cdot k}$
We need to show no such large tree survives.

Here is the win:

If $T$ is an arbitrary subtree of size $u$, we have $(m)$ choices of $T$, need to show all don't survive (lots of term in union bound)

Here $T$ is an arbitrary subtree of size $u$ in a graph $G(u)$ of degree $\leq 8D(d+1)^2$.

Expected # Trees of size $u$ that survive

$$\leq m \cdot (4D)^u \cdot (d+1)^u \cdot \frac{1}{2} \cdot \sum \frac{1}{2^u} \leq m \frac{8D(d+1)^2}{2^{1/2}}$$

$$\Rightarrow \text{if } u \geq \Omega(\log m) \text{ this term is } o(1)$$

If $o(1)$ $u$-trees survive in expectation, then Markov's $\Rightarrow$

$$\Pr[\text{more than } k \cdot o(1) \text{ trees survive }] < \frac{1}{k}$$

Pick $K$ so that this is $\leq 1$

So $\Pr[\text{any } u \text{-tree survives }] < 1/k$ (note: if no $u$-tree survives $\Rightarrow$ no $u$-tree survives)