6.5420 Randomness and Computation	April 18, 2026

Homework 5

Lecturer: Ronitt Rubinfeld — *Due Date: May 1, 2026*

1. **Noise sensitivity.** Show that any function $f : \{\pm 1\}^n \to \{\pm 1\}$ satisfies

$$NS_\epsilon(f) = \frac{1}{2} - \frac{1}{2}\sum_S (1-2\epsilon)^{|S|}\hat{f}(S)^2.$$

2. **Influence of monotone functions.** For $x = (x_1, \ldots, x_n) \in \{\pm 1\}^n$, let $x^{\oplus i}$ be x with the i-th bit flipped, that is,

$$x^{\oplus i} = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n).$$

The *influence of the i-th variable on* $f : \{\pm 1\}^n \to \{\pm 1\}$ is

$$\mathrm{Inf}_i(f) = \Pr_x\left[f(x) \neq f\left(x^{\oplus i}\right)\right].$$

The *total influence* of f is

$$\mathrm{Inf}(f) = \sum_{i=1}^n \mathrm{Inf}_i(f).$$

A function $f : \{\pm 1\}^n \to \{\pm 1\}$ is *monotone* if $f(x) \leq f(y)$ whenever $x \leq y$ in the product order (i.e. $x_i \leq y_i$ for each i).

(a) Show that for a monotone function $f : \{\pm 1\}^n \to \{\pm 1\}$, the influence of the i^{th} variable is equal to the value of the Fourier coefficient of $\{i\}$; that is $\mathrm{Inf}_i(f) = \widehat{f}(\{i\})$.

(b) Show that the majority function $\mathrm{maj}(x) = \mathrm{sign}(\sum_i x_i)$ maximizes the total influence among monotone functions $\{\pm 1\}^n \to \{\pm 1\}$, for n odd.

3. **Influence and testing.**

(a) Given a function $f : \{\pm 1\}^n \to \{\pm 1\}$, we define the probability distribution $\mathcal{S}_f$ on the subsets of $[n]$ via

$$\Pr_{S \sim \mathcal{S}_f}[S = T] = \widehat{f}(T)^2.$$

Briefly explain why $\mathcal{S}_f$ is a well-defined probability distribution, and prove that

$$\mathrm{Inf}(f) = \mathbf{E}_{S \sim \mathcal{S}_f}[|S|]$$

for any function f.

(b) Show that any function $f : \{\pm 1\}^n \to \{\pm 1\}$ is $\frac{\mathrm{Inf}(f)}{\epsilon}$-Fourier concentrated.

(c) Show that the class of monotone functions can be learned to accuracy ϵ and confidence $\delta = 1\%$ using $n^{\Theta(\sqrt{n}/\epsilon)} = 2^{\tilde{O}(\sqrt{n}/\epsilon)}$ samples under the uniform distribution. *Hint:* You can use Problem 2.

4. **(Almost k-wise independent random variables)** Let $\epsilon \in (0,1)$ and $k \in [n]$. A random vector $(X_1, \dots, X_n) \in \{\pm 1\}^n$ is said to be (ϵ, k)-wise independent if the restriction of $(X_1, \dots, X_n)$ to any subset of k coordinates in $[n]$ is ϵ-close[1] to the uniform distribution on $\{\pm 1\}^k$. Note that $(0, k)$-wise independence coincides with our usual notion of k-wise independence. The goal of this problem is to show that any (ϵ, k)-wise independent distribution is $O(\epsilon n^k)$-close to some k-wise independent distribution. If μ is the probability mass function (pmf) of a distribution with support $\{\pm 1\}^n$, we denote its Fourier transform by $\widehat{\mu} : 2^{[n]} \to [-1, +1]$.

   (a) Show that if $(X_1, \dots, X_n)$ is an (ϵ, k)-wise independent random vector with pmf μ, then $|\widehat{\mu}(S)| \le \epsilon / 2^{n-1}$ for every non-empty $S \subseteq [n]$ with $|S| \le k$.

   (b) We say a function on $\{\pm 1\}^n$ is a *k-junta* if its output only depends on k of the n variables. Prove that if $h : \{\pm 1\}^n \to [-1, 1]$ is a k-junta, then $\sum_{S \subseteq \{\pm 1\}^n} |\widehat{h}(S)| \le 2^{k/2}$.

   (c) Show that if $(X_1, \dots, X_n)$ is a random vector with pmf μ satisfying $|\widehat{\mu}(S)| \le \beta$ for every non-empty $S \subseteq [n]$ with $|S| \le k$, then $(X_1, \dots, X_n)$ is $(\beta \cdot 2^{n+k/2}, k)$-wise independent.

   Hint: Use Plancherel's identity and part (b) to bound $\left| \Pr_{X \sim \mu}[X_S \in A] - \frac{|A|}{2^{|S|}} \right|$.

   (d) Show that if $(X_1, \dots, X_n)$ is an (ϵ, k)-wise independent random vector, then it is $O(\epsilon n^k)$-close to some k-wise independent distribution.

   Hint: Using part (c), what condition on the non-empty low-degree Fourier coefficients of μ would imply k-wise independence? How can you modify μ in order to make it satisfy this condition for one particular Fourier coefficient? (Try taking a mixture of μ with another distribution.) How are the other non-empty Fourier coefficients affected?

[1] Two distributions p and q over $\{\pm 1\}^k$ are ϵ-close if the total variation distance $d_{TV}(p, q) = \frac{1}{2} \sum_{x \in \{\pm 1\}^k} |p(x) - q(x)| = \max_{A \subseteq \{\pm 1\}^k} |p(A) - q(A)|$ is at most ϵ.