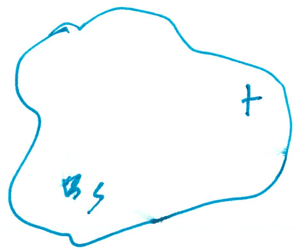
Approximating random walks with no randomness

H 

$H = (V, E)$ $|V| = n$

$L \le n$ walk length

$\ell = \lg L$

H is two-out-regular $v \xrightarrow{0} H[v,0]$, $v \xrightarrow{1} H[v,1]$

Goal: estimate $\Pr_{x \sim U_L}[H[s,x] = t] \pm \frac{1}{L}$

1) deterministically

2) using $o(n)$ workspace

In fact, WalkEst $\in$ RL → randomized logsp

Today: WalkEst $\in L^2$ → det. $\log^2 n$ space

via "INW PRG"

$G : \{0,1\}^{\#} \to \{0,1\}^L$ s.t. for all H, s, t

$$\Pr_{\sigma \sim U_{\#}}[H[s, G(\sigma)] = t] \approx_\varepsilon \Pr[H[s, U_L] = t]$$

additive error

Note: above gives WalkEst $\in$ Space[$O(\#)$]

For now: Assume we're halfway there

$G' : \{0,1\}^{\#'} \to \{0,1\}^{L/2}$ st $\forall$ H, s, t ↑ holds

of size n

How do we go from this to length-L walks?

Ideal: "doubling PRG"

$$G_1(\sigma_1 \sigma_2) = G'(\sigma_1) G'(\sigma_2)$$

Seed len: $2d'$

Is it a good PRG? Fix H, s, t

$$\cancel{\mathbb{E}[\mathbb{1}[H[s, G'(U_{s'}) G(U_{s'})] = t]]} \qquad H_{st}(x) = \begin{cases} 1 & H[s,x] = t \\ 0 & \text{else} \end{cases}$$

$$\cancel{\mathbb{E}_{r_1 r_2} \mathbb{1}[H[s}$$

$$\mathbb{E}_{r_1 r_2} H_{st}(r_1 r_2) = \mathbb{E}_{r_1}\Big[\underbrace{\mathbb{E}_{r_2} H_{st}(r_1 r_2)}\Big] \quad v = H[s, r_1]$$

$$\mathbb{E}_{r_2}[H_{vt}(r_2)]$$

$$\approx_\varepsilon \mathbb{E}_{r_1} \mathbb{E}_{\sigma_2} H_{st}(r_1 G'(\sigma_2))$$

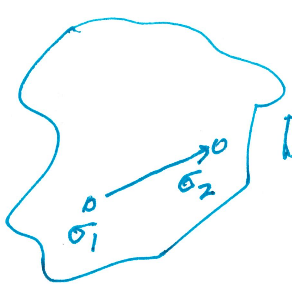
$$= \mathbb{E}_{\sigma_2} \mathbb{E}_{r_1} H_{st}(r_1 G'(\sigma_2))$$

$$\approx_\varepsilon \mathbb{E}_{\sigma_1 \sigma_2} H_{st}(G'(\sigma_1) G'(\sigma_2))$$

$\Rightarrow$ doubling PRG G' is a 2ε-PRG for length L walks

can we do better than seed length $2d'$?

Heuristic: "choose ~~two~~ indep ~~things~~" = "take a step on complete graph"



D on $\{0,1\}^{d'}$

choose vtx: d' bits

choose nbr: d' bits :(

Heuristic 2: complete graph $\approx$ expander graph!

sps D is λ expander w/ deg $m << d'$

Then $(\sigma_1, D[\sigma_1, U_m])$ takes $d' + \log(m)$ bits to sample

$$G_2(\sigma_1, e_1) = G'(\sigma_1) G'(D[\sigma_1, e_1])$$

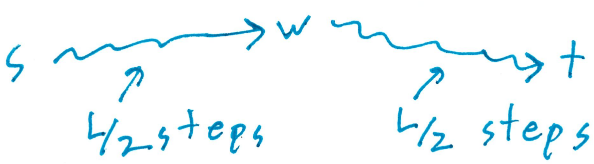
How to analyze G_2? Show its a 3ε-PRG?

Idea: compare to doubling PRG

$\forall$ H, s, t

$$\mathop{\mathbb{E}}_{\sigma_1, \sigma_2}[H_{st}(G_1(\sigma_1, \sigma_2))] \approx \mathop{\mathbb{E}}_{\sigma_1, e_1}[H_{st}(G_2(\sigma_1, e_1))]$$

Consider midpoint of walk



$$A_w = \{\sigma : H_{sw}(G'(\sigma))\}$$

$$B_w = \{\sigma : H_{wt}(G'(\sigma))\}$$

$$\mathop{\mathbb{E}}_{\sigma_1 \sigma_2}[H_{st}(G_1(\sigma_1, \sigma_2))] = \sum_w \frac{|A_w|}{2^{d'}} \cdot \frac{|B_w|}{2^{d'}} = \sum_w p_w$$

Clm: $\forall w$, $(\sigma_1, D[\sigma_1, e_1]) \in A_w \times B_w$ with approximately correct density

$$\Pr_{\sigma_1, e_1}[\sigma_1 \in A_w, D[\sigma_1, e_1] \in B_w] = \frac{|A_w|}{2^{d'}} \cdot \frac{|\Gamma(A_w) \cap B}{m}$$

By "expander mixing lemma" $\approx_\lambda p_w$

Thus G_2 $n \cdot \lambda$-appx G_1 which 2ε-appx U_L

$\Rightarrow$ G_2 $(n\lambda + 2\varepsilon)$-appx U_L

Take $\lambda \approx 1/nL$ & recurse construction ℓ times

$\Rightarrow$ final PRG w/ seed $\log(\deg(D_1)) + \dots \log \deg D_\ell$

$= \ell \cdot \log \operatorname{poly}(nL) = O(\log(n) \log(nL))$