6.5420 Randomness and Computation April 23, 2026

Lecture 19

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This lecture covers:

- Weak learning of monotone functions
- Canonical Path Argument

Note: The 0,1 and ±1 conventions for Booleans are used interchangeably here.

1 Monotonicity

Definition 1.
Partial order: For vectors x and y, write $x \leq y$ if and only if $x_i \leq y_i$ for all i.
Monotone function: f is monotone if $x \leq y \implies f(x) \leq f(y)$

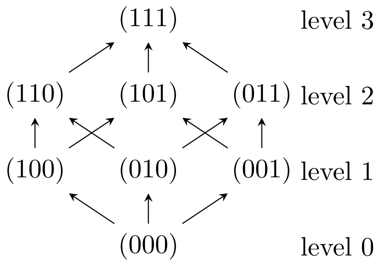


Figure 1: The Boolean Hypercube

Disclaimer: AI was used in transcribing the diagrams to LaTeX.

- Each node is labeled by an n-bit string, with a directed edge $x \to y$ if we can flip a "0" in x to a "1" to get y (e.g. $000 \to 010$).
- Each node in level k has k ones and $(n-k)$ zeroes.
- The "Balloon area" is $[\frac{n}{2} - c\sqrt{n}, \frac{n}{2} - c\sqrt{n}]$ where c is some constant. The rest is a negligible fraction ($\leq \epsilon$).
- Since each string is n bits, there are 2^n nodes.
- There are $\frac{n2^n}{2}$ edges total.

  For the number of edges, if you ignore directions, it is actually regular! For each "1" in the node, we have an in-edge, and for each "0" we have an out-edge. Thus each node always has n edges.
- Level k has $\binom{n}{k}$ nodes, since we choose n of the bits to be "1."

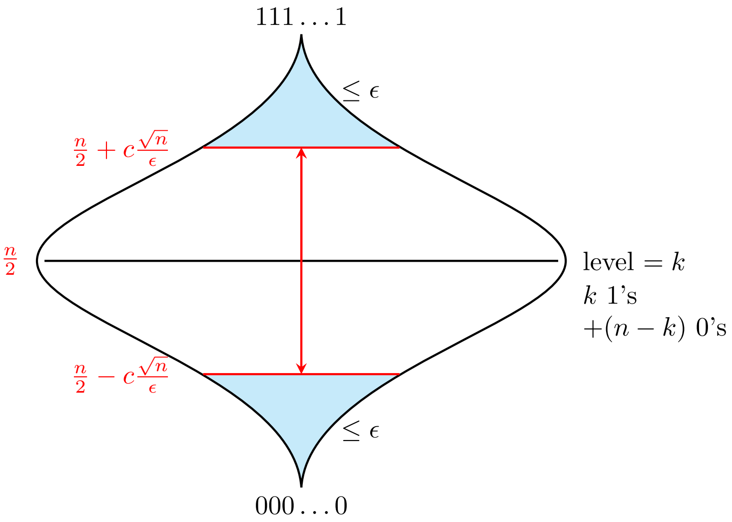


Figure 2: Concentration of Boolean n-cube

1.1 Verifying Monotonicity

Let $\mathcal{C}$ be the class of monotone functions.
Occam's Razor says $\log|\mathcal{C}|$ samples suffices (ignoring runtime).

We can lower bound $|\mathcal{C}|$: Consider the case where $n/2+1$ and above are 1 (red) and $n/2-1$ and below are 0 (blue). Any setting for the $n/2$ layer yields a monotone "middle slice" function. There are $2^{\binom{n}{n/2}}$ of these (which is a lot). Occam's yields the bound $\binom{n}{n/2}$ samples.

Claim 2. *It can be shown that $2^{\tilde{O}(\sqrt{n})}$ is enough over the uniform distribution (HW problem).*

1.2 Monotone functions and Dictator functions

We will show that all monotone functions have a weak agreement with some dictator function ($f(x) = x_i$).

Theorem 3. *For all monotone functions f, there exists some $g \in \{\pm 1, x_1, x_2, \ldots, x_n\} = \mathcal{S}$ such that*

$$\Pr_{x \in \text{unif dist}}[f(x) = g(x)] \geq \frac{1}{2}\Omega\left(\frac{1}{n}\right)$$

Proof. We take some samples of f and test against elements of $\mathcal{S}$ until we find one that works (this is essentially equivalent to learning weight 1 Fourier coefficients).

We first introduce a definition and some other theorems:

Definition 4. *The **influence** of f*

$$\inf(f) = \frac{\#\text{red-blue edges in } i\text{th direction}}{2^{n-1}} = \Pr_x[f(x) \neq f(x^{\oplus i})]$$

Theorem 5. *If f is monotone,*

$$\inf(f) = \hat{f}(\{i\})$$

Theorem 6. *The majority function*

$$f(x) = \text{sign}\left(\sum x_i\right)$$

for odd n maximizes influence among monotone functions.

Theorems 5 and 6 will be proved on the homework.

Case 1: $f(x)$ has weak agreement with $+1$ or -1. Then we are done.

Case 2: Otherwise,

$$\Pr[f(x) = 1] \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Then use the canonical path argument to lower bound the number of red-blue edges.

Canonical Path Argument

i. For every red-blue pair of nodes, define a canonical path which must cross at least one red-blue edge (ignoring the direction of edges)

ii. Show an upper bound on number of canonical paths passing through any edge (in particular, any red-blue edge)

iii. Conclude a lower bound on the number of red-blue edges

Part I

For all x red and y blue, define the "canonical path from x to y" as follows:

Scan bits from left to right, taking an edge to flip a bit when needed.

For example:

$$\begin{array}{llcccc}
\text{dimension} & & 1 & 2 & 3 & 4 \\
\text{red} & x = & -1 & +1 & +1 & +1 \\
 & & \downarrow & & & \\
 & w = & +1 & +1 & +1 & +1 \\
 & & & \downarrow & & \\
 & z = & +1 & -1 & +1 & +1 \\
 & & & & & \downarrow \\
\text{blue} & y = & +1 & -1 & +1 & -1
\end{array}$$

Because we are in Case 2,

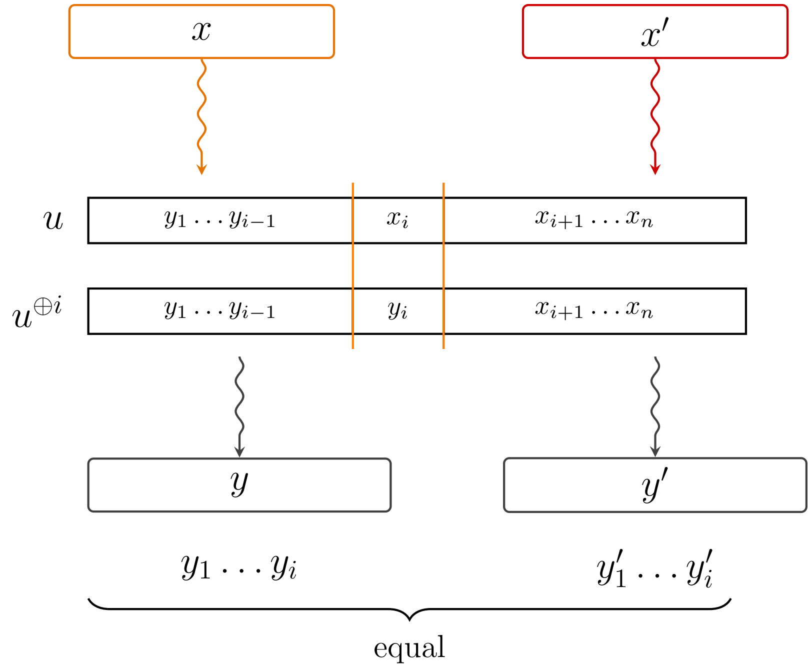
$$\#\text{reds} \geq \frac{1}{4} \cdot 2^n, \quad \#\text{blues} \geq \frac{1}{4} \cdot 2^n$$

We have a path between any pair of red and blue nodes, so

$$\#\text{paths} \geq \left(\frac{1}{4} \cdot 2^n\right)^2 = \frac{1}{16} \cdot 2^{2n}$$

Part II

Any pair of x, x' with any prefix, but the same suffix can use the edge. Likewise, any pair of y, y' with the same prefix can use the edge.



Then there are $\leq 2^n$ total settings of prefix of x ($\leq 2^i$ settings) and suffix of y ($\leq 2^{n-i}$ settings) consistent with the edge. This means there are $\leq 2^n$ canonical paths that can use that edge.

Part III

Note that the number of red-blue canonical paths is at most the product of the number of red-blue edges and the maximum number of canonical paths that can use that edge:

$$\#\text{red-blue paths} \leq (\#\text{red-blue edges}) \cdot (\text{max \# of paths that can use that edge})$$

This means that

$$\#\text{red-blue edges} \geq \frac{\frac{1}{16} \cdot 2^{2n}}{2^n} = \frac{1}{16} \cdot 2^n.$$

Then there must exists i such that the number of red-blue edges in direction i is at least $\frac{2^n}{16n}$ by averaging.

Hence

$$\inf_i(f) \geq \frac{\frac{2^n}{16n}}{2^{n-1}} = \frac{1}{8n}.$$

From Theorem 5 we have

$$\inf(f) = \hat{f}(\{i\}) = 2 \cdot \Pr[f(x) = \chi_{\{i\}}(x)] - 1 = 2\Pr[f(x) = x_i] - 1$$

so

$$\inf_i(f) \geq \frac{1}{n} \implies \Pr[f(x) = x_i] \geq \frac{1}{2} + \Omega\left(\frac{1}{n}\right).$$

Thus

$$\Pr[f(x) = x_i] \geq \frac{1}{2} + \frac{1}{16n} = \frac{1}{2} + \Omega\left(\frac{1}{n}\right).$$

$\square$