6.045

Lecture 10: Undecidability, Unrecognizability, and Reductions

Next Thursday (3/19)

Your Midterm: 2:35-3:55pm, 32-144 + 155

No pset this week!

Just an optional (not graded) practice midterm

Solutions to practice midterm will come out with the practice midterm. Also all HW solutions.

When you see the practice midterm…

DON'T PANIC!

Practice midterm will be harder than midterm

Next Thursday (3/19)

Your Midterm: 2:35-3:55pm, 32-144 + 155

No pset this week!

Just an optional (not graded) practice midterm

FAQ: What material is on the midterm?

Everything up to Thursday (Lectures 1-11)

FAQ: Can I bring notes?

Yes, one single-sided sheet of notes, US letter paper

A TM *M* *recognizes* a language L
if *M* **accepts** exactly those strings in L

A language L is *recognizable*
(a.k.a. recursively enumerable)
if some TM **recognizes** L

A TM *M* *decides* a language L if *M* **accepts** all
strings in L and **rejects** all strings not in L

A language L is *decidable (a.k.a. recursive)*
if some TM **decides** L

L(M) := set of strings M accepts

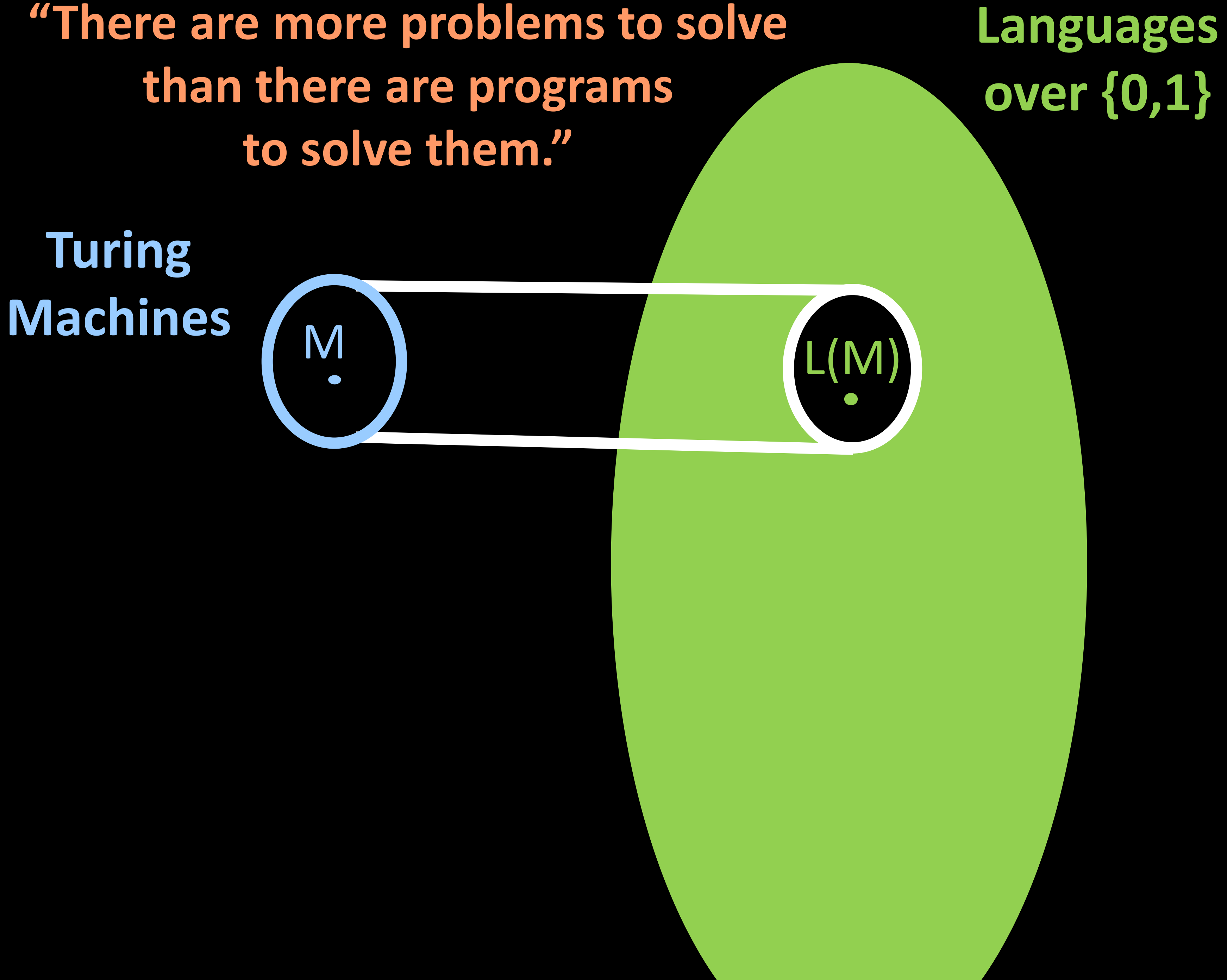
Thm: There are *unrecognizable* languages

Assuming the Church-Turing Thesis, this means there are problems that *NO* computing device will *ever* solve!

The proof will be very NON-CONSTRUCTIVE:
We will prove there is no *onto* function from the set of all Turing Machines to the set of all languages over {0,1}.
(But the proof will work for any *finite* Σ)

Therefore, the function mapping every TM M to its language L(M), *fails to cover all possible languages*

“There are more problems to solve than there are programs to solve them.”



$f : A \to B$ is *not* onto $\Leftrightarrow$ $(\exists b \in B)(\forall a \in A)[f(a) \neq b]$

Let L be any set and 2^L be the power set of L

Theorem: There is *no* onto function from L to 2^L

Proof: Let $f : L \to 2^L$ be arbitrary

Define $S = \{ x \in L \mid x \notin f(x) \} \in 2^L$

Claim: For all $x \in L$, $f(x) \neq S$

For all $x \in L$, observe that

$x \in S$ if and only if $x \notin f(x)$ [by definition of S]

Therefore $f(x) \neq S$:

the element x is in *exactly one* of those sets!

Therefore f is *not* onto!

What does this mean?

No function from L to 2^L
can "cover" all the elements in 2^L

No matter what the set L is,
the power set 2^L *always* has
strictly larger cardinality than L
(and all subsets of L)

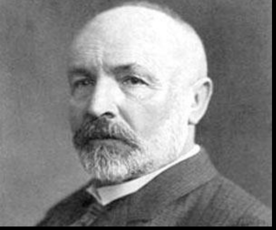
Thm: There are *unrecognizable* languages

Proof: Suppose all languages are recognizable.
Then for all L, there's a TM M that recognizes L.
Thus the function R: {Turing Machines} $\rightarrow$ {Languages}
defined by M $\mapsto$ L(M) is an onto function.

{Turing Machines} $\subseteq$ {0,1}* = Set T

{Languages over {0,1}} $\leftrightarrow$ {All possible subsets of {0,1}*} = Set of all subsets of T: 2^T

But we showed there is *no* onto function from
{Turing Machines} $\subseteq$ T to its power set 2^T. Contradiction!



Theorem: There is no onto function from the positive integers $\mathbb{Z}^+$ to the real numbers in (0, 1)

{0,1}* Languages over {0,1}

Proof:

Suppose f is such a function. Then we can make a list:

1 ⟶ 0.28347279…
2 ⟶ 0.88388384…
3 ⟶ 0.77635284…
4 ⟶ 0.11111111…
5 ⟶ 0.12345678…
⋮ ⋮

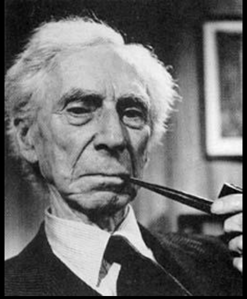
Diagonalization

Define: $r \in (0, 1)$

$$[\text{ n-th digit of } r\text{ }] = \begin{cases} 1 & \text{if [n-th digit of } f(n)\text{]} \neq 1 \\ 2 & \text{otherwise} \end{cases}$$

$f(n) \neq r$ for all n (Here, r = 0.11121...)

r is never output by f!



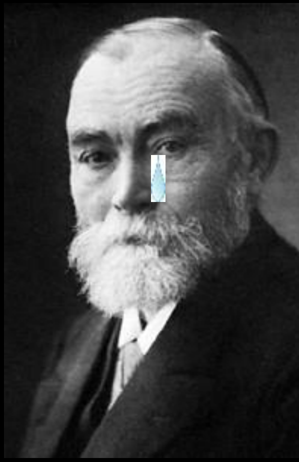
Russell's Paradox in Set Theory

In the early 1900's, logicians were trying to define consistent foundations for mathematics.

Suppose X = "Universe of all possible sets"

Frege's Axiom: Let $f : X \to \{0,1\}$

Then $\{S \in X \mid f(S) = 1\}$ is a set.



Russell: Define $F = \{ S \in X \mid S \notin S \}$

Suppose $F \in F$. Then by definition, $F \notin F$.

So $F \notin F$ and by definition $F \in F$.

This logical system is inconsistent!

Thm: There are *unrecognizable* languages

A Concrete Undecidable Problem: The Acceptance Problem for TMs

$A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM that accepts string } w \}$

Given: code of a Turing machine M and
an input w for that Turing machine,
Decide: Does M accept w?

Theorem [Turing]:
A_{TM} is recognizable but NOT decidable

A_{TM} = { ⟨M,w⟩ | M is a TM that accepts string w }

Thm. A_{TM} is undecidable: (proof by contradiction)

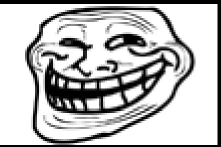
Assume H is a machine that decides A_{TM}

$$H(\langle M,w\rangle) = \begin{cases} \text{Accept} & \text{if } M \text{ accepts } w \\ \text{Reject} & \text{if } M \text{ does not accept } w \end{cases}$$

Define a new TM D with the following spec:

$\boxed{\text{D(⟨M⟩): Run H on ⟨M,M⟩ and output the } \textit{opposite} \text{ of H}}$

$$D(\langle D\rangle) = \begin{cases} \text{Reject} & \text{if } D \text{ accepts } \langle D\rangle \\ \text{Accept} & \text{if } D \text{ does not accept } \langle D\rangle \end{cases}$$



Set M=D?

The table of outputs of H on $\langle x,y \rangle$

x \ y	w_1	w_2	w_3	w_4 …	D
M_1	accept	accept	accept	reject	accept
M_2	reject	accept	reject	reject	reject
M_3	accept	reject	reject	accept	accept
M_4	accept	reject	reject	reject	accept
:					
D	reject	reject	accept	accept	?

$M_1, M_2, \ldots$ and $w_1, w_2, \ldots$ are both ordered lists of all binary strings

The table of outputs of H on ⟨x,y⟩

x \ y	w_1	w_2	w_3	w_4	…	D
M_1	accept	accept	accept	reject		accept
M_2	reject	accept	reject	reject		reject
M_3	accept	reject	reject	accept		accept
M_4	accept	reject	reject	reject		accept
:						
D	reject	reject	accept	accept		?

D on ⟨x⟩ outputs the *opposite* of H on ⟨x,x⟩

The behavior of D(x) is a *diagonal* on this table

	w_1	w_2	w_3	w_4 …	D
M_1	**reject**	accept	accept	reject	accept
M_2	reject	**reject**	reject	reject	reject
M_3	accept	reject	**accept**	accept	accept
M_4	accept	reject	reject	**accept**	accept
:					
D	reject	reject	accept	accept	?

D on ⟨x⟩ outputs the *opposite* of H on ⟨x,x⟩

D on ⟨D⟩ outputs the *opposite* of D on ⟨D⟩

$A_{TM} = \{ \langle M,w \rangle \mid M \text{ is a TM that accepts string } w \}$

Thm. A_{TM} **is undecidable.** (a constructive proof)

Let U be a machine that **recognizes** A_{TM}

$$U(\langle M,w \rangle) = \begin{cases} \text{Accept} & \text{if } M \text{ accepts } w \\ \text{Rejects or loops} & \text{otherwise} \end{cases}$$

Define a new TM D_U as follows:

$D_U(\langle M \rangle)$: Run U on $\langle M,M \rangle$ until it halts.
Output the opposite answer

$$D_U(\langle D_U \rangle) = \begin{cases} \text{Reject if } D_U \text{ accepts } \langle D_U \rangle \\ \text{(i.e. if } H(D_U, D_U) = \text{Accept)} \\ \\ \text{Accept if } D_U \text{ rejects } \langle D_U \rangle \\ \text{(i.e. if } H(D_U, D_U) = \text{Reject)} \\ \\ \text{Loops if } D_U \text{ loops on } \langle D_U \rangle \\ \text{(i.e. if } H(D_U, D_U) \text{ loops)} \end{cases}$$

Note: There is **no** contradiction here!

D_U must run forever on $\langle D_U \rangle$

We have an input $\langle D_U, D_U \rangle$ which is *not* in A_{TM} but U infinitely loops on $\langle D_U, D_U \rangle$!

In summary:

Given the code of any machine U that *recognizes* A_{TM} (i.e. a Universal Turing Machine) we can effectively construct an input $\langle D_U, D_U \rangle$, where:

1. **$\langle D_U, D_U \rangle \notin A_{TM}$ (D_U does not accept D_U)**
2. **U *runs forever* on the input $\langle D_U, D_U \rangle$**

Therefore U cannot *decide* A_{TM}

Given any universal Turing machine, we can efficiently construct an input on which the program hangs!

Note how generic this argument is: it does not depend on Turing machines!

A Concrete Unrecognizable Problem: The "Non-Acceptance Problem" for TMs

A_{TM} = { ⟨M, w⟩ | M encodes a TM over some Σ,
w encodes a string over Σ
and M accepts w}

We choose a decoding of pairs, TMs, and strings so that *every* binary string decodes to *some* TM M and string w

If $z \in \{0,1\}^*$ doesn't decode to ⟨M, w⟩ in the usual way, then we *define* that z decodes to a TM D and ε
where D is a "dummy" TM that accepts nothing.

Then, $\neg A_{TM}$ = { z | z decodes to M and w
such that M does not accept w }

A Concrete Unrecognizable Problem: The "Non-Acceptance Problem" for TMs

A TM *M* *recognizes* a language L
if *M* accepts exactly those strings in L
(but could run forever on other strings)

A TM *M* *decides* a language L if *M* accepts all strings in L and rejects all strings not in L

Theorem: L is decidable

$\Leftrightarrow$ L and $\neg$L are recognizable

Recall: Given $L \subseteq \Sigma^*$, define $\neg L := \Sigma^* \setminus L$

Theorem: L is **decidable**

$\Leftrightarrow$ L and $\neg L$ are **recognizable**

($\Leftarrow$) Given: a TM M_1 that recognizes L and

a TM M_2 that recognizes $\neg L$,

we want to build a new machine M that *decides* L

How? Any ideas?

Hint: M_1 always accepts x, when x is in L

M_2 always accepts x, when x isn't in L

Recall: Given $L \subseteq \Sigma^*$, define $\neg L := \Sigma^* \setminus L$

Theorem: **L** is **decidable**

$\Leftrightarrow$ **L** and $\neg L$ are **recognizable**

($\Leftarrow$) Given: a TM M_1 that recognizes L and

a TM M_2 that recognizes $\neg L$,

we want to build a new machine M that *decides* L

M(x): Run M_1 (x) and M_2 (x) on separate tapes.

Alternate between simulating one step

of M_1 , and one step of M_2 .

If M_1 ever accepts, then accept

If M_2 ever accepts, then reject

Theorem: A_{TM} is recognizable but **NOT** decidable

Corollary: $\neg A_{TM}$ is not recognizable!

Proof: Suppose $\neg A_{TM}$ is recognizable.
Then $\neg A_{TM}$ and A_{TM} are both recognizable…
But that would mean they're both decidable!
Contradiction!

The Halting Problem [Turing]

$HALT_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM that } \textit{halts} \text{ on string } w\}$

Theorem: $HALT_{TM}$ is undecidable

Proof: Assume (for a contradiction) there is a TM H that decides $HALT_{TM}$

Idea: Use H to construct a TM M' that *decides* A_{TM}

M'($\langle M, w\rangle$): Run H($\langle M, w\rangle$)

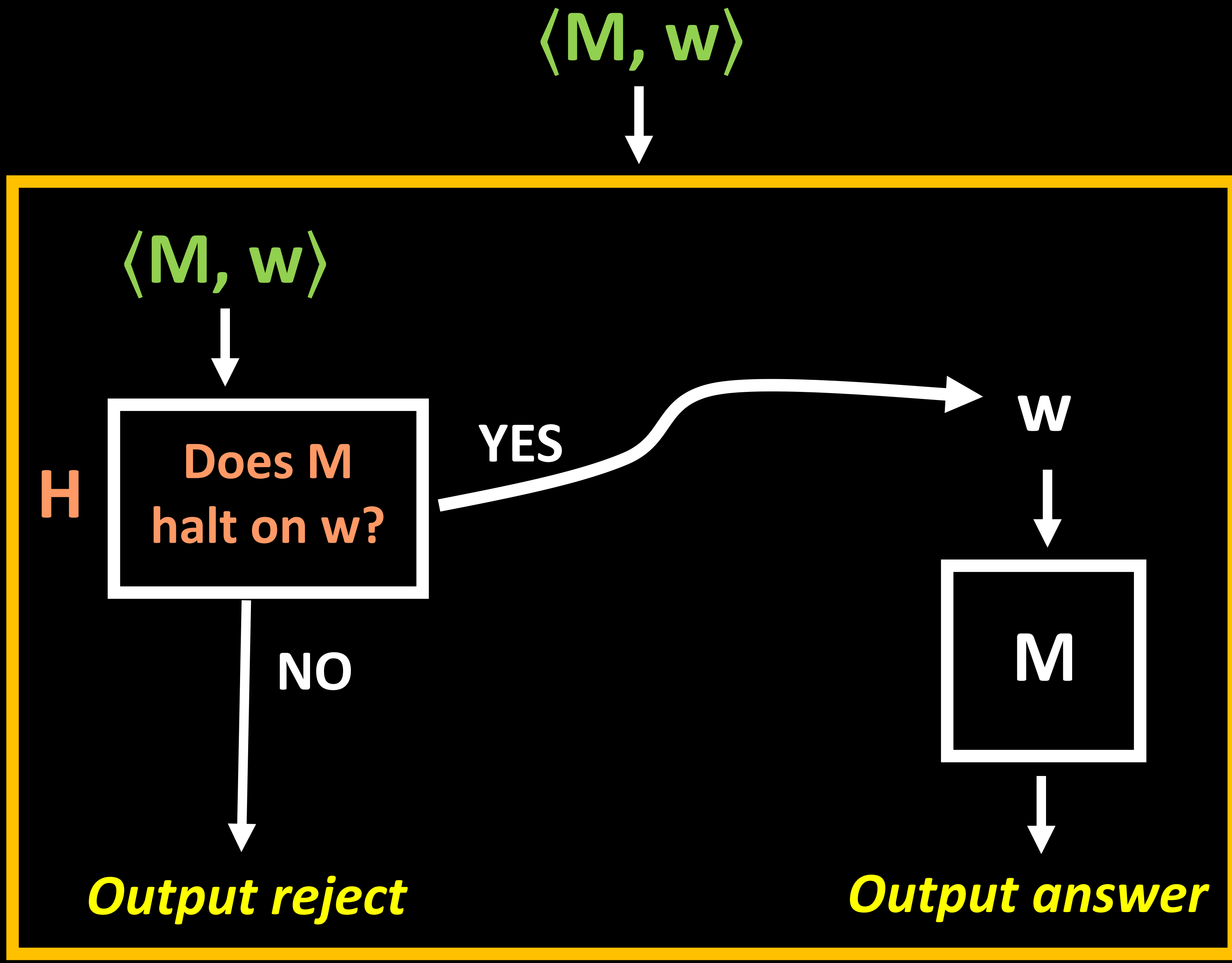
If H rejects then *reject*

If H accepts, run M on w until it halts:

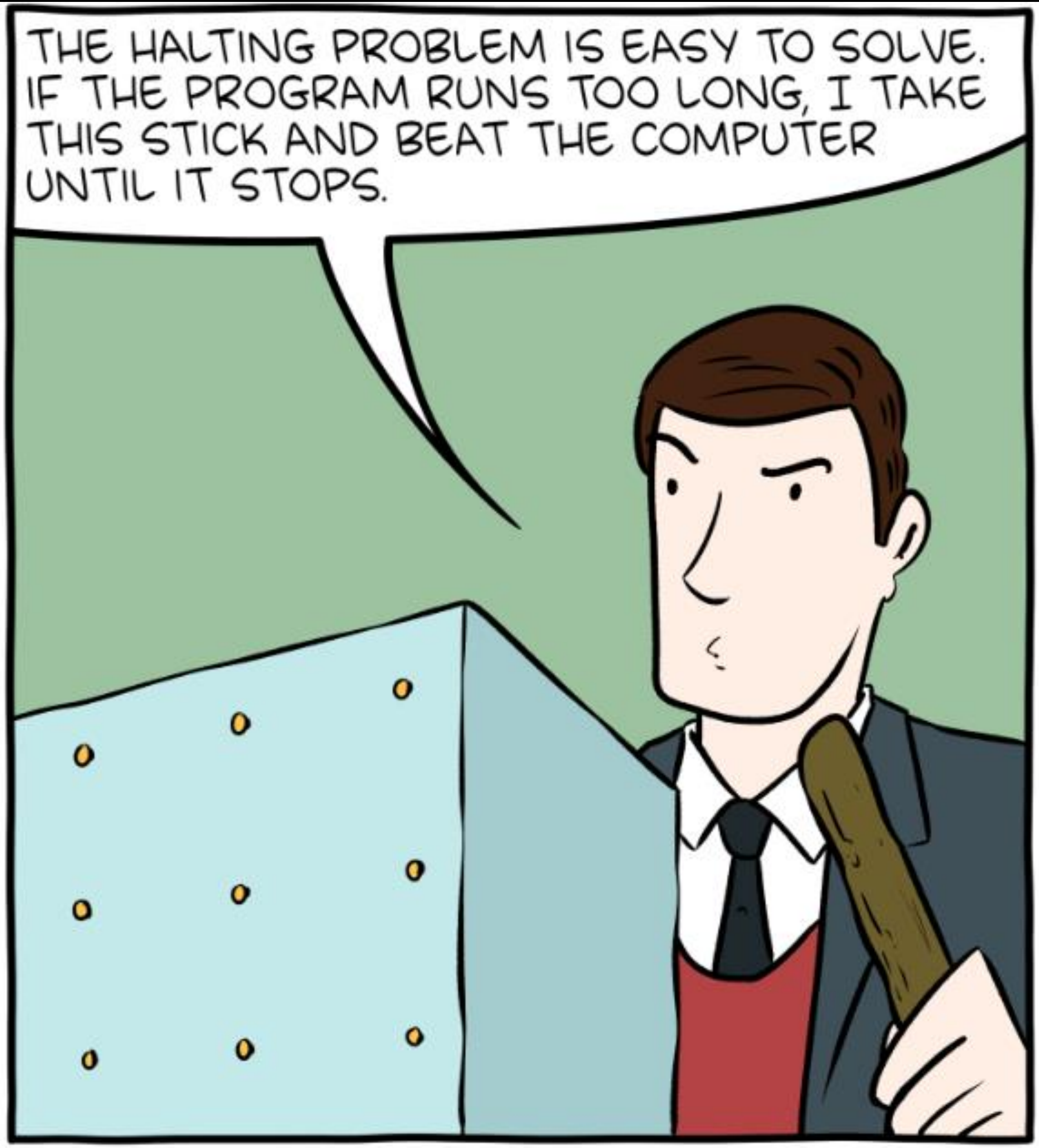
If M accepts, then *accept*

If M rejects, then *reject*

Claim: If H exists, then M' decides A_{TM} $\Rightarrow$ H does not exist!



M’ decides A_{TM}



What if Alan Turing had been an engineer?

http://smbc-comics.com/comic/halting

Public Health Announcement

R. Ryan Williams
@rrwilliams

6.045 health reminder: wash your hands for the time it takes to prove that the Halting problem is undecidable.

10:55 AM · Mar 10, 2020 · Twitter for Android

R. Ryan Williams @rrwilliams · 8m
Replying to @rrwilliams

"Suppose Turing machine H can decide, given any string (M,w), whether TM M halts on w. Define a TM D which, on input (M), runs H on (M,M) and halts iff H rejects. So D on (D) halts iff H on (D,D) rejects iff D on (D) does not halt. D cannot both halt and not halt. Contradiction!"

The previous proof is one example of a MUCH more general phenomenon.

Can often prove a language L is undecidable by proving: "If L is decidable, then so is A_{TM}"

We *reduce* A_{TM} to the language L:

$$A_{TM} \leq L$$

Intuition: L is "at least as hard as" A_{TM}

Given the ability to solve problem L, we can solve A_{TM}

Motivating Example

Theorem [Turing]: $HALT_{TM}$ is undecidable

Proof: Assume some TM H decides $HALT_{TM}$

We'll make an M' that decides A_{TM}

M'(⟨M, w⟩): Run H on ⟨M, w⟩

If H rejects then *reject*

If H accepts, run M on w until it halts:

If M accepts, then *accept*

If M rejects, then *reject*

This is called a TURING REDUCTION:

Using a TM for deciding $HALT_{TM}$ we could decide A_{TM}

Reducing One Problem to Another

$f : \Sigma^* \to \Sigma^*$ is a **computable function** if
there is a Turing machine M that halts with
just $f(w)$ written on its tape, for every input w

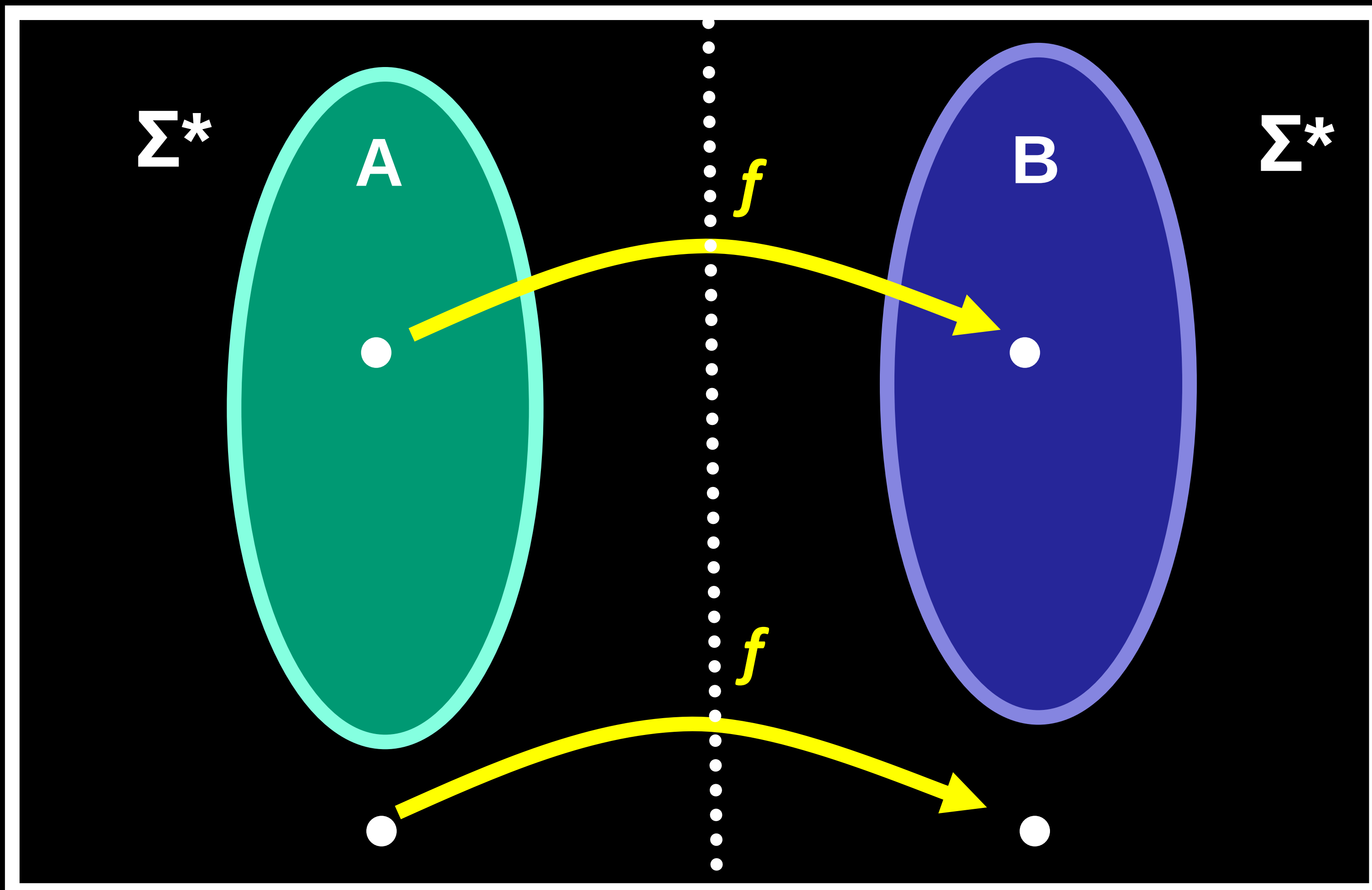
A language A is *mapping reducible* to language B,
written as $A \leq_m B$, if there is a computable
$f : \Sigma^* \to \Sigma^*$ such that for every $w \in \Sigma^*$,

$$w \in A \Leftrightarrow f(w) \in B$$

f is called a mapping reduction
(or many-one reduction) from A to B

Let $f : \Sigma^* \rightarrow \Sigma^*$ be a **computable function**

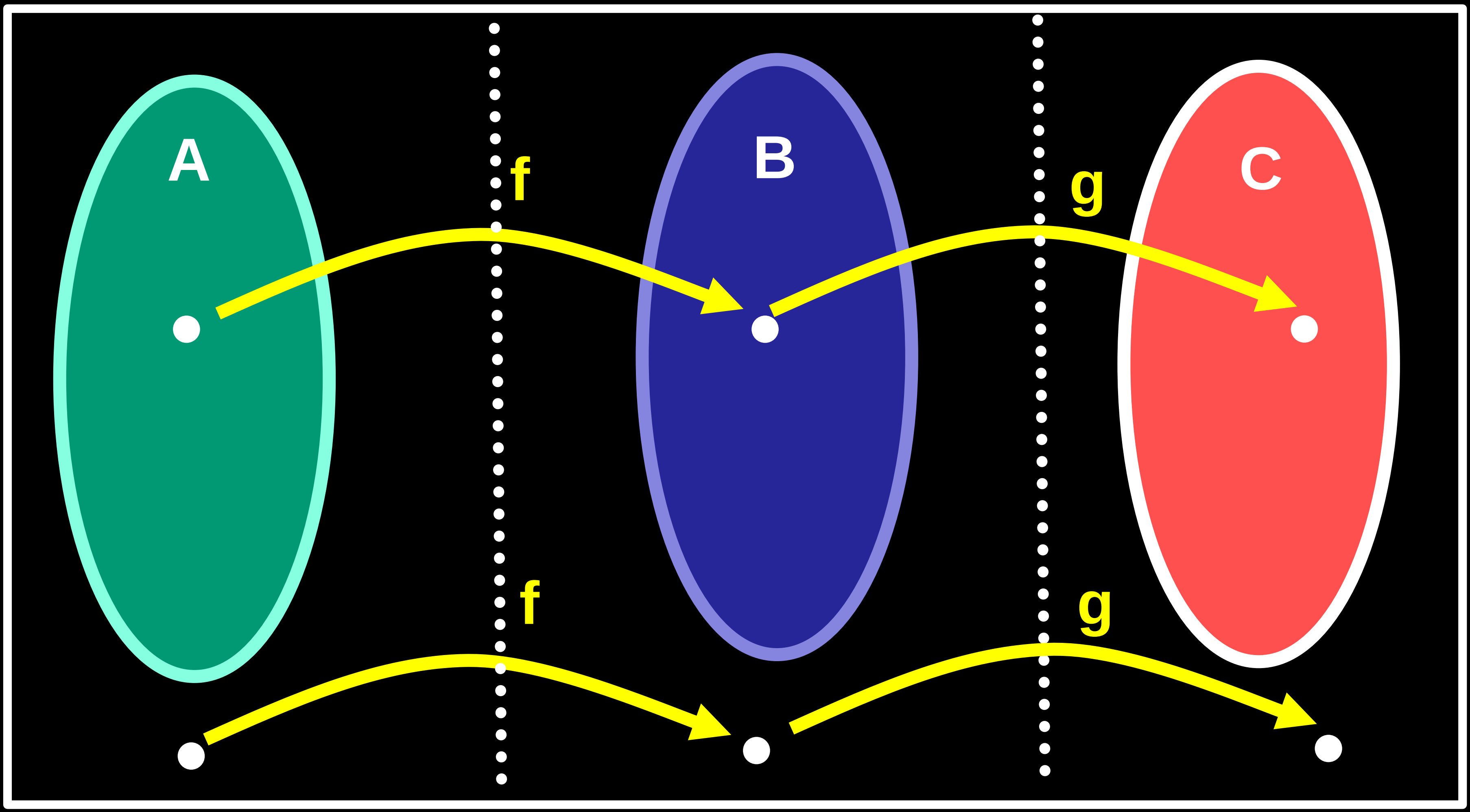
such that **for all $w \in \Sigma^*$, $w \in A \Leftrightarrow f(w) \in B$**



Say: "A is mapping reducible to B"

Write: $A \leq_m B$

Theorem: If $A \leq_m B$ and $B \leq_m C$, then $A \leq_m C$



$$w \in A \Leftrightarrow f(w) \in B \Leftrightarrow g(f(w)) \in C$$

Some (Simple) Examples

$A_{DFA} = \{ \langle D, w \rangle \mid D$ encodes a DFA over some Σ, and D accepts $w \in \Sigma^* \}$

$A_{NFA} = \{ \langle N, w \rangle \mid N$ encodes an NFA, N accepts $w \}$

Theorem: For every regular language L', $L' \leq_m A_{DFA}$

For every regular L', there's a DFA D for L'.

So here's a mapping reduction f from L' to A_{DFA}:

$f(w) :=$ Output $\langle D,w \rangle$

Then, $w \in L' \Leftrightarrow D$ accepts $w \Leftrightarrow f(w) = \langle D, w \rangle \in A_{DFA}$

So f is a mapping reduction from L' to A_{DFA}

Some (Simple) Examples

A_{DFA} = { $\langle D, w\rangle$ | D encodes a DFA over some Σ, and D accepts $w \in \Sigma^*$ }

A_{NFA} = { $\langle N, w\rangle$ | N encodes an NFA, N accepts w }

Theorem: $A_{DFA} \leq_m A_{NFA}$

Every DFA can be trivially written as an NFA.

So here's a reduction f from A_{DFA} to A_{NFA} :

f($\langle D, w\rangle$) := Write down NFA N equivalent to D
Output $\langle N, w\rangle$

Theorem: $A_{NFA} \leq_m A_{DFA}$

f($\langle N, w\rangle$) := Use subset construction to convert NFA N into an equivalent DFA D. Output $\langle D, w\rangle$

Theorem: If $A \leq_m B$ and B is decidable,
then A is decidable

"If A is as hard as B, and B is decidable, then A is decidable"

Proof: Let M decide B.
Let f be a mapping reduction from A to B

We build a machine M' deciding A as follows:

$M'(w)$:

1. Compute $f(w)$
2. Run M on $f(w)$, output its answer

Then: $w \in A \Leftrightarrow f(w) \in B$ [since f reduces A to B]

$\Leftrightarrow M$ accepts $f(w)$ [since M decides B]

$\Leftrightarrow M'$ accepts w [by def of M']

Theorem: If $A \leq_m B$ and B is recognizable, then A is recognizable

Proof: Let M recognize B.
Let f be a mapping reduction from A to B

To *recognize* A, we build a machine M'

M'(w):

1. Compute f(w)
2. Run M on f(w), output its answer if you ever receive one

Theorem: If $A \leq_m B$ and B is decidable, then A is decidable

Corollary: If $A \leq_m B$ and A is undecidable, then B is undecidable

Theorem: If $A \leq_m B$ and B is recognizable, then A is recognizable

Corollary: If $A \leq_m B$ and A is unrecognizable, then B is unrecognizable