6.045

Lecture 11: Fun With Undecidability!

Announcements

- If MIT is denying you resources you need and you're running out of options, please contact me personally.

- Pset now due Monday March 30: we will release pset solutions immediately after that
- For now, midterm is still Thursday April 2
- Practice midterm + solutions out tonight!
- There is candy!

The Acceptance Problem for TMs

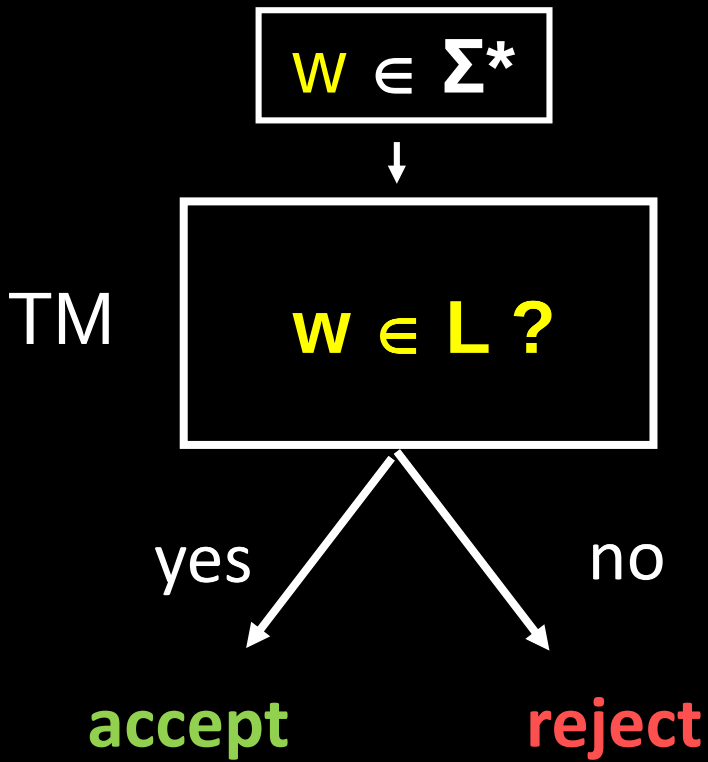
A_{TM} = { ⟨M, w⟩ | M is a TM that accepts string w }

Given: code of a Turing machine M and
an input w for that Turing machine,

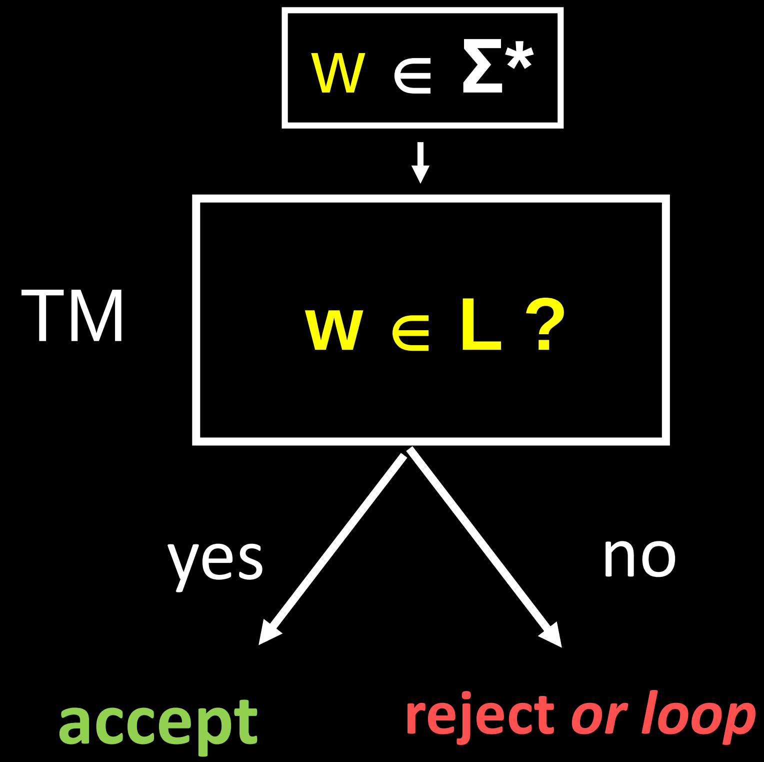
Decide: Does M accept w?

A_{TM} decidable ⇒ There is an algorithm ALG which,
given *any* code and input,
ALG determines in finite time
if the code will stop and accept the input

Theorem [Turing]:
A_{TM} is recognizable, but NOT decidable!



L is decidable



L is recognizable

Theorem: L is decidable
iff both L and ¬L are recognizable

Theorem: L is **decidable**
iff both L and $\neg$L are **recognizable**

Theorem: A_{TM} is recognizable but NOT decidable

Corollary: $\neg A_{TM}$ is not recognizable!

Theorem: $HALT_{TM}$ is not decidable

Reducing One Problem to Another

$f : \Sigma^* \rightarrow \Sigma^*$ is a **computable function** if
there is a Turing machine M that halts with
just $f(w)$ written on its tape, for every input w

A language A is *mapping reducible* to language B,
written as $A \leq_m B$, if there is a computable
$f : \Sigma^* \rightarrow \Sigma^*$ such that for every $w \in \Sigma^*$,

$$w \in A \Leftrightarrow f(w) \in B$$

f is called a mapping reduction
(or many-one reduction) from A to B

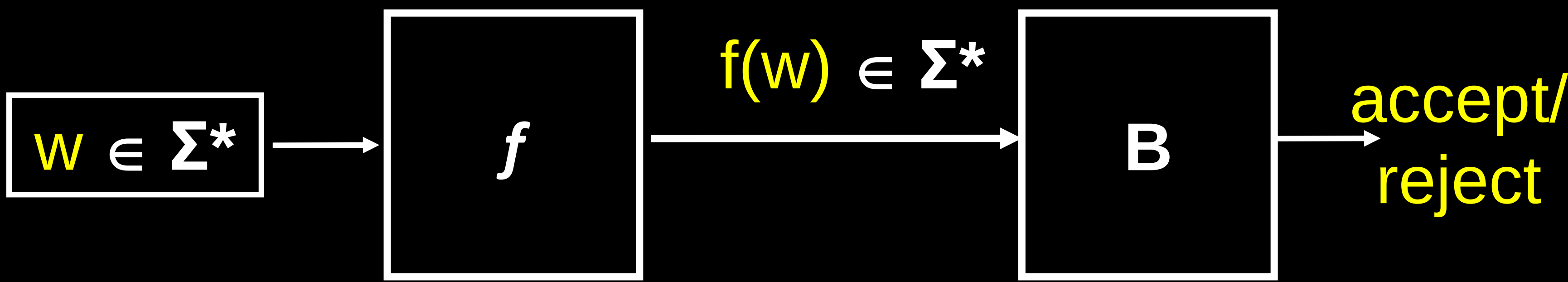
Theorem: If $A \leq_m B$ and B is decidable, then A is decidable

Corollary: If $A \leq_m B$ and A is undecidable, then B is undecidable

Theorem: If $A \leq_m B$ and B is recognizable, then A is recognizable

Corollary: If $A \leq_m B$ and A is unrecognizable, then B is unrecognizable

Theorem: If $A \leq_m B$ and B is decidable, then A is decidable



$$w \in A \Leftrightarrow f(w) \in B$$

A recipe for proving undecidability!

To prove B is undecidable, find undecidable A and a mapping reduction from A to B.

A mapping reduction from A_{TM} to $HALT_{TM}$

Theorem: $A_{TM} \leq_m HALT_{TM}$

f(z) := Decode z into a pair $\langle M, w \rangle$. Write down the description of a TM M' with the spec:

"M'(w) = Run M on w.

If M accepts, then *accept,* else *loop forever*"

Output the encoding $\langle M', w \rangle$

Then, $z = \langle M, w \rangle \in A_{TM} \Leftrightarrow$ M accepts w

$\Leftrightarrow$ M' halts on w $\Leftrightarrow \langle M', w \rangle \in HALT_{TM}$

Corollary: $HALT_{TM}$ is undecidable

Theorem: $A_{TM} \leq_m HALT_{TM}$

Corollary: $\neg A_{TM} \leq_m \neg HALT_{TM}$

Corollary: $\neg HALT_{TM}$ is unrecognizable!

Proof: If $\neg HALT_{TM}$ were recognizable, then $\neg A_{TM}$ would also be recognizable, because $\neg A_{TM} \leq_m \neg HALT_{TM}$. But $\neg A_{TM}$ is not!

Question: $A_{TM} \leq_m \neg A_{TM}$?

Theorem: $HALT_{TM} \leq_m A_{TM}$

Theorem: $\mathrm{HALT}_{\mathrm{TM}} \leq_m A_{\mathrm{TM}}$

Proof: Define a mapping reduction f:

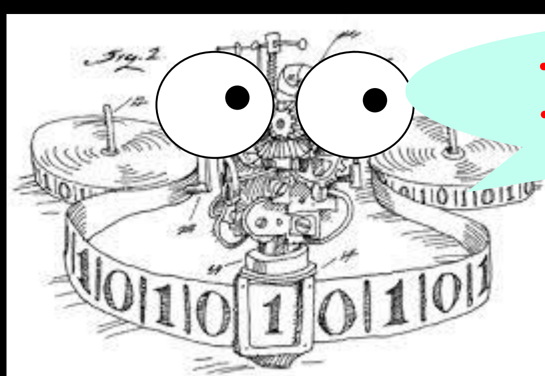
f(z) := Decode z into a pair $\langle M, w \rangle$
Write down a TM M' with the specification:
"M'(w) = Run M on w. If M halts, *accept*"
Output $\langle M', w \rangle$

Observe $z = (M, w) \in \mathrm{HALT}_{\mathrm{TM}} \Leftrightarrow (M', w) \in A_{\mathrm{TM}}$

Corollary: $HALT_{TM} \equiv_m A_{TM}$

Yo, T.M.! I can give you the magical power to either compute the halting problem, or the acceptance problem. Which do you want?

Wow, hm, so hard to choose…



I can't decide!

The Emptiness Problem for TMs

$\text{EMPTY}_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a TM such that } L(M) = \varnothing \}$

Given a program, does it reject or loop on all inputs?

Theorem: EMPTY_{TM} is *unrecognizable*

Proof: Show that $\neg A_{\text{TM}} \leq_m \text{EMPTY}_{\text{TM}}$

f(z) := Decode z into $\langle M, w \rangle$. Output code of the TM:

"M'(x) := if (x = w) then run M(w) and output answer, else reject"

Observe: EITHER $L(M') = \varnothing$ OR $L(M') = \{w\}$

$z=(M,w) \notin A_{\text{TM}} \Leftrightarrow$ M doesn't accept w

$\Leftrightarrow L(M') = \varnothing$

$\Leftrightarrow \langle M' \rangle \in \text{EMPTY}_{\text{TM}} \Leftrightarrow f(z) \in \text{EMPTY}_{\text{TM}}$

The Emptiness Problem for Other Models

$\text{EMPTY}_{\text{DFA}} = \{ \langle M \rangle \mid M \text{ is a DFA such that } L(M) = \varnothing \}$

Given a DFA, does it reject every input?

Theorem: $\text{EMPTY}_{\text{DFA}}$ is decidable

Why?

$\text{EMPTY}_{\text{NFA}} = \{ \langle M \rangle \mid M \text{ is a NFA such that } L(M) = \varnothing \}$

$\text{EMPTY}_{\text{REX}} = \{ \langle R \rangle \mid M \text{ is a regexp such that } L(M) = \varnothing \}$

Moral:

Analyzing Programs is Really, Really Hard for Programs to Do.

(Sometimes)

Computing With Oracles: Another Kind of Reduction

*We do not condone smoking. Don't do it. It's bad. Kthxbye

Oracle Turing Machines

q_{YES}

Is (M, w) in A_{TM}?

yes!

A	N	P	U	T					

INFINITE TAPE

Now leaving reality for a moment….

Oracle Turing Machines

An oracle Turing machine M is equipped with a set $B \subseteq \Gamma^*$ and a special oracle tape, on which M may ask membership queries about B

Formally, M enters a special state $q_?$ to ask a query

and the TM receives a query answer in one step

[Formally, the transition function on $q_?$ is defined in terms of the *entire oracle tape*:

State $q_?$ changes to q_{YES}

if the string y written on the oracle tape is in B,

else $q_?$ changes to q_{NO}]

This notion makes sense even if B is not decidable!

How to Think about Oracles?

Think in terms of Turing Machine pseudocode!

An oracle Turing machine M with oracle $B \subseteq \Gamma^*$ lets you include the following kind of if-then statement:

```
"if (z in B) then <do something>
              else <do something else>"
```

where `z` is some string defined earlier in pseudocode.
We define the oracle TM to that it can always check the condition `(z in B)` in one step

This notion makes sense even if B is not decidable!

Deciding one problem with another

Definition: A is decidable with B
if there is an *oracle TM M with oracle B*
that accepts strings in A and rejects strings not in A

Language A "Turing-Reduces" to B

$$A \leq_T B$$

A_{TM} is decidable with $HALT_{TM}$ ($A_{TM} \leq_T HALT_{TM}$)

We can decide if M accepts w
using an ORACLE for the Halting Problem:

On input (M,w),
If (M,w) is in $HALT_{TM}$ then
run M(w) and output its answer.
else REJECT.

(This is exactly like our proof that
$HALT_{TM}$ is undecidable, from last lecture!)

$HALT_{TM}$ is decidable with A_{TM} ($HALT_{TM} \leq_T A_{TM}$)

On input (M,w), decide if M halts on w as follows:

1. If (M,w) is in A_{TM} then ACCEPT

2. Else, swap the accept and reject states of M to get a machine M′. If (M′,w) is in A_{TM} then ACCEPT

3. REJECT

Theorem: If $A \leq_T B$ and B is decidable, then A is decidable

Corollary: If $A \leq_T B$ and A is undecidable, then B is undecidable

Proof: Exactly the same proof as the one for mapping reductions!

If $A \leq_T B$ then there is a TM M with oracle B that decides A. If B is decidable, then we can replace every oracle call to B with a TM that decides B. Now M is a TM with no oracle!

$\leq_T$ versus $\leq_m$

Theorem: If $A \leq_m B$ then $A \leq_T B$

Proof (Sketch):

$A \leq_m B$ means there is a computable function $f : \Sigma^* \rightarrow \Sigma^*$, where for every w,

$$w \in A \Leftrightarrow f(w) \in B$$

To decide A on an input w with oracle B, just compute f(w), then call B on f(w) and return answer

Theorem: $\neg A_{TM} \leq_T A_{TM}$

D ($\langle M,w \rangle$): If ($\langle M,w \rangle$ in A_{TM}) then *reject* else *accept*

Theorem: $\neg A_{TM} \not\leq_m A_{TM}$

The Busy Beaver Function

How much work can a little TM do?

The Busy Beaver Function

Define a simple Turing machine to be one with input alphabet {1}, tape alphabet {1,□}, and a "halt state". Besides the "halt state", our TMs have n other states.

Define BB(n) to be the maximum number of steps taken on input ε by any n-state simple TM that halts.

BB(1) = 1: For a 1-state TM running on blank tape, it either halts in the first step, or it runs forever!

BB(2) = 6

BB(3) = 21

BB(4) = 107

BB(5) $\geq$ 47,176,870

BB(6) > $10^{36,534}$

BB(7) > $10^{10^{10^{10^{18,000,000}}}}$

The Busy Beaver Function

Theorem: BB(n) is not computable!

BB(n) grows so ridiculously fast that no computable function whatsoever *(no function you have ever seen)* can even *upper bound* it!!

First Idea: If you could compute BB(n), then you could solve the Halting problem for simple TMs running on blank tape!

Second Idea: It is impossible to decide that Halting problem

The Busy Beaver Function

Theorem: BB(n) is not computable!

Theorem: Assuming there is a TM computing BB(n), we can solve the Halting problem for simple TMs on ε.

Proof: Here's pseudocode for the Halting problem:

On the input ⟨M⟩ [code of a TM M]

Count the number of states in M, call it n

Compute t = BB(n) [in binary or unary]

Run M on blank tape for t steps.

If it halts, then accept. Otherwise, reject!

Theorem: There is NO computable function $f : \mathbb{N} \to \mathbb{N}$ such that for all n, $f(n) \geq$ BB(n).

The Busy Beaver Function

You can encode arbitrary math conjectures in simple TMs!

Theorem: There is a 1919-state simple TM that halts iff ZFC (set theory) is inconsistent!

There is a 744-state simple TM that halts iff the Riemann hypothesis is false.

There is a 43-state simple TM that halts iff Goldbach's conjecture is false

Good luck verifying if those halt!

Two Problems

Problem 1 Undecidable

{ (M, w) | M is a TM that on input w, tries to move its head past the left end of the tape at some point }

Problem 2 Decidable

{ (M, w) | M is a TM that on input w, moves its head left at some point}

Problem 1 Undecidable

L’ = { (M, w) | M is a TM that on input w, tries to move its head past the left end of the tape }

Proof: Reduce A_{TM} to L’

On input (M,w),
make a TM N that shifts w over one cell,
puts a special symbol # on the leftmost cell,
then simulates M(w) on its tape.
If M’s head moves to the cell with # but has *not yet accepted*, N moves the head back to the right.
If M accepts, N tries to move its head past the #.

(M,w) is in A_{TM} if and only if (N,w) is in L’

Problem 2 **Decidable**

{ (M, w) | M is a TM that on input w, moves its head left at some point}

On input (M,w), run M on w for $|Q| + |w| + 1$ steps,

where $|Q|$ = number of states of M

Accept If M's head moved left at all

Reject Otherwise

(Why does this work?)

Thank you all …

… see you in June, hopefully?

Be safe!