6.045

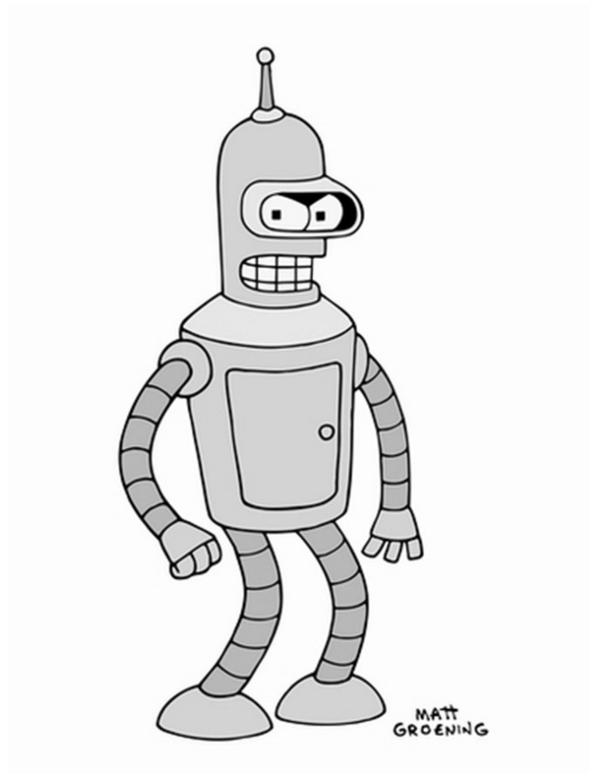
Lecture 3: Nondeterminism and Regular Expressions

6.045

Announcements:

- **Pset 0 is out, due tomorrow 11:59pm**
  - **Latex source of hw on piazza**
  - **Pset 1 coming out tomorrow**
- **No class next Tuesday** ***(…because next week Monday classes will be on Tuesday)***

Deterministic **F**inite **A**utomata



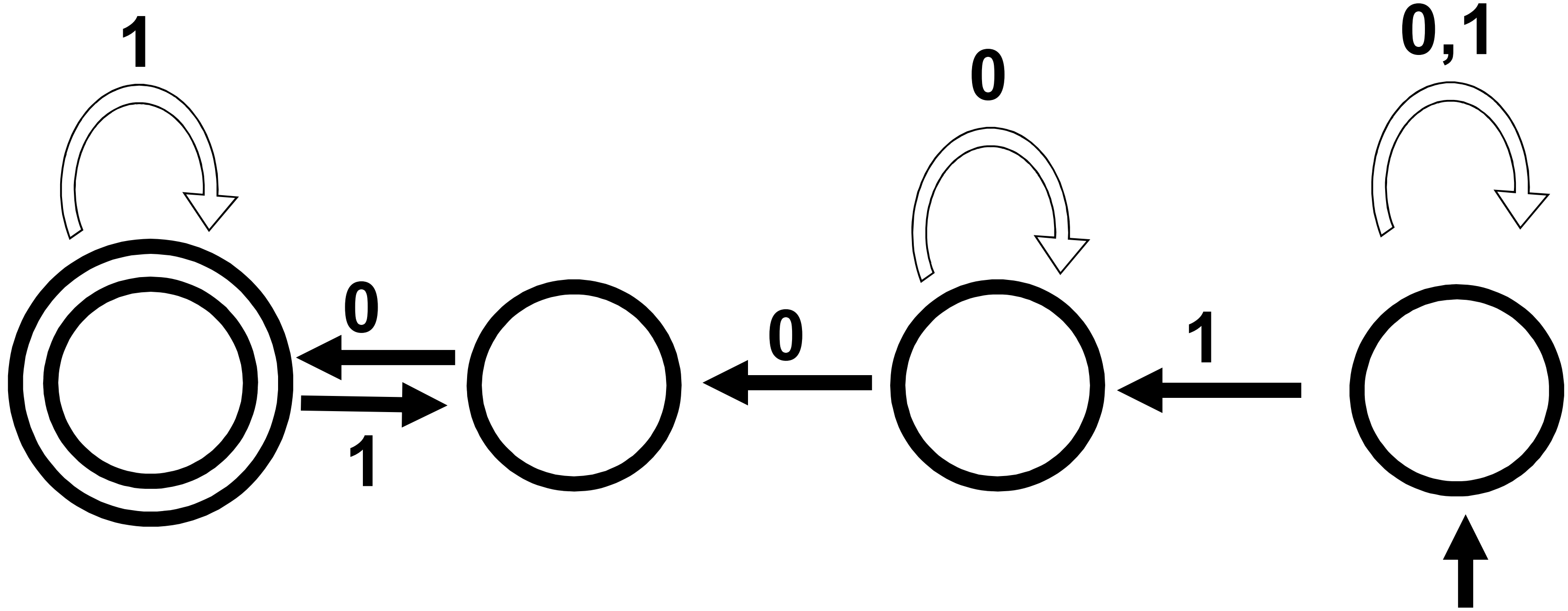
Computation with finite memory

Non-Deterministic **F**inite **A**utomata

Computation with finite memory

and magical guessing

Non-deterministic Finite Automata (NFA)



This NFA recognizes: {w | w contains 100}

An NFA accepts string x
if *there is some path reading in x* that
reaches *some accept state* from *some start state*

Every NFA can be perfectly simulated by some DFA!

Theorem: For every NFA N, there is a DFA M such that L(M) = L(N)

Corollary: A language A is regular if and only if A is recognized by an NFA

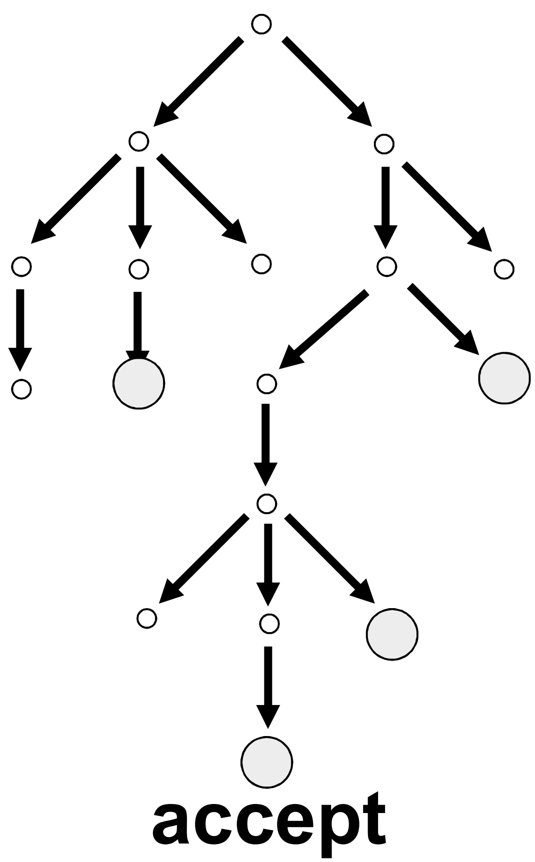
Corollary: A is regular iff A^R is regular

left-to-right DFAs $\equiv$ right-to-left DFAs

From NFAs to DFAs

Input: NFA N = $(Q, \Sigma, \delta, Q_0, F)$

Output: DFA M = $(Q', \Sigma, \delta', q_0', F')$



To learn if NFA N accepts, we could do the computation of N *in parallel*, maintaining the set of *all* possible states that can be reached

Idea:

Set $Q' = 2^Q$

From NFAs to DFAs: Subset Construction

Input: NFA $N = (Q, \Sigma, \delta, Q_0, F)$

Output: DFA $M = (Q', \Sigma, \delta', q_0', F')$

$$Q' = 2^Q$$

$$\delta' : Q' \times \Sigma \rightarrow Q'$$

For $S \in Q'$, $\sigma \in \Sigma$: $\delta'(S,\sigma) = \bigcup_{q \in S} \varepsilon(\delta(q,\sigma))$ *

$$q_0' = \varepsilon(Q_0)$$

$$F' = \{ S \in Q' \mid f \in S \text{ for } \textit{some } f \in F \}$$

* For **$S \subseteq Q$**, the **ε-closure of S is**
$\varepsilon(S)$ = {**r** ∈ **Q** reachable from some **q** ∈ **S**
by taking zero or more **ε**-transitions}

Reverse Theorem for Regular Languages

The reverse of a regular language
is also a regular language

If **a language can be recognized by a DFA that reads strings from *right* to *left*,**
then **there is an "normal" DFA that accepts the same language**

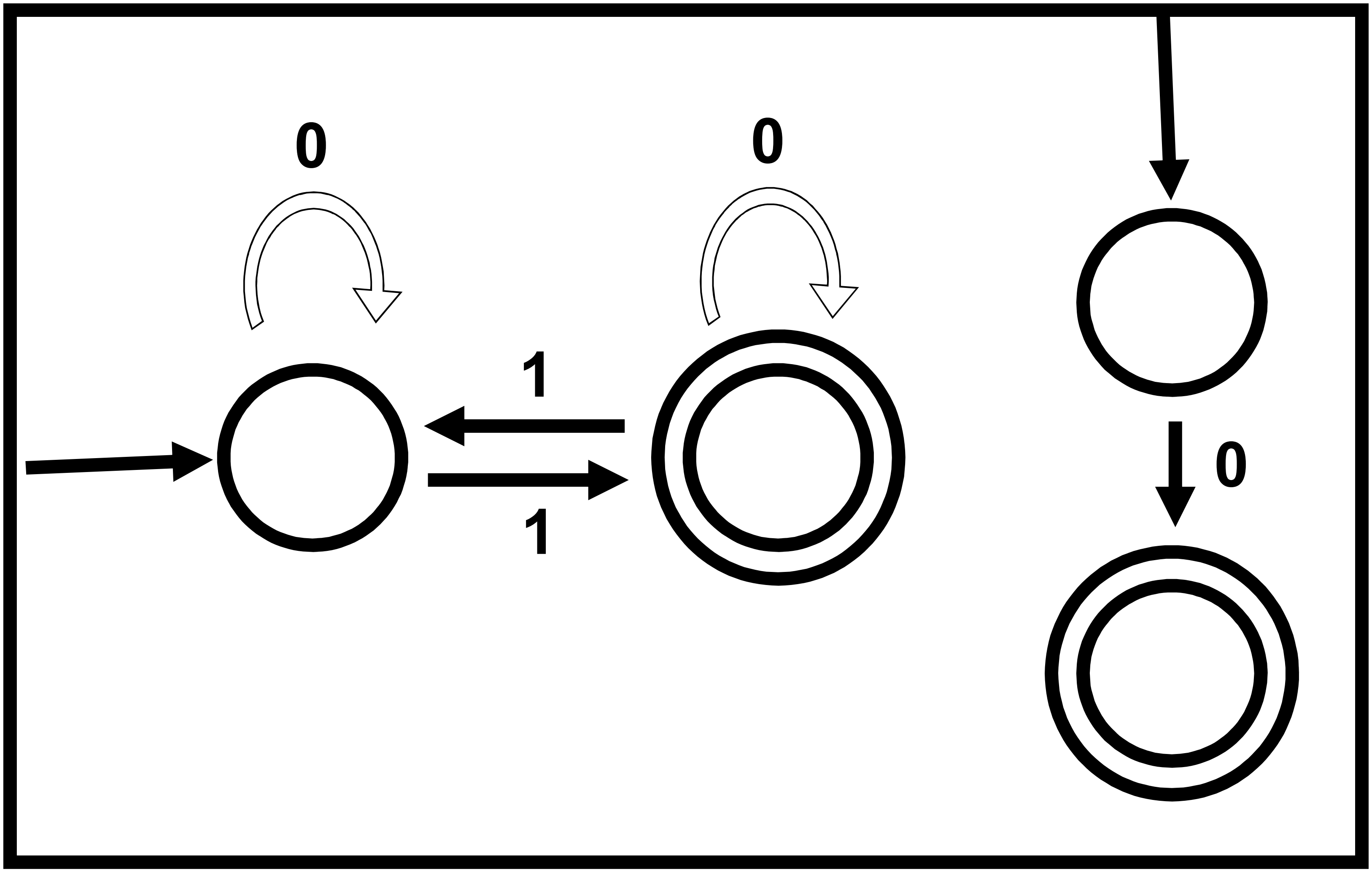
Proof Sketch?

Given a DFA for a language L, "reverse" its arrows, and flip its start and accept states, getting an NFA. Convert that NFA back to a DFA!

Using NFAs in place of DFAs can make proofs about regular languages *much* easier!

Remember this on homework/exams!

Union Theorem using NFAs?



Some Operations on Languages

- Union: $A \cup B = \{ w \mid w \in A \text{ or } w \in B \}$
- Intersection: $A \cap B = \{ w \mid w \in A \text{ and } w \in B \}$
- Complement: $\neg A = \{ w \in \Sigma^* \mid w \notin A \}$
- Reverse: $A^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in A, w_i \in \Sigma \}$

Concatenation: $A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \}$

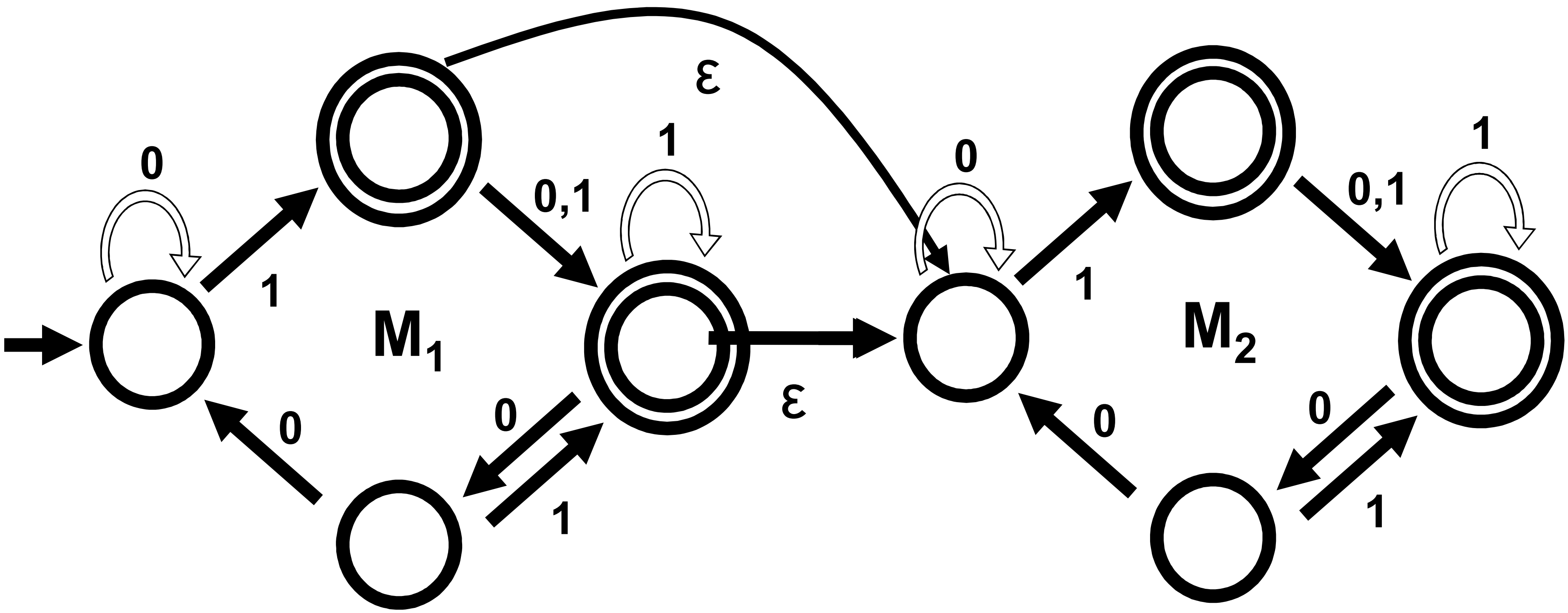
Star: $A^* = \{ s_1 \ldots s_k \mid k \geq 0 \text{ and each } s_i \in A \}$

A^* = set of all strings over alphabet A

Regular Languages are closed under concatenation

Concatenation: $A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \}$

Given DFAs M_1 for A and M_2 for B, connect the accept states of M_1 to the start state of M_2



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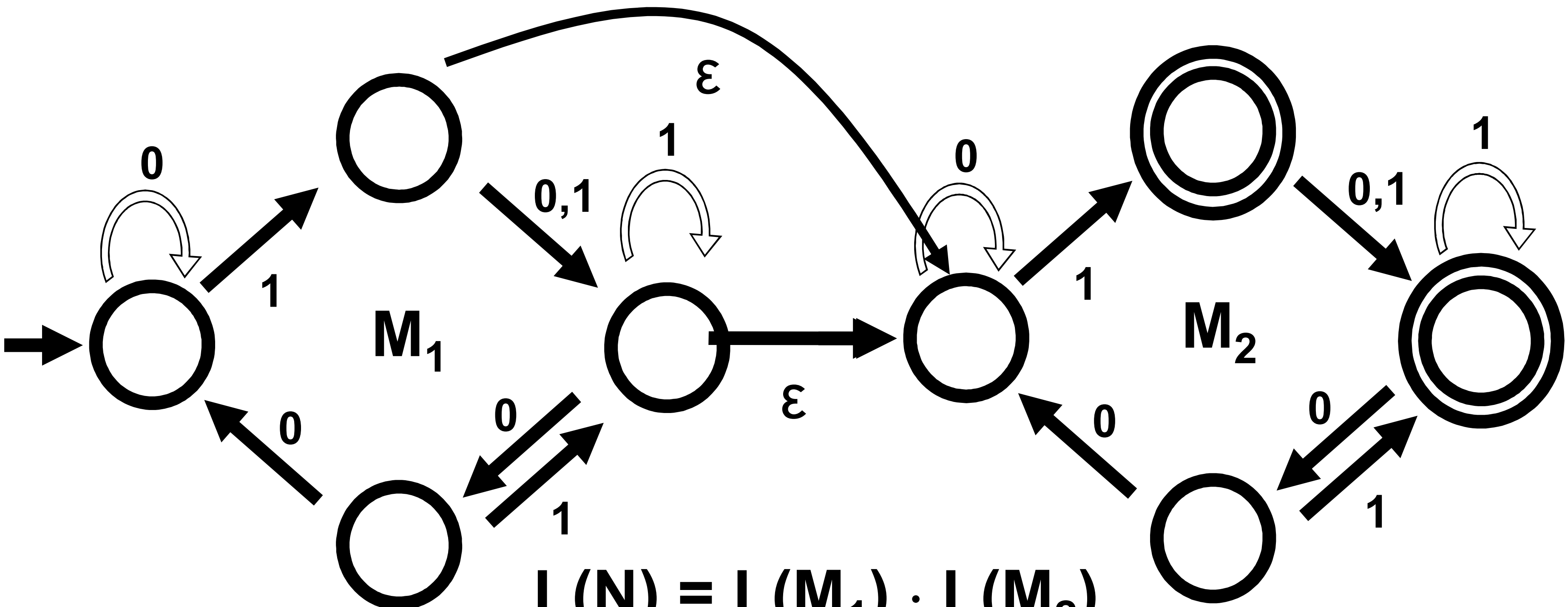


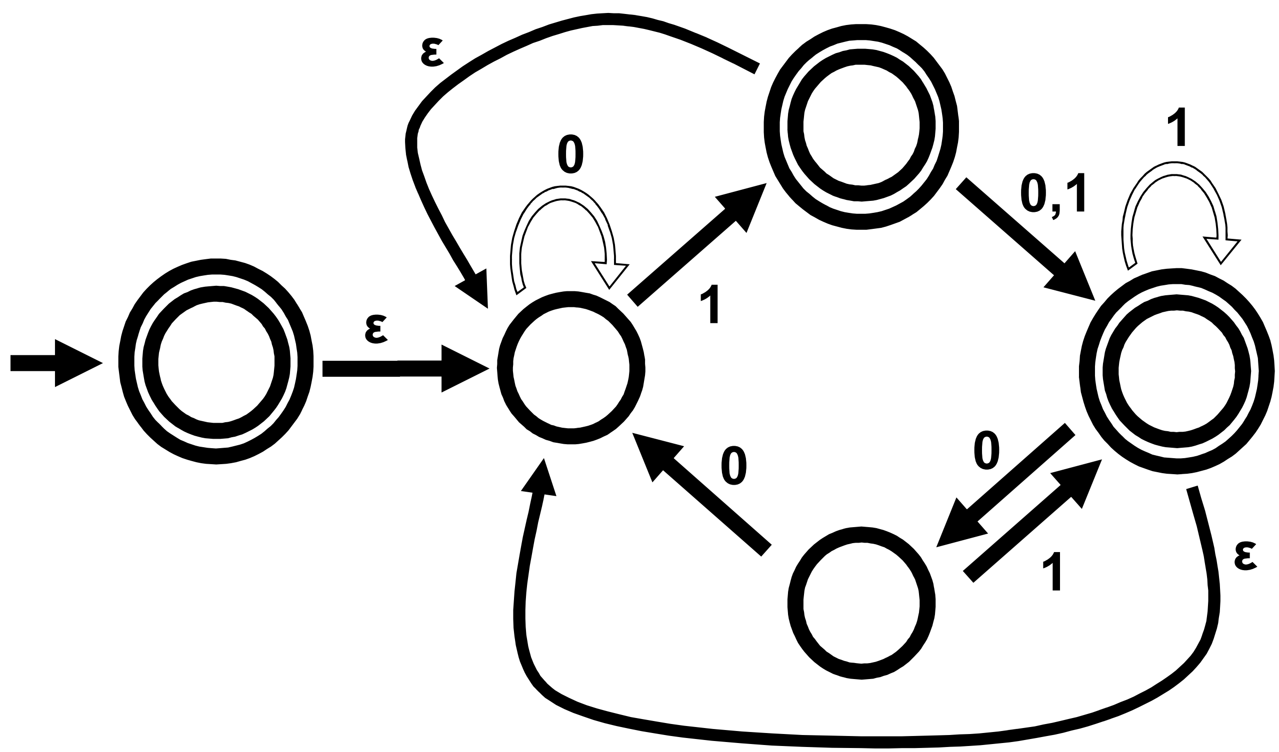
$L(N) = L(M_1) \cdot L(M_2)$

Regular Languages are closed under star

$A^* = \{ s_1 \ldots s_k \mid k \geq 0 \text{ and each } s_i \in A \}$

Let M be a DFA

We construct an NFA N that recognizes L(M)*



Formally, the construction is:

Input: DFA $M = (Q, \Sigma, \delta, q_1, F)$

Output: NFA $N = (Q', \Sigma, \delta', \{q_0\}, F')$

$$Q' = Q \cup \{q_0\}$$

$$F' = F \cup \{q_0\}$$

$$\delta'(q,a) = \begin{cases} \{\delta(q,a)\} & \text{if } q \in Q \text{ and } a \neq \varepsilon \\ \{q_1\} & \text{if } q \in F \text{ and } a = \varepsilon \\ \{q_1\} & \text{if } q = q_0 \text{ and } a = \varepsilon \\ \varnothing & \text{if } q = q_0 \text{ and } a \neq \varepsilon \\ \varnothing & \text{else} \end{cases}$$

Regular Languages are closed under star

How would we *prove* that the NFA construction works? 

Want to show: L(N) = L(M)*

1. $L(N) \supseteq L(M)^*$

2. $L(N) \subseteq L(M)^*$

1. $L(N) \supseteq L(M)^*$

Let $w = w_1 \cdots w_k$ be in $L(M)^*$ where $w_1, \ldots, w_k \in L(M)$

We show: N accepts w by induction on k

Base Cases:

- ✓ **$k = 0$ ($w = \varepsilon$)**
- ✓ **$k = 1$ ($w \in L(M)$ and $L(M) \subseteq L(N)$)**

Inductive Step: Let $k \geq 1$ be an integer

I.H. N accepts all strings $v = v_1 \cdots v_k \in L(M)^*$, $v_i \in L(M)$

Let $u = u_1 \cdots u_k u_{k+1} \in L(M)^*$, $u_j \in L(M)$

N accepts $u_1 \cdots u_k$ (by I.H.) **and M accepts u_{k+1}**

imply that N also accepts u

(since N has ε-transitions from final states to start state of M!)

2. $L(N) \subseteq L(M)^*$

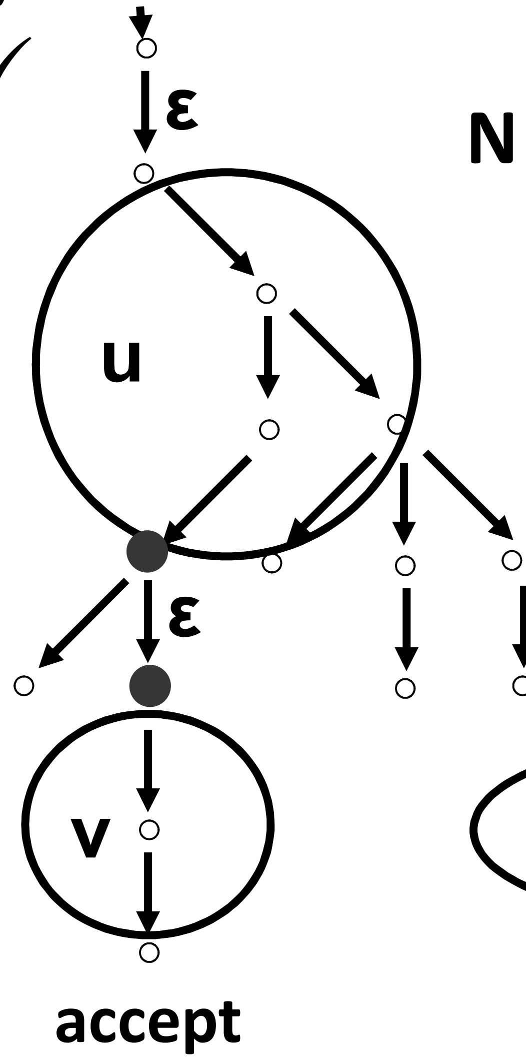
Let w be accepted by N; we want to show $w \in L(M)^*$

If $w = \varepsilon$, then $w \in L(M)^*$ ✓

I.H. N accepts u and takes *k ε-transitions* $\Rightarrow u \in L(M)^*$

Let w be accepted by N with k+1 *ε-transitions*.

Write w as w=uv, where v is the substring read after the *last* ε-transition



N accepts u, so

$u \in L(M)^*$

By I.H.

$w = uv \in L(M)^*$

$v \in L(M)$

Regular Languages are closed under all of the following operations:

Union: $A \cup B = \{ w \mid w \in A \text{ or } w \in B \}$

Intersection: $A \cap B = \{ w \mid w \in A \text{ and } w \in B \}$

Complement: $\neg A = \{ w \in \Sigma^* \mid w \notin A \}$

Reverse: $A^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in A, w_i \in \Sigma \}$

Concatenation: $A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \}$

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Regular Expressions: Computation as Description

A different way of thinking about computation:

What is the complexity of describing the strings in the language?

Syntax

Inductive Definition of Regexp

Let Σ be an alphabet. We define the regular expressions over Σ inductively:

For all $\sigma \in \Sigma$, σ is a regexp

ε is a regexp

$\varnothing$ is a regexp

If R_1 and R_2 are both regexps, then

(R_1R_2), $(R_1 + R_2)$, and $(R_1)^*$ are regexps

Examples: ε, 0, (1)*, (0+1)*, ((((0)*1)*1) + (10))

Precedence Order: $*$

then $\cdot$

then $+$

Example: $R_1{*}R_2 + R_3 = ((R_1{*}) \cdot R_2) + R_3$

Semantics

Definition: Regexps Describe Languages

The regexp $\sigma \in \Sigma$ *represents* the language $\{\sigma\}$

The regexp ε represents $\{\varepsilon\}$

The regexp $\varnothing$ represents $\varnothing$

If R_1 and R_2 are regular expressions representing L_1 and L_2 then:

(R_1R_2) represents $L_1 \cdot L_2$

$(R_1 + R_2)$ represents $L_1 \cup L_2$

$(R_1)^*$ represents L_1^*

Example: $(10 + 0^*1)$ represents $\{10\} \cup \{0^k1 \mid k \geq 0\}$

Regexps Describe Languages

For every regexp R,
define L(R) to be the language that R represents

A string $w \in \Sigma^*$ is *accepted by R*
(or, *w matches R)* if $w \in L(R)$

Examples: 0, 010, and 01010 match (01)*0

110101110101100 matches (0+1)*0

$\boxed{\textbf{Assume } \Sigma = \{0,1\}}$

{ w | w has exactly a single 1 }

0*10*

{ w | w contains 001 }

(0+1)*001(0+1)*

$$\boxed{\text{Assume } \Sigma = \{0,1\}}$$

What language does the regexp $\varnothing^*$ represent?

$\{\varepsilon\}$

Assume $\Sigma = \{0,1\}$

$\{ w \mid w$ has length ≥ 3 and its 3rd symbol is 0 $\}$

$$(0+1)(0+1)0(0+1)^*$$

Assume $\Sigma = \{0,1\}$

$\{ w \mid w = \varepsilon$ or every odd position in w is a 1 $\}$

$$(1(0 + 1))^*(1 + \varepsilon)$$

How expressive are regular expressions?

During the “nerve net” hype in the 1950s…

U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

REPRESENTATION OF EVENTS IN NERVE NETS AND
FINITE AUTOMATA

S. C. Kleene

RM-704

15 December 1951

DFAs $\equiv$ NFAs $\equiv$ Regular Expressions!

L can be represented by some regexp

$\Leftrightarrow$ L is regular

L can be represented by some regexp
$\Rightarrow$ L is regular

Induction Step: Suppose every regexp of length $< k$ represents some regular language.

Consider a regexp R of length $k > 1$

Three possibilities for R:

$R = R_1 + R_2$

$R = R_1 R_2$

$R = (R_1)^*$

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Three possibilities for R:

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$R = R_1 R_2$

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By induction, R_1 and R_2 represent some regular languages, L_1 and L_2

But $L(R) = L(R_1 + R_2) = L_1 \cup L_2$

so L(R) is regular, by the union theorem!

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Consider a regexp R of length k > 1

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$R = R_1 + R_2$

$R = R_1 R_2$

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By induction, R_1 and R_2 represent some regular languages, L_1 and L_2

But $L(R) = L(R_1 \cdot R_2) = L_1 \cdot L_2$

Thus L(R) is regular because regular languages are closed under concatenation

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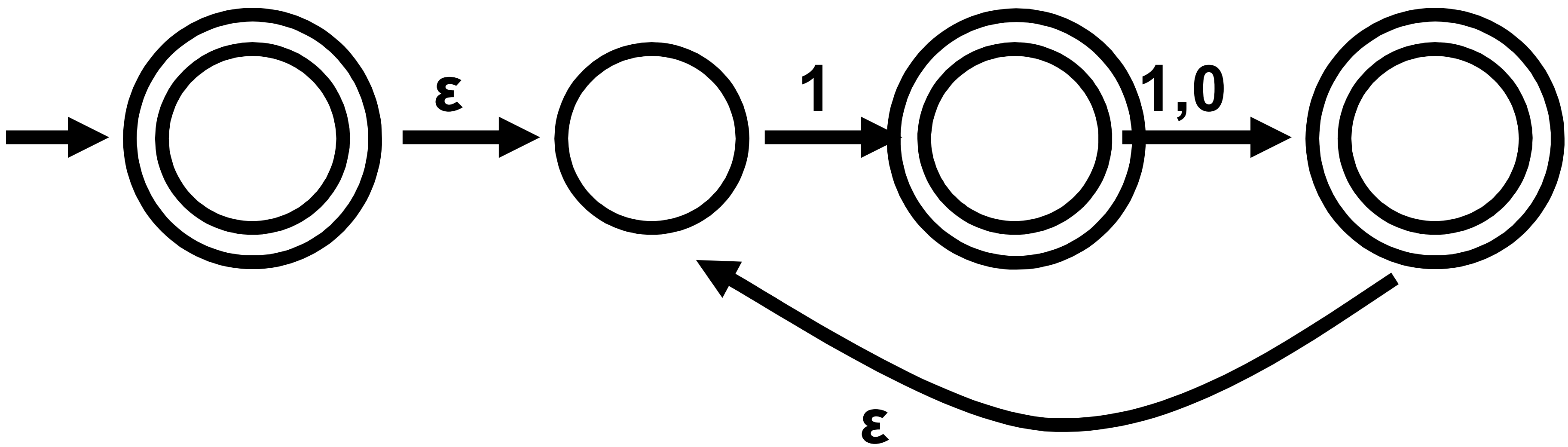
But $L(R) = L(R_1^*) = L_1^*$

Thus L(R) is regular because regular languages are closed under star

Therefore: If L is represented by a regexp, then L is regular!

Give an NFA that accepts the language represented by (1(0 + 1))*



Regular expression: (1 (0+1))*

Generalized NFAs (GNFA)

L can be represented by a regexp

$\Leftarrow$

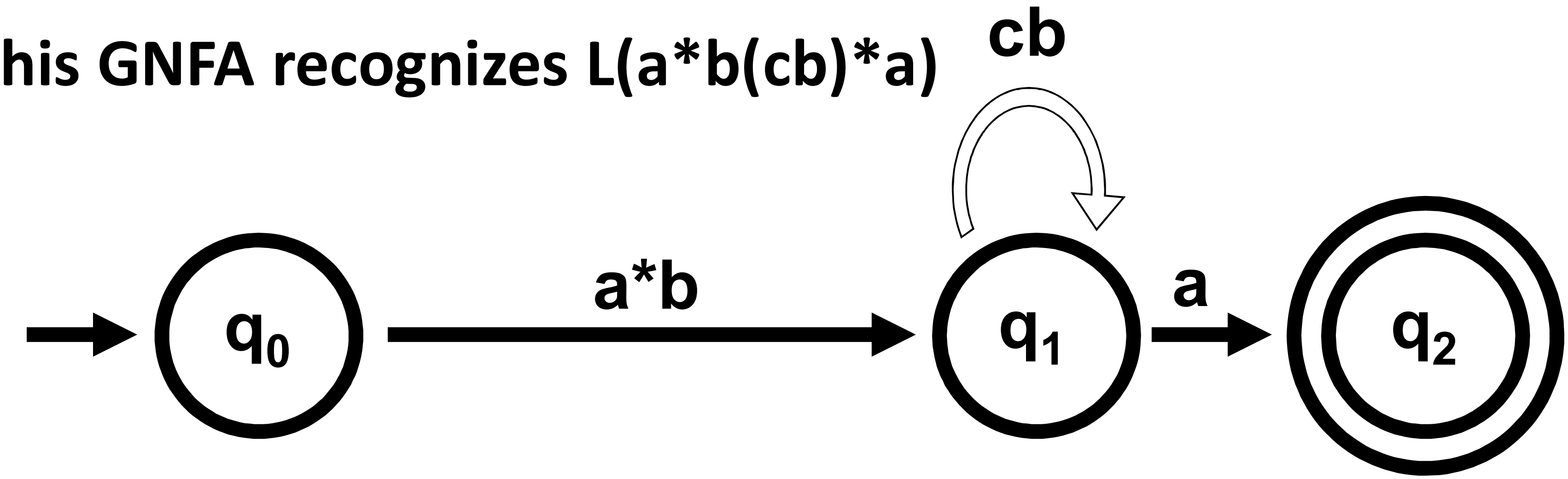
L is a regular language

Idea: Transform a DFA for L into a regular expression by *removing states* and re-labeling the arcs with *regular expressions*

Rather than reading in just 0 or 1 letters from the string on an arc, we can read in *entire substrings*

Generalized NFA (GNFA)

This GNFA recognizes L(a*b(cb)*a)



Accept string x ⇔ there is *some path* of regexps $R_1, \dots, R_k$ from start state to final state such that x matches $R_1 \cdots R_k$

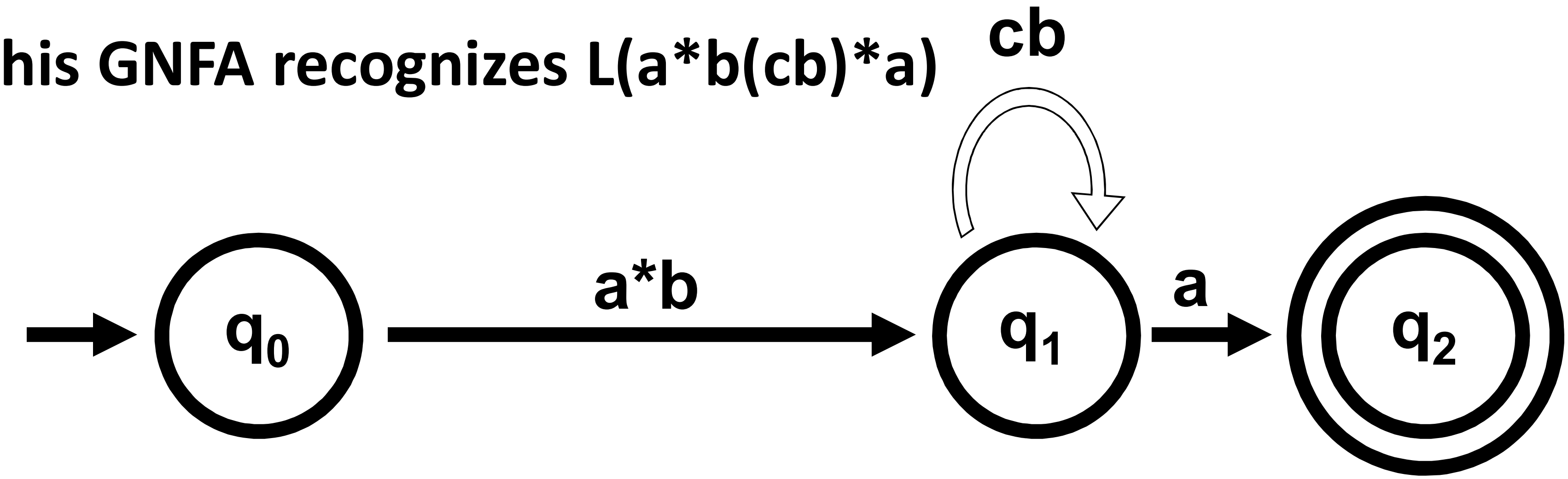
Is aaabcbcba accepted or rejected?

Is bba accepted or rejected?

Is bcba accepted or rejected?

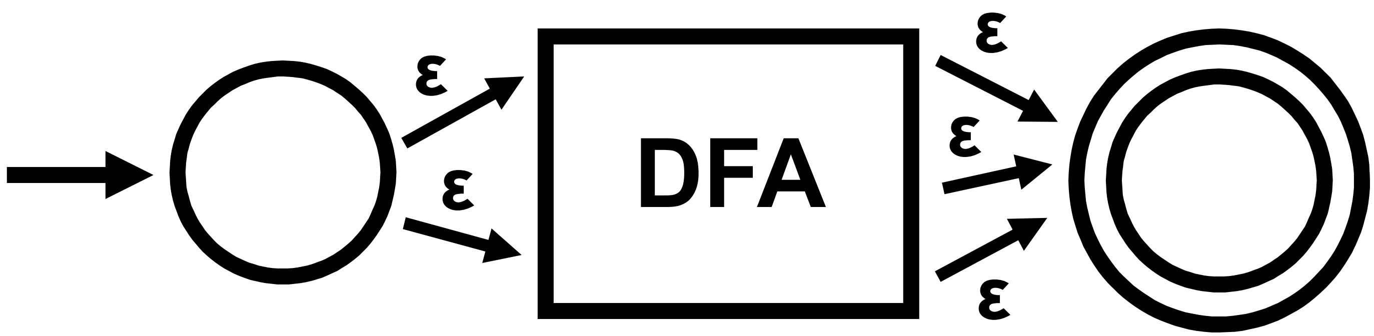
Generalized NFA (GNFA)

This GNFA recognizes L(a*b(cb)*a)

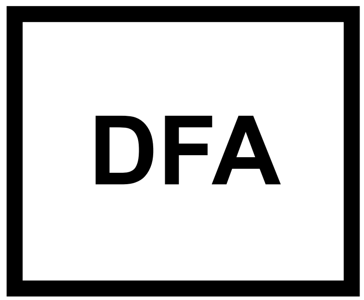


Accept string x ⇔ there is *some path* of regexps $R_1, \ldots, R_k$ from start state to final state such that x matches $R_1 \cdots R_k$

Every NFA is also a GNFA.
Every regexp can be converted into a GNFA with just two states!



Add unique start and accept states

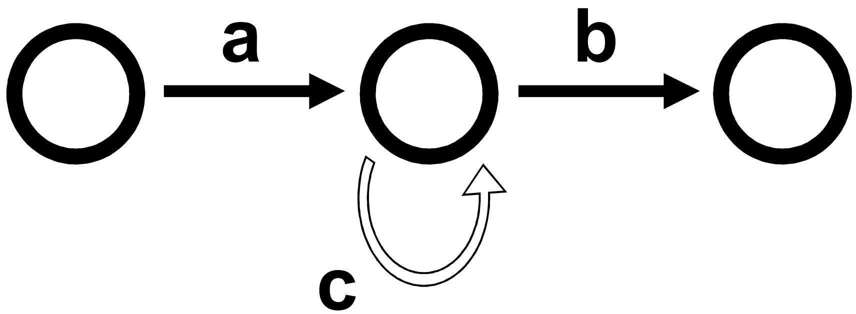
Goal: Replace 

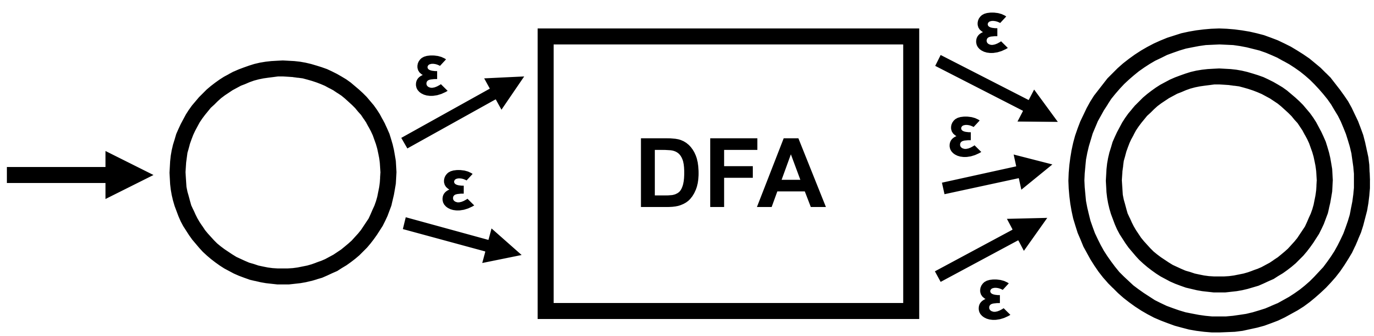
with a single regexp R

Then, L(R) = L(DFA)

While the machine has more than 2 states:

Pick an internal state, rip it out and re-label the arrows with regexps, to account for paths through the missing state





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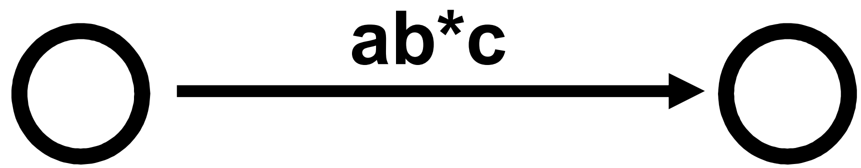
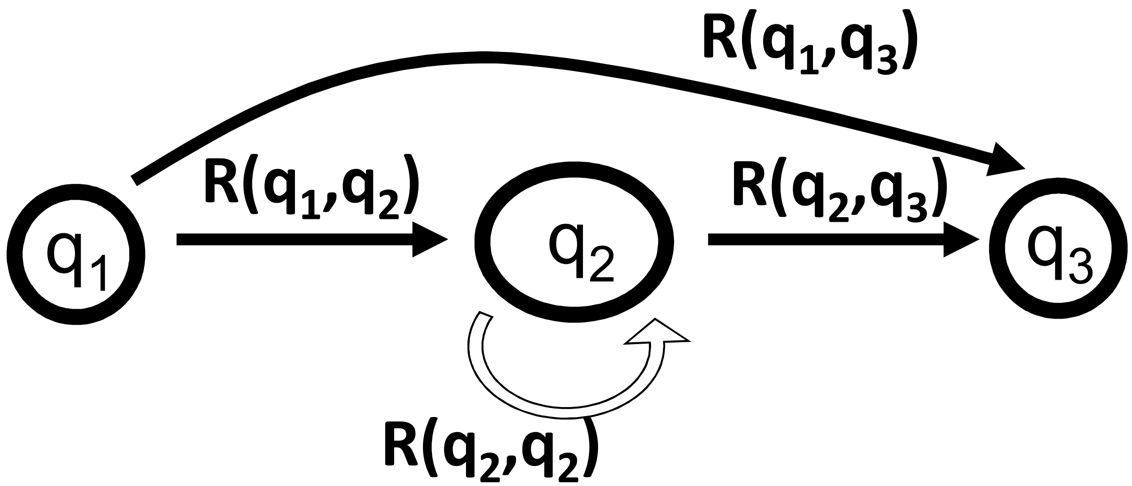


While the machine has more than 2 states:

In general:



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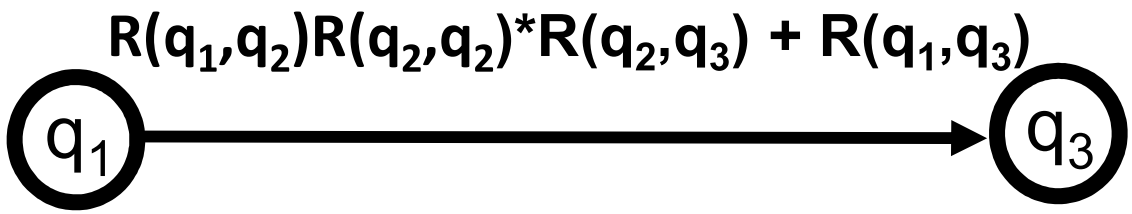
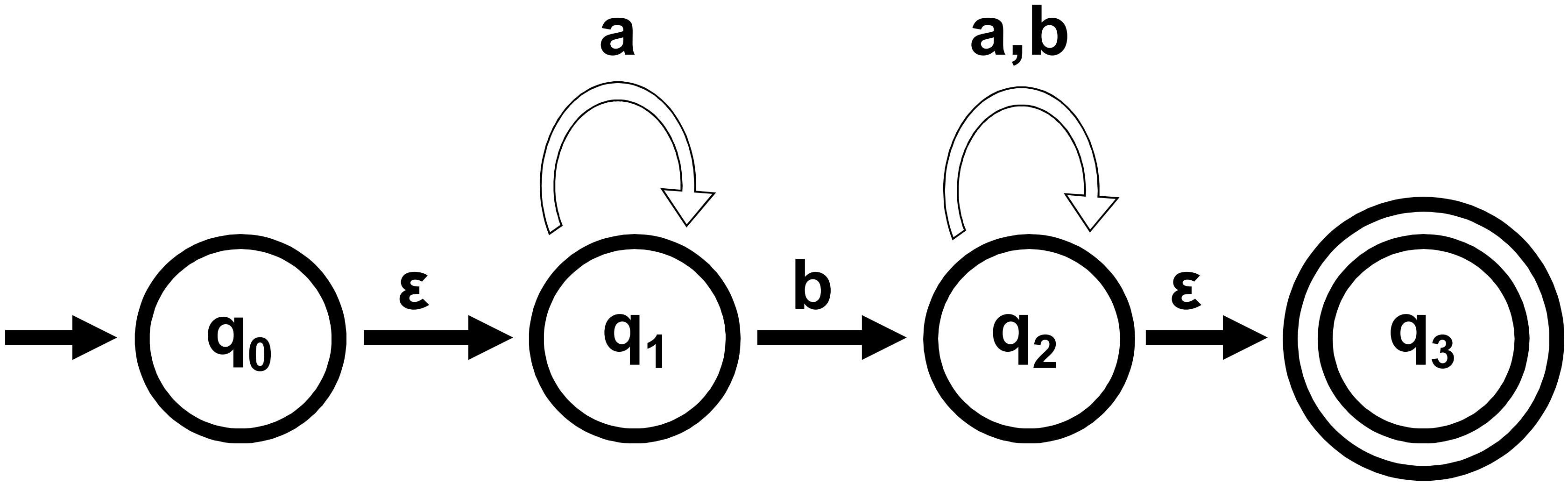
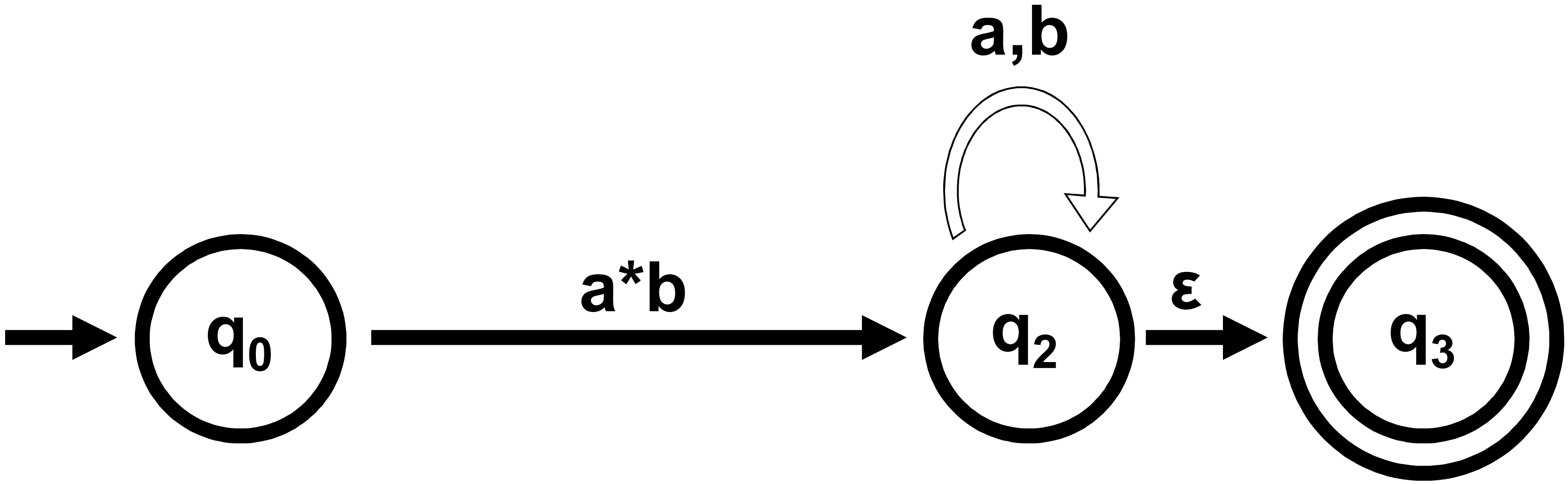




$R(q_0,q_3) = (a^*b)(a+b)^*$

represents L(N)



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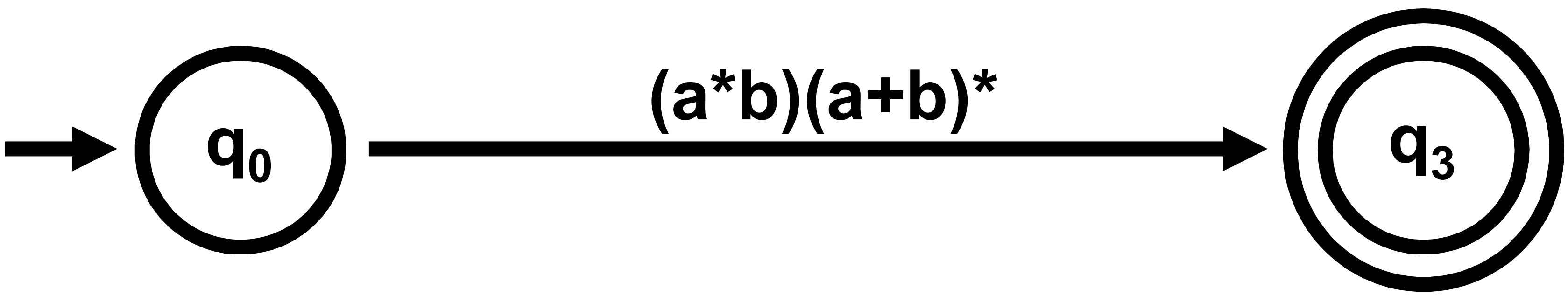

$R(q_0,q_3) = (a^*b)(a+b)^*$

represents L(N)

Formally: Given a DFA M, add q_{start} and q_{acc} to create G

For all $q, q' \in Q$, define $R(q,q') = \sigma_1 + \cdots + \sigma_k$ s.t. $\delta(q,\sigma_i) = q'$

CONVERT(G): *(Takes a GNFA, outputs a regexp)*

If #states = 2 return $R(q_{start}, q_{acc})$

If #states > 2

pick $q_{rip} \in Q$ different from q_{start} and q_{acc}

define $Q' = Q - \{q_{rip}\}$

define R' on $Q'-\{q_{acc}\} \times Q'-\{q_{start}\}$ as:

$$R'(q_i,q_j) = R(q_i,q_{rip})R(q_{rip},q_{rip})^*R(q_{rip},q_j) + R(q_i,q_j)$$

return CONVERT(G')

defines a new GNFA G'

Theorem: Let R = CONVERT(G). Then L(R) = L(M).

Claim: $L(G') = L(G)$ *[Sipser, p.73-74]*