6.045

Lecture 6: The Myhill-Nerode Theorem and Streaming Algorithms

DFA Minimization Theorem:

For every regular language A, there is a **unique** (up to re-labeling of the states) minimal-state DFA M* such that A = L(M*).

Furthermore, there is an *efficient algorithm* which, given any DFA M, will output this unique M*.

If such algorithms existed for more general models of computation, that would be an engineering breakthrough!!

How could we show whether two regular expressions are equivalent?

Claim: **There is an algorithm which given regular expressions R and R', determines whether L(R) = L(R').**

The Myhill-Nerode Theorem:

For *every* language L:

Either there's a DFA for L

or there's a set of strings that "trick"

every possible DFA trying to recognize L

In DFA Minimization, we defined an equivalence relation between states of a DFA.

We can also define a similar equivalence relation over *strings* in a *language*:

Let $L \subseteq \Sigma^*$ and $x, y \in \Sigma^*$

$x \equiv_L y$ means: for all $z \in \Sigma^*$, $xz \in L \Leftrightarrow yz \in L$

Def. x and y are indistinguishable to L iff $x \equiv_L y$

Claim: $\equiv_L$ ("L-equivalent") is an equivalence relation

Let $L \subseteq \Sigma^*$ and $x, y \in \Sigma^*$

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Reflexive:

$x \equiv_L x$: for all $z \in \Sigma^*$, $xz \in L \Leftrightarrow xz \in L$

Symmetric:

$x \equiv_L y$: for all $z \in \Sigma^*$, $xz \in L \Leftrightarrow yz \in L$

Equivalent to: for all $z \in \Sigma^*$, $yz \in L \Leftrightarrow xz \in L$, $y \equiv_L x$

Transitive:

$x \equiv_L y$: for all $z \in \Sigma^*$, $xz \in L \Leftrightarrow yz \in L$

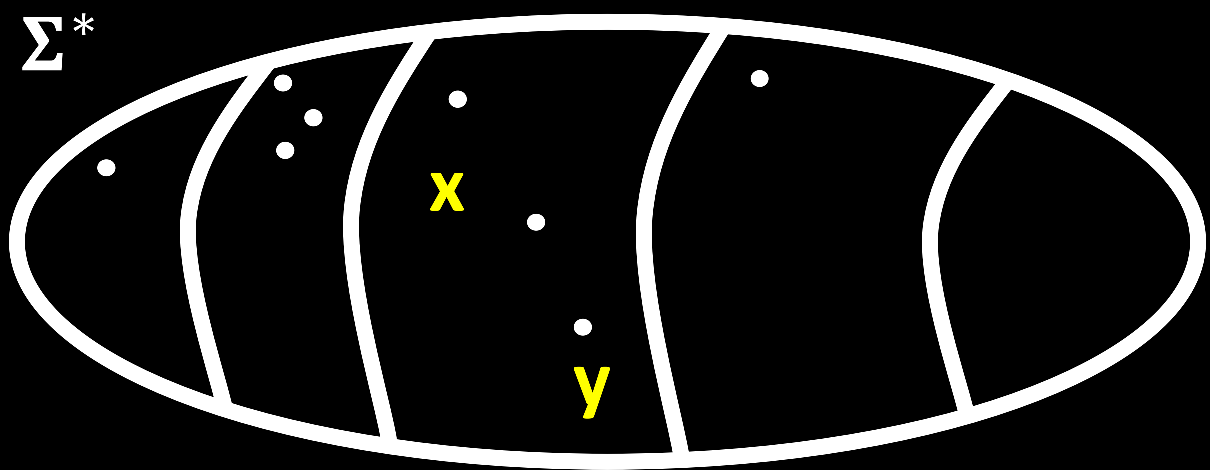
$y \equiv_L w$: for all $z \in \Sigma^*$, $yz \in L \Leftrightarrow wz \in L$

Implies for all $z \in \Sigma^*$, $xz \in L \Leftrightarrow wz \in L$, $x \equiv_L x$

Let $L \subseteq \Sigma^*$ and $x, y \in \Sigma^*$

$x \equiv_L y$ means: for all $z \in \Sigma^*$, $xz \in L \Leftrightarrow yz \in L$

Suppose we partition all strings in Σ^* into equivalence classes under $\equiv_L$



The Myhill-Nerode Theorem:

If the number of parts is finite → can construct a DFA!

If the number of parts is infinite → there is no DFA!

Mapping strings to DFA states

Given DFA $M = (Q, \Sigma, \delta, q_0, F)$, we define a function $\Delta : \Sigma^* \to Q$ as follows:

$$\Delta(\varepsilon) = q_0$$

$$\Delta(\sigma) = \delta(q_0, \sigma)$$

$$\Delta(\sigma_1 \cdots \sigma_{k+1}) = \delta(\Delta(\sigma_1 \cdots \sigma_k), \sigma_{k+1})$$

$\Delta(w)$ = ***the state of M reached after reading in w***

Note: $\Delta(w) \in F \Leftrightarrow$ M accepts w

Let $L \subseteq \Sigma^*$ and $x, y \in \Sigma^*$

$x \equiv_L y$ means: for all $z \in \Sigma^*$, $xz \in L \Leftrightarrow yz \in L$

The Myhill-Nerode Theorem:
A language L is regular *if and only if* the number of equivalence classes of $\equiv_L$ is *finite*.

Proof $(\Rightarrow)$ Let $M = (Q, \Sigma, \delta, q_0, F)$ be any DFA for L.

Define the relation: $x \approx_M y \Leftrightarrow \Delta(x) = \Delta(y)$

Claim: $\approx_M$ is an equivalence relation with $|Q|$ classes

Claim: If $x \approx_M y$ then $x \equiv_L y$

Proof: $x \approx_M y$ implies for all $z \in \Sigma^*$, xz and yz reach the *same state* of M. So $xz \in L \Leftrightarrow yz \in L$, and $x \equiv_L y$

Corollary: The number of $\equiv_L$ classes is *at most* the number of $\approx_M$ classes (which is $|Q|$)

Let $L \subseteq \Sigma^*$ and $x, y \in \Sigma^*$

$x \equiv_L y$ means: for all $z \in \Sigma^*$, $xz \in L \Leftrightarrow yz \in L$

The Myhill-Nerode Theorem:
A language L is regular *if and only if* the number of equivalence classes of $\equiv_L$ is *finite*.

Claim: If $x \approx_M y$ then $x \equiv_L y$

Corollary: The number of $\equiv_L$ classes is *at most* the number of $\approx_M$ classes (which is $|Q|$)

Proof: Let $S = \{x_1, x_2, \dots\}$ be distinct strings, one from every $\equiv_L$ class. $|S|$ = number of $\equiv_L$ classes.

Thus for all $i \neq j$, $x_i \not\equiv_L x_j$. By the claim: $x_i \not\approx_M x_j$.

So each $x_i \in S$ is in a distinct $\approx_M$ equivalence class.

$\Rightarrow$ The number of $\approx_M$ classes is *at least* $|S|$.

Let $L \subseteq \Sigma^*$ and $x, y \in \Sigma^*$

$x \equiv_L y$ means: for all $z \in \Sigma^*$, $xz \in L \Leftrightarrow yz \in L$

($\Leftarrow$) If the number of equivalence classes of $\equiv_L$ is k then there is a DFA for L with k states

Idea: Build a DFA whose *states* are the *equivalence classes* of $\equiv_L$

Define a DFA M where:

Q is the set of equivalence classes of $\equiv_L$

$q_0 = [\varepsilon] = \{y \mid y \equiv_L \varepsilon\}$

for all $x \in \Sigma^*$, $\delta([x], \sigma) = [x\,\sigma]$ *(well-defined??)*

$F = \{[x] \mid x \in L\}$

Claim: M accepts x if and only if $x \in L$

Define a DFA M where:

Q is the set of equivalence classes of $\equiv_L$

q_0 = $[\varepsilon] = \{y \mid y \equiv_L \varepsilon\}$

$\delta([x], \sigma) = [x\,\sigma]$

$F = \{[x] \mid x \in L\}$

Claim: M accepts x if and only if $x \in L$

Proof: Let M run on $x = x_1 \cdots x_n \in \Sigma^\star$, for $x_i \in \Sigma$.
M starts in state $[\varepsilon]$, reads x_1 and moves to $[x_1]$, reads x_2 and moves to $[x_1\, x_2]$, …, and ends in state $[x_1 \cdots x_n]$.
So, M accepts $x_1 \cdots x_n \Leftrightarrow [x_1 \cdots x_n] \in F$
By definition of the set F, $[x_1 \cdots x_n] \in F \Leftrightarrow x \in L$

The **Myhill-Nerode Theorem** gives us a *new* way to prove that a given language is not regular:

L is not regular
if and only if
there are infinitely many equiv. classes of $\equiv_L$

L is not regular
if and only if
There are infinitely many strings $w_1, w_2, \ldots$ so that
for all $w_i \neq w_j$, w_i and w_j are distinguishable to L:
there is a $z \in \Sigma^*$ such that
***exactly one* of $w_i z$ and $w_j z$ is in L**

$\boxed{\text{Distinguishing set for L}}$

Distinguishing set for L

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if and only if
There are infinitely many strings $w_1, w_2, \ldots$ so that
for all $w_i \neq w_j$, w_i and w_j are distinguishable to L

To prove that L is regular, we have to show that a special finite object (DFA/NFA/regex) exists.

To prove that L is not regular, it is sufficient to show that a special infinite set of strings exists!

We can prove the nonexistence of a DFA/NFA/regex by proving the existence of this special string set!

Using Myhill-Nerode to prove non-regularity:

Theorem: $L = \{0^n 1^n \mid n \geq 0\}$ is not regular.

Proof: Consider the infinite set of strings

$$S = \{0, 00, 000, \ldots, 0^n, \ldots\}$$

Claim: S is a distinguishing set for L.

Take any pair $(0^m, 0^n)$ of distinct strings in S

Let $z = 1^m$

Then $0^m 1^m$ is in L, but $0^n 1^m$ is *not* in L

So all pairs of strings in S are distinguishable to L

Hence there are infinitely many equivalence classes of $\equiv_L$, and L is not regular!

Theorem: $PAL = \{x\,x^R \mid x \in \{0, 1\}^*\}$ is not regular.

Proof: Consider the infinite set of strings

$$S = \{01^k0 \mid k \geq 1\}$$

Claim: S is a distinguishing set for L.

Take any pair $(01^k0, 01^j0)$ of strings where $j \neq k$

Let $z = 1^k0$

Then $01^k0\ 1^k0$ is in PAL, but $01^j0\ 1^k0$ is *not* in PAL

So all pairs of strings in S are distinguishable to PAL

Hence there are infinitely many equivalence classes of $\equiv_L$, and L is not regular (by the Myhill-Nerode theorem)

Streaming Algorithms

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Streaming Algorithms

Have three components

Initialize:

<variables and their assignments>

When next symbol seen is σ:

<pseudocode using σ and vars>

When stream stops (end of string):

<*accept/reject* condition on vars>

(or: <pseudocode for output>)

Algorithm A computes $L \subseteq \Sigma^\star$ if

A accepts the strings in L, rejects strings not in L

Streaming Algorithms

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Streaming algorithms differ from DFAs in several significant ways:

1. Streaming algorithms could output more than one bit

2. The "memory" or "space" of a streaming algorithm can (slowly) *increase* as it reads longer strings

Can recognize non-regular languages!

3. Could also make multiple passes over the input, could be randomized

L = {x | x has more 1's than 0's}

Initialize: C := 0 and B := 0

When next symbol seen is σ:

If (C = 0) then B := σ, C := 1

If (C $\neq$ 0) and (B = σ) then C := C + 1

If (C $\neq$ 0) and (B $\neq$ σ) then C := C – 1

When stream stops:

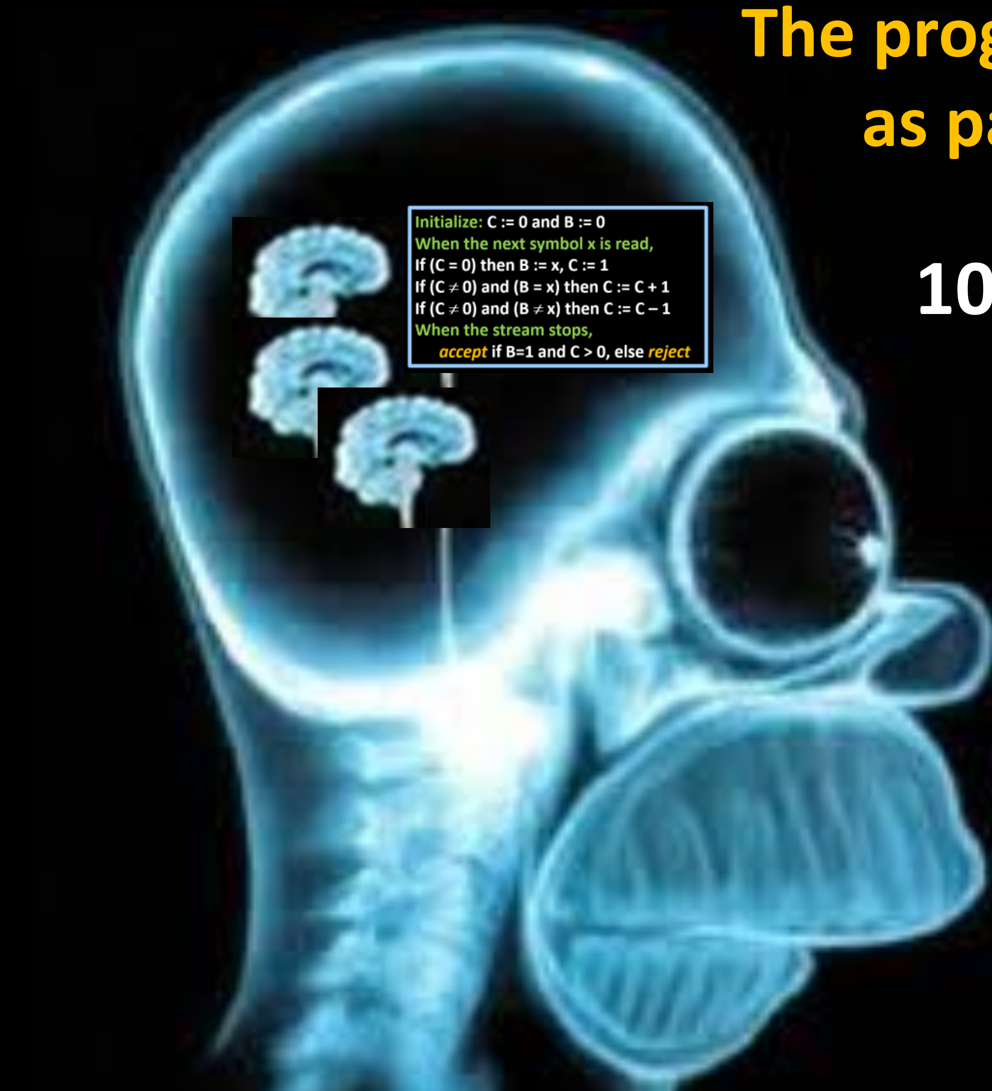
accept if B=1 and C > 0, else *reject*

B = the majority bit

C = how many more times B appears

On all strings of length n, the algorithm uses $(\log_2 n)+O(1)$ bits of space *(to store B and C)*

How to think of memory usage



The program is *not considered* as part of the memory

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Space usage of A:
$S(n)$ = maximum # of bits used to store vars in A, over all inputs of length *up to* n

L = $\{0^n 1^n \mid n \geq 0\}$

Initialize: z := 0, s := false, fail := false

When next symbol seen is σ:

If (not s) and (σ = 0) then z := z + 1

If (not s) and (σ = 1) then s := true; z:=z-1

If (s) and (σ = 0) then fail := true

If (s) and (σ = 1) then z := z – 1

When stream stops:

accept if and only if (not fail) and (z=0)

z = how many more times
0 appears than 1

s = "Started reading 1s yet?"

fail = "Reject for certain?"

On all strings of length n, uses $(\log_2 n)+O(1)$ space

DFAs and Streaming

Thm: Let L' be recognized by DFA M with $\leq 2^p$ states.
Then L' is computable by a streaming algorithm A using $\leq$ p bits of space.

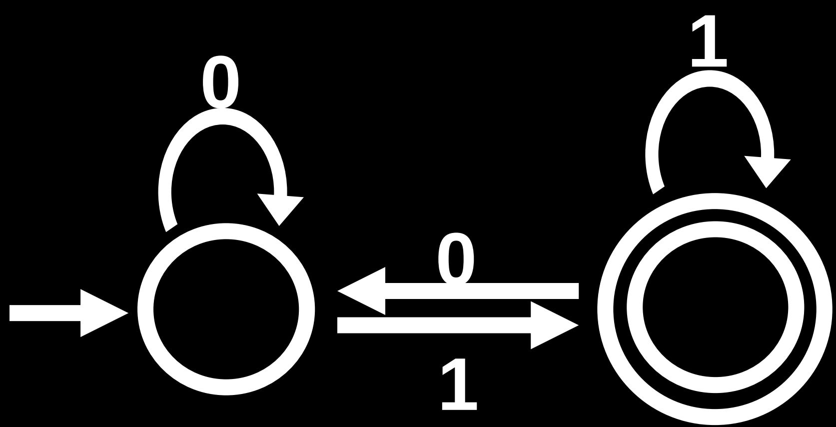
Proof Idea: Define algorithm A as follows.

Initialize: Encode the *start state* of M in memory.
When next symbol seen is σ:
  Update state of M using M's transition function
When stream stops:
  Accept if current state of M is final, else *reject*

DFAs and Streaming

Thm: Let L’ be recognized by DFA M with ≤ 2^p states.

Then L’ is computable by a streaming algorithm A using ≤ p bits of space.



Initialize: B = 0

When reading σ:

Set B := σ

When stream stops:

Accept iff B = 1

Uses 1 bit of space

DFAs and Streaming

For any $A \subseteq \Sigma^*$ define $A_n = \{x \in A \mid |x| \leq n\}$

Theorem: Let L' be computable by streaming algorithm A using $\leq S(n)$ bits of space on all strings of length up to n.
Then for all n, there is a DFA M with $< 2^{S(n)+1}$ states such that $L'_n = L(M)_n$

That is, for all streaming algorithms A using $S(n)$ space, there's a DFA M of $< 2^{S(n)+1}$ states such that A and M agree on all strings of length up to n.

Note: L'_n is always regular! (It's finite!)

DFAs and Streaming

For any $A \subseteq \Sigma^*$ define $A_n = \{x \in A \mid |x| \leq n\}$

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Then for all n, there is a DFA M with $< 2^{S(n)+1}$ states such that $L'_n = L(M)_n$

Proof Idea: States of M = at most $2^{S(n)+1} - 1$ possible memory configurations of A, over strings of length up to n

Start state of M = Initialized memory of A

Transition function = Mimic how A updates its memory

Final states of M = Subset of memory configurations in which A would accept, if the string ended there

Streaming Lower Bounds via DFAs

For any $A \subseteq \Sigma^*$ define $A_n = \{x \in A \mid |x| \leq n\}$

Theorem: Let L' be computable by streaming algorithm A using $S(n)$ bits of space on all strings of length up to n. Then for all n, there is a DFA M with $< 2^{S(n)+1}$ states such that $L'_n = L(M)_n$

Corollary: Suppose for some n, every DFA M agreeing with L'_n requires at least $Q(n) := 2^{S(n)+1}$ states. Then L' is ***not computable*** by a streaming algorithm using $S(n) = \log_2(Q(n)/2) = \log_2(Q(n))-1$ space!

That is, L' requires at least $\log_2(Q(n))$ space for some n.