6.045

Lecture 7:
Streaming Algorithms and Communication Complexity

6.045

Announcements:

- Pset 2 is due tonight, 11:59pm
- Pest 3 is out!
  Due next Wednesday

L is regular

if and only if

$(\exists$ DFA M$)(\forall$ strings $x)$[M acc. x $\Leftrightarrow$ x $\in$ L]

"M gives the correct output on all strings"

L is NOT regular

if and only if

$(\forall$ DFA M$)(\exists$ string $x_M)$[M acc. x_M $\Leftrightarrow$ x $\notin$ L]

"M gives the wrong output on x_M"

So the problem of proving L is NOT regular can be viewed as a problem about designing "bad inputs"

Distinguishing set for L

L is not regular
if and only if
There are infinitely many strings $w_1, w_2, \ldots$ so that
for all $i \neq j$, there's a string z such that
exactly one of w_iz and w_jz is in L

To prove that L is regular, we have to show that a special finite object (DFA/NFA/regex) exists.

To prove that L is not regular, it is sufficient to show that a special infinite set of strings exists!

We can prove the nonexistence of a DFA/NFA/regex by proving the existence of this special string set!

Streaming Algorithms

Have three components

Initialize:

<variables and their assignments>

When next symbol seen is σ:

<pseudocode using σ and vars>

When stream stops (end of string):

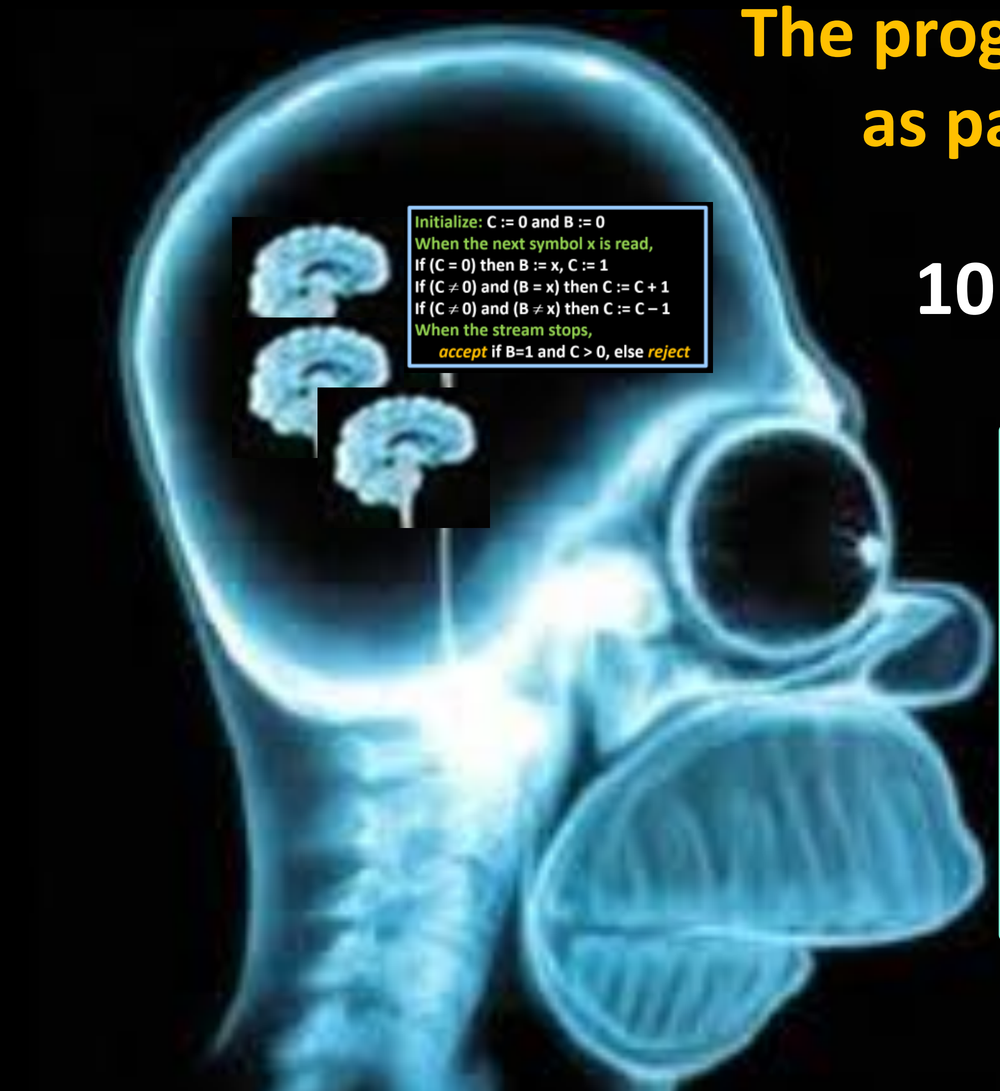
<*accept/reject* condition on vars>

(or: <pseudocode for output>)

Algorithm A computes $L \subseteq \Sigma^{\star}$ if

A accepts the strings in L, rejects strings not in L

How to think of memory usage



The program is *not considered* as part of the memory

1010101111101011111110101

Space Usage of A:
$S(n)$ = maximum # of bits used to store vars in A, over all inputs of length *up to* n

DFAs and Streaming

For any $A \subseteq \Sigma^*$ define $A_n = \{x \in A \mid |x| \leq n\}$

Theorem: Let L' be computable by streaming algorithm A with space usage $\leq S(n)$. Then for all n, there is a DFA M with $< 2^{S(n)+1}$ states such that $L'_n = L(M)_n$

For all streaming algorithms A using $S(n)$ space, and all n, there's a DFA M of $< 2^{S(n)+1}$ states such that A and M *agree* on all strings of length up to n.

Note: L'_n is always regular!
(It's a finite set!)

DFAs and Streaming

For any $A \subseteq \Sigma^*$ define $A_n = \{x \in A \mid |x| \leq n\}$

Theorem: Let L' be computable by streaming algorithm A with space usage $\leq S(n)$.
Then for all n, there is a DFA M with $< 2^{S(n)+1}$ states such that $L'_n = L(M)_n$

Proof Idea: States of M = The set of (at most) $2^{S(n)+1} - 1$ memory configurations of A, over strings of length up to n

(Why $2^{S(n)+1} - 1$?)

Start state of M = Initialized memory of A

Transition function = Mimic how A updates its memory

Final states of M = Subset of memory configurations in which A would accept, if the string ended there

Distinguishing set for L

L is not regular

if and only if

There are infinitely many strings $w_1, w_2, \ldots$ so that for all $i \neq j$, there's a string z such that

exactly one of $w_i z$ and $w_j z$ is in L

In fact, Myhill-Nerode shows that the size of a distinguishing set for L is a *lower bound* on the number of states in a DFA for L.

In other words, if S is a distinguishing set for L, then any DFA for L must have at least |S| states.

We can use similar ideas to prove lower bounds on streaming algorithms!

For any $L \subseteq \Sigma^*$ define $L_n = \{x \in L \mid |x| \leq n\}$

A **streaming distinguisher for** L_n is a subset D_n of Σ^*: for all distinct $x, y \in D_n$, there is a z in Σ^* such that $|xz| \leq n$, $|yz| \leq n$, and *exactly one* of xz, yz is in L.

Streaming Theorem: Suppose for all n, there is a streaming distinguisher D_n for L_n with $|D_n| \geq 2^{S(n)}$. Then all streaming algs for L must use at least $S(n)$ space!

Idea: Use the set D_n to show that every streaming algorithm for L must enter at least $2^{S(n)}$ different memory states, over all inputs of length at most n.

But if there are at least $2^{S(n)}$ distinct memory states, Then the alg must be using at least $S(n)$ bits of space!

$L = \{ 0^k 1^k \mid k \geq 0 \}$

Is there a streaming algorithm for L using *less than* $(\log_2 n)$ space?

Theorem: For all n, every streaming algorithm computing L must to use at least $(\log_2 n)$ bits of space.

Idea: Show there is a streaming distinguisher D_n for $L_n = \{ 0^k 1^k \mid 0 \leq k \leq n \}$ with $|D_n| = n/2+1$.
By the Streaming Theorem, it follows that all streaming algs for L need $\geq \log_2 (n/2+1)$ space!

$L = \{ 0^k 1^k \mid k \geq 0 \}$

Theorem: For all (even) n, every streaming algorithm computing L needs at least $(\log_2 n)$ bits of space.

Proof: For even n, let $D_n = \{0^i \mid i = 0, \ldots, n/2\}$

Claim: For all n, D_n is a *streaming distinguisher* for L_n

Let $x=0^a$ and $y=0^b$ be distinct strings in D_n. Set $z = 1^b$.
Then $yz \in L$, $xz \notin L$, and $|xz| \leq n$, $|yz| \leq n$. QED

Since $|D_n| = n/2+1$, Streaming Thm says: every streaming algorithm for L needs $\geq \log_2 (n/2+1)$ space.

Note $\log_2 (n/2+1) > \log_2 (n/2) = \log_2 (n) - 1$

“heavy hitters”

Finding Frequent Items

A streaming algorithm for

$L = \{x \mid x \text{ has more 1's than 0's}\}$

tells us if 1’s occur more frequently than 0’s.

What if the alphabet is *more* than just 1’s and 0’s?

And what if we want to find the “top 10” symbols?

FREQUENT ITEMS: Given k and a string $x = x_1 \ldots x_n \in \Sigma^n$, output the set $S = \{\sigma \in \Sigma \mid \sigma \text{ occurs} > n/k \text{ times in } x\}$

(Question: How large can the set S be?)

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Theorem: There is a two-pass streaming algorithm for FREQUENT ITEMS using $(k-1)(\log|\Sigma| + \log n)$ space!

1st pass: Initialize a set $T \subseteq \Sigma \times \{1,\ldots,n\}$ (originally empty)

When the next symbol σ is read:

If $(\sigma,m) \in T$, then $T := T - \{(\sigma,m)\} + \{(\sigma,m+1)\}$

Else if $|T| < k-1$ then $T := T + \{(\sigma,1)\}$

Else for all $(\sigma',m') \in T$,

$T := T - \{(\sigma',m')\} + \{(\sigma',m'-1)\}$

If $m' = 0$ then $T := T - \{(\sigma',m')\}$

Claim: At end, T contains all σ occurring $> n/k$ times in x

2nd pass: Count occurrences of all σ' appearing in T to determine those occurring $> n/k$ times

Claim: At end, T contains all σ occurring > n/k times in x

1st pass: Initialize a set $T \subseteq \Sigma \times \{1,\ldots,n\}$ (originally empty)

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If $(\sigma,m) \in T$, then $T := T - \{(\sigma,m)\} + \{(\sigma,m+1)\}$

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If $m' = 0$ then $T := T - \{(\sigma',m')\}$

2nd pass: Count occurrences of all σ' appearing in T

to determine those occurring > n/k times

Claim: At end, if σ is not in T then σ occurs $\leq$ n/k times

Idea: Have k-1 containers, n colored balls, and a trash can.

For each ball colored *σ*: either add it to a container, or *throw it in the trash along with k-1 other balls*, one from each container.

If there were m balls colored *σ*, and no balls of color *σ* are in containers at the end, there must be $k \cdot m \leq n$ balls in the trash!

1st pass: Initialize a set $T \subseteq \Sigma \times \{1,\ldots,n\}$ (originally empty)

When the next symbol σ is read:

If $(\sigma,m) \in T$, then $T := T - \{(\sigma,m)\} + \{(\sigma,m+1)\}$

Else if $|T| < k-1$ then $T := T + \{(\sigma,1)\}$

Else for all $(\sigma',m') \in T$,

$T := T - \{(\sigma',m')\} + \{(\sigma',m'-1)\}$

If $m' = 0$ then $T := T - \{(\sigma',m')\}$

When this happens

Decrement k-1 counters

2nd pass: Count occurrences of all σ' appearing in T to determine those occurring > n/k times

Number of Distinct Elements

Distinct Elements (DE):

Input: $x \in \{1,\ldots,2^k\}^*$, $n = |x| < 2^{k/2}$

Output: The *number* of different elements appearing in x; call this DE(x)

Observation: There is a streaming algorithm for DE using $O(k\ n)$ space

Theorem: Every streaming algorithm for DE requires $\Omega(k\ n)$ space!

Theorem: **Every streaming algorithm for DE requires Ω(k n) space**

Theorem: Every streaming algorithm for DE requires $\Omega(k\ n)$ space

Say $x, y \in \Sigma$ are *length-n DE distinguishable* if $(\exists z \in \Sigma^*)[DE(xz) \neq DE(yz)]$ & $|xz| \leq n$, $|yz| \leq n]$

Lemma: Let $S \subseteq \Sigma^*$ be such that every pair of strings in S is length-n DE distinguishable. Then, streaming algs for DE need $\geq \log_2 |S|$ bits of space (on inputs of length $\leq n$)

Proof Sketch: Let algorithm A use $< \log_2 |S|$ space. We show A cannot compute DE on all inputs of length $\leq n$. By the pigeonhole principle, there are distinct x, y in S that lead A to the *same memory state*.
So A gives the *same output* on both xz and yz.
But $DE(xz) \neq DE(yz)$, so A does not compute DE.

Theorem: Every streaming algorithm for DE requires $\Omega(k\ n)$ space

Lemma: Let $S \subseteq \Sigma^*$ be such that every pair of strings in S is length-n DE distinguishable. Then every streaming algorithm for DE needs $\geq \log_2 |S|$ bits of space.

Claim: For all n, there is a such a set S with $|S| \geq 2^{k\,n/4}$

Proof: For each subset T of Σ of size $n/2$,

define x_T to be any concatenation of the symbols in T

For *distinct* sets T and T', x_T and $x_{T'}$ are distinguishable:

$x_T\, x_T$ contains exactly $n/2$ distinct elements

$x_{T'}\, x_T$ has more than $n/2$ distinct elements

The total number of such subsets T is

$\binom{2^k}{n/2} \geq 2^{k\,n/2}/(n/2)^{n/2} \geq 2^{k\,n/4}$, for $n < 2^{k/2}$

Theorem: Every streaming algorithm for *approximating the number of* DE to within +- 20% error also requires Ω(k n) space!

See Lecture Notes.

Randomized Algorithms Help!

Distinct Elements (DE)
Input: $x \in \{1,\ldots,2^k\}^*$, $n = |x| < 2^{k/2}$
Output: The number of different elements appearing in x

Theorem: There is a *randomized* streaming algorithm that w.h.p. approximates DE to within 0.1% error, using $O(k + \log n)$ space!

Recall: *Deterministic* streaming algorithms require at least $\Omega(kn)$ space.

Communication Complexity

Communication Complexity

A theoretical model of distributed computing

- **Function** $f : \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}$
  - Two inputs, $x \in \{0,1\}^*$ and $y \in \{0,1\}^*$
  - **We assume $|x|=|y|=n$. Think of n as HUGE**
- **Two computers: Alice** and **Bob**
  - **Alice** *only* knows x, **Bob** *only* knows y
- **Goal: Compute $f(x, y)$ by communicating as few bits as possible between Alice and Bob**

We do not count computation cost. We *only* care about the number of bits communicated.

Alice and Bob Have a Conversation

f(x,y)=0

A(x,ε) = 0 →

← B(y,0) = 1

A(x,01) = 1 →

← B(y,011) = 0

A(x,0110) = STOP →

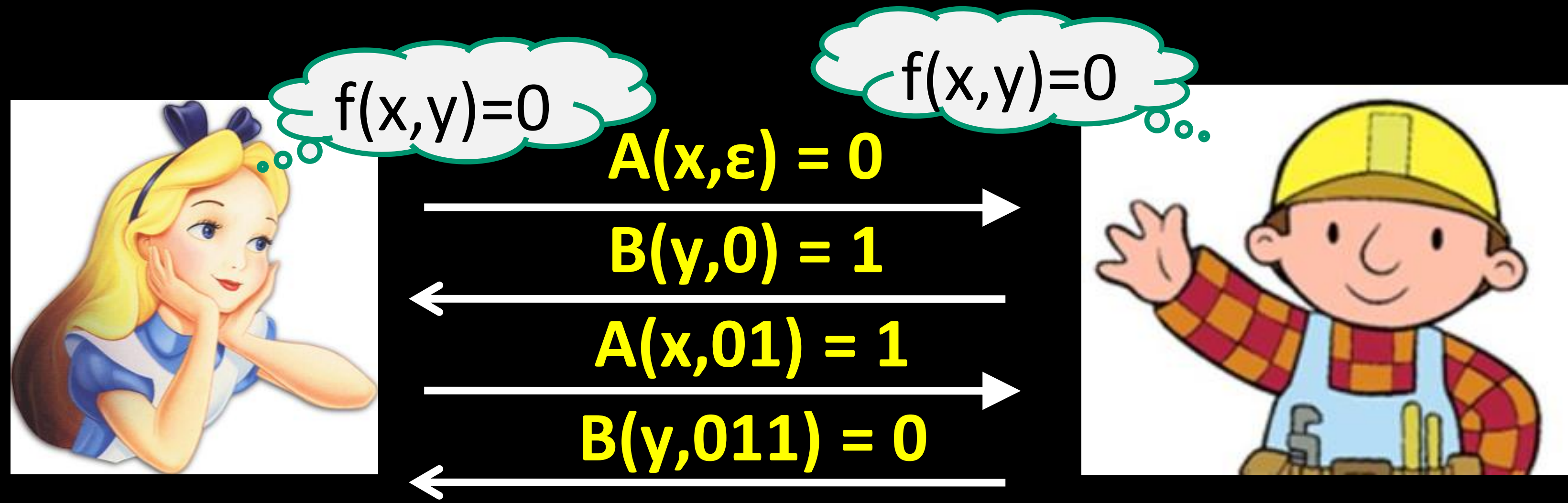
f(x,y)=0

x **y**

In every step: A **bit** or **STOP** is sent, which is a function of the party's input and all the bits communicated so far.

Communication cost = number of bits communicated
= 4 (in the example)

We assume Alice and Bob alternate in communicating, and the last BIT sent is the value of *f*(x,y)



x y

Def. A *protocol* for a function *f* is a pair of functions

A, *B* : {0,1}* × {0,1}* → {0, 1, STOP} with the semantics:

On input (x, y), let r := 0, b_0 := ε.

While ($b_r \neq$ STOP),

$r++$

If r is odd, Alice sends $b_r = A(x, b_1 \cdots b_{r-1})$

else Bob sends $b_r = B(y, b_1 \cdots b_{r-1})$

Output b_{r-1} = *f*(x, y). Number of *rounds* $= r - 1$

f(x,y)=0

f(x,y)=0

A(x,ε) = 0 →

← B(y,0) = 1

A(x,01) = 1 →

← B(y,011) = 0

x

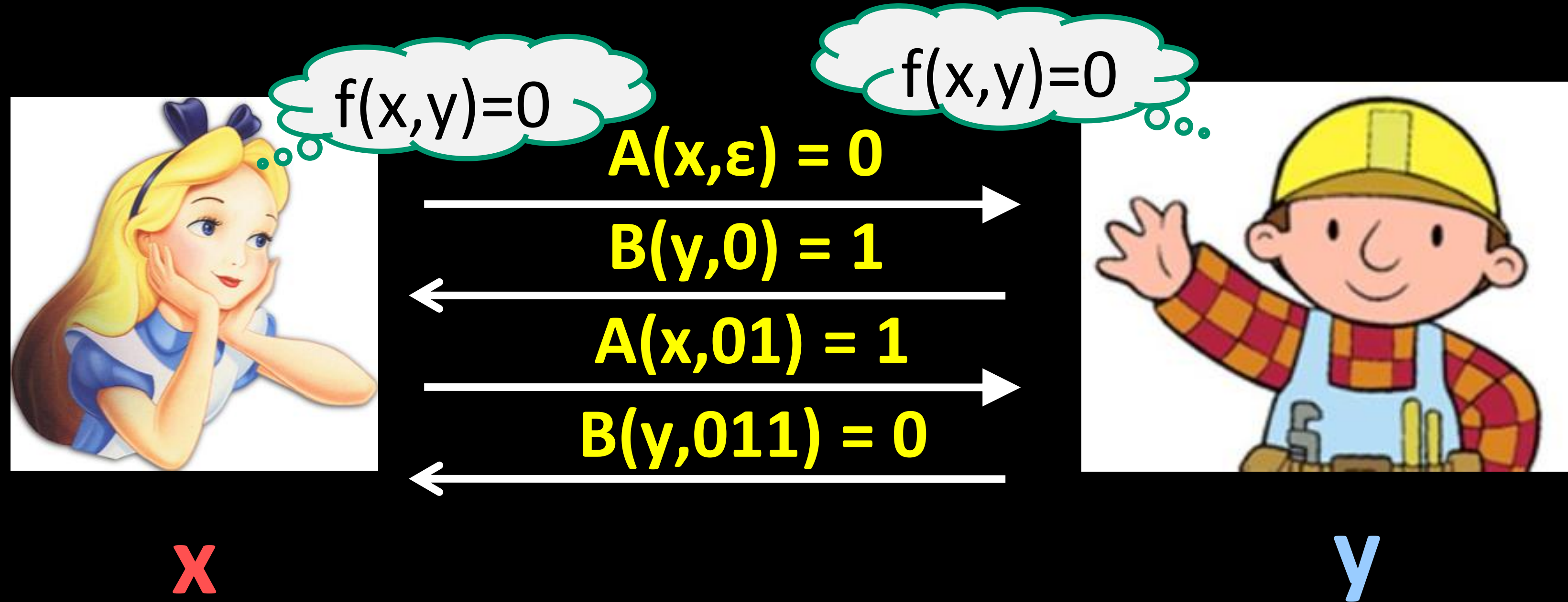
y

Def. The *cost of a protocol* (A,B) *on n-bit strings is*

$$\max_{x,y\,\in\{0,1\}^n} \text{[number of rounds taken by (A,B) on } (x,y)]$$

The *communication complexity* of *f* on n-bit strings, *cc(f)*, is *min* cost over *all protocols* computing *f* on n-bit strings

= the minimum number of rounds used by any protocol computing *f*(x,y), over all n-bit x,y



Example. Let f : {0,1}* $\times$ {0,1}* → {0,1} be arbitrary

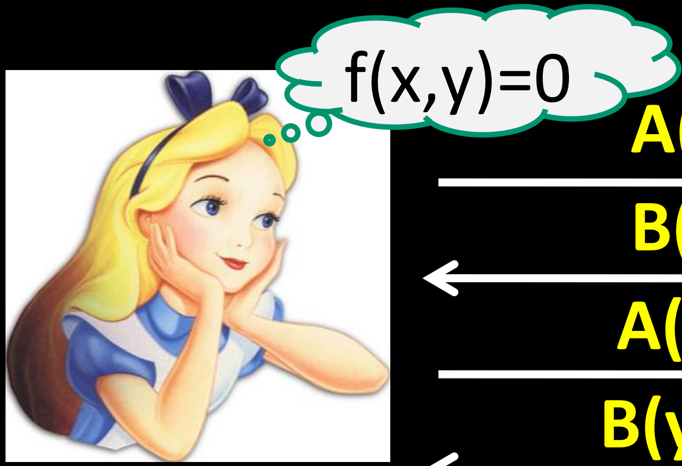
There is always a "trivial" protocol for f:

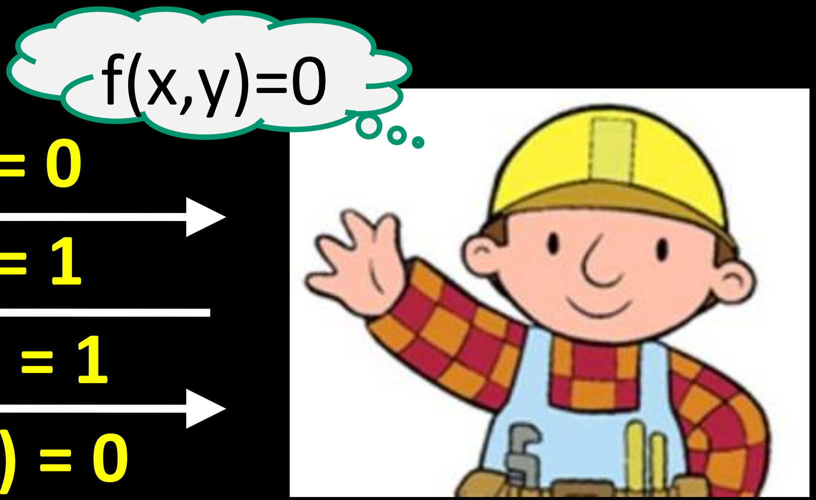
Alice sends the bits of her x in odd rounds

Bob sends whatever bit he wants in even rounds

After $2n-1$ rounds, Bob knows x and can send $f(x,y)$

Proposition: For every f, cc(f) $\leq 2n$





A(x,ε) = 0 →

← B(y,0) = 1

A(x,01) = 1 →

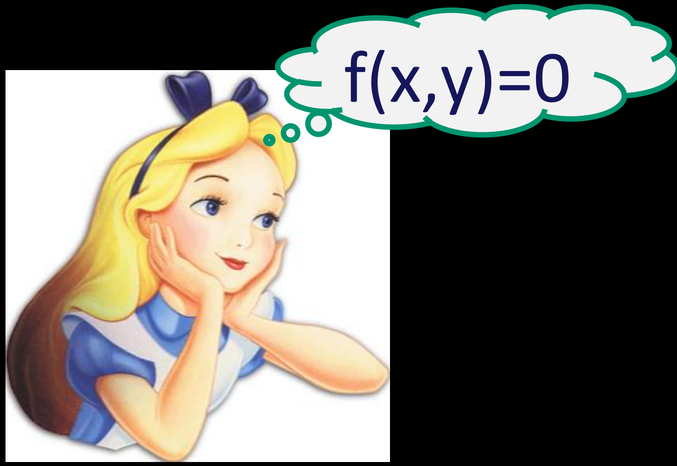
← B(y,011) = 0

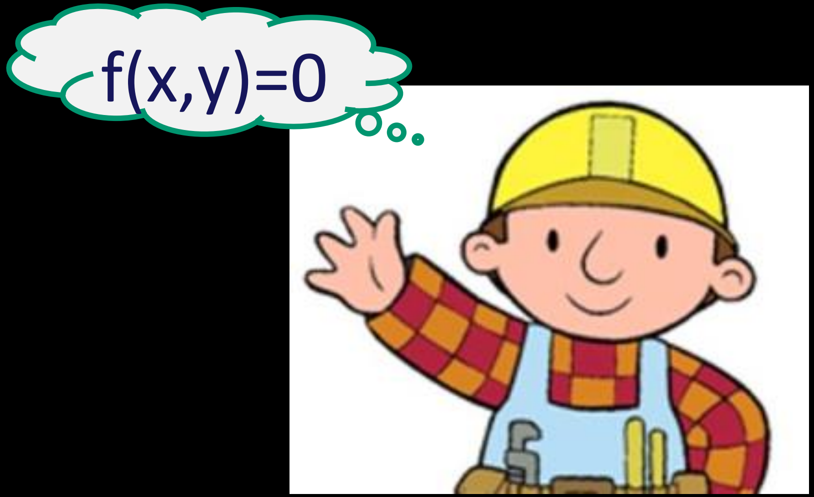
x **y**

Example. **PARITY(x, y) = $\sum_i x_i$ + $\sum_i y_i$ mod 2.**

What's a good protocol for computing PARITY?

Proposition: cc(PARITY) = 2





x

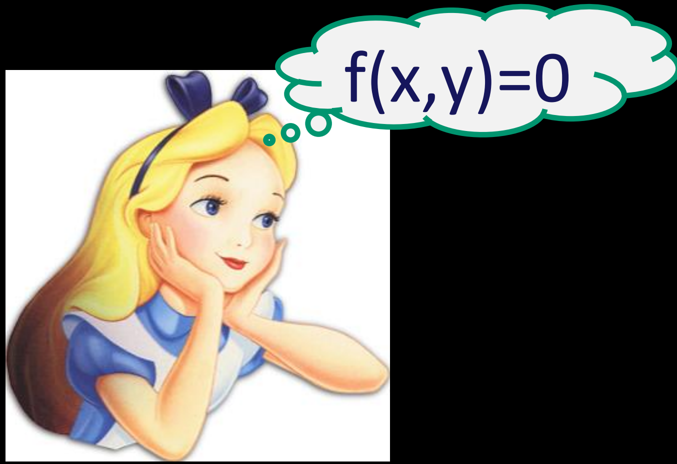
y

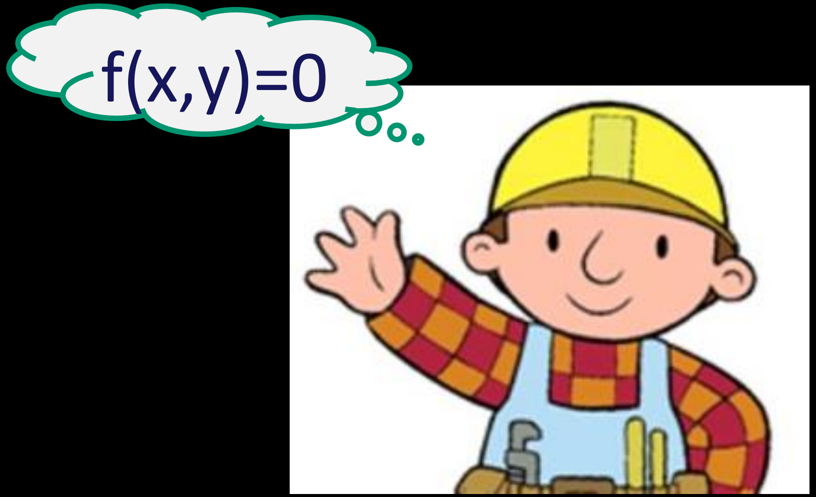
Example. MAJORITY(x, y) = most frequent bit in xy

Models voting in two "remote" locations; they want to determine a winner

What's a good protocol for computing MAJORITY?

Proposition: cc(MAJORITY) $\leq$ O(log $\mathbf{n}$)





x **y**

Example. EQUALS(x, y) = 1 $\Leftrightarrow$ $x = y$

Useful for checking consistency of two far-apart databases!

What's a good protocol for computing EQUALS?

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