6.1400

Lecture 19:
Finish NP-Completeness,
coNP and Friends

Definition: A language B is **NP-complete** if:

1. $B \in NP$
2. Every A in NP is poly-time reducible to B

   That is, $A \le_P B$

   When this is true, we say "B is NP-hard"

Last time: We showed

$3SAT \le_P CLIQUE \le_P IS \le_P VC \le_P SUBSET\text{-}SUM \le_P KNAPSACK$

All of them are in NP, and 3SAT is NP-complete, so all of these problems are NP-complete!

The Subset Sum Problem

Given: Set S = $\{a_1, \ldots, a_n\}$ of positive integers and a positive target integer t

Is there an a subset of S that sums to the target?

SUBSET-SUM = {(S, t) | $\exists\, A \subseteq \{1, \ldots, n\}$ s.t. $t = \sum_{i \in A} a_i$}

A simple summation problem!

Theorem: SUBSET-SUM is NP-complete

The Partition Problem

Input: Set $S = \{a_1, \ldots, a_n\}$ of positive integers

Decide: Is there an $S' \subseteq S$ where $(\sum_{i \in S'} a_i) = (\sum_{i \in S - S'} a_i)$?

(Formally: PARTITION is the set of all encodings of sets S such that the answer to the question is yes.)

In other words, is there a way to partition S into two parts, so that both parts have equal sum?

A problem in Fair Division:

Think of a_i as "value" of item i. Want to divide a set of items into two parts S' and $S - S'$, of the same total value.
Give S' to one party, and $S - S'$ to the other.

Theorem: PARTITION is NP-complete

PARTITION is NP-complete

(1) PARTITION is in NP

(2) SUBSET-SUM $\leq_P$ PARTITION

Input: Set $S = \{a_1, \ldots, a_n\}$ of positive integers and a positive integer t

Reduction: First, let $A := \sum_i a_i$

If $t > A$ then output $\{1,2\}$

Else output $T := \{a_1, \ldots, a_n, 2A-t, A+t\}$

Claim: $(S,t) \in$ SUBSET-SUM $\Leftrightarrow T \in$ PARTITION

That is, S has a subset that sums to t

$\Leftrightarrow$ T can be partitioned into two sets with equal sums

Easy case: $t > A = \sum_i a_i$

Input: Set $S = \{a_1,\ldots, a_n\}$ of positive integers, positive t

Output: $T := \{a_1,\ldots, a_n, 2A-t, A+t\}$, where $A := \sum_i a_i$

Claim: $(S,t) \in$ SUBSET-SUM $\Leftrightarrow$ $T \in$ PARTITION

What's the sum of all numbers in T? **4A**

Therefore: $T \in$ PARTITION

$\Leftrightarrow$ There is a $T' \subseteq T$ that sums to 2A.

Proof of $(S,t) \in$ SUBSET-SUM $\Rightarrow$ $T \in$ PARTITION:

If $(S,t) \in$ SUBSET-SUM, then let $S' \subseteq S$ sum to t.

The set $S' \cup \{2A-t\}$ sums to 2A, so $T \in$ PARTITION

Input: Set $S = \{a_1,\ldots, a_n\}$ of positive integers, positive t

Output: $T := \{a_1,\ldots, a_n, 2A-t, A+t\}$, where $A := \sum_i a_i$

Remember: sum of all numbers in T is 4A.

Claim: $(S,t) \in$ SUBSET-SUM $\Leftrightarrow T \in$ PARTITION

$T \in$ PARTITION $\Leftrightarrow$ There is a $T' \subseteq T$ that sums to 2A.

Proof of: $T \in$ PARTITION $\Rightarrow (S,t) \in$ SUBSET-SUM

If $T \in$ PARTITION, let $T' \subseteq T$ be a subset that sums to 2A.

Observation: Exactly *one* of $\{2A-t, A+t\}$ is in T'.

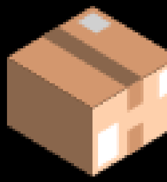
If $(2A-t) \in T'$, then $T' - \{2A-t\}$ sums to t. By Observation, the set $T' - \{2A-t\}$ is a subset of S. So $(S,t) \in$ SUBSET-SUM.

If $(A+t) \in T'$, then $(T - T') - \{2A-t\}$ sums to $(2A - (2A-t)) = t$

By Observation, $(T - T') - \{2A-t\}$ is a subset of S.

Therefore $(S,t) \in$ SUBSET-SUM in this case as well.

The Bin Packing Problem



Input: Set S = {a_1,…, a_n} of positive integers,
a bin capacity **B**, and a number of bins **K**.

Decide: Can S be partitioned into disjoint subsets $S_1, \ldots, S_K$ such that each S_i sums to at most B?

Think of a_i as the capacity of item i.
Is there a way to pack the items of S into K bins, where each bin has capacity B?

Ubiquitous problem in shipping and optimization!

Theorem: BIN PACKING is NP-complete

BIN PACKING is NP-complete

(1) BIN PACKING is in NP (Why?)

(2) PARTITION $\leq_P$ BIN PACKING

Proof: Given an instance $S = \{a_1, \ldots, a_n\}$ of PARTITION, output an instance of BIN PACKING with:

$$S = \{a_1, \ldots, a_n\}$$
$$B = (\textstyle\sum_i a_i)/2$$
$$K = 2$$

Then, $S \in$ PARTITION $\Leftrightarrow$ $(S,B,k) \in$ BIN PACKING:

There is a partition of S into two equal sums iff there is a solution to this Bin Packing instance!

P vs NP is Subtle

Let G denote a graph, and s and t denote nodes.

Recall: a simple path is a walk with no cycles

SHORTEST PATH

= {(G, s, t, k) |

G has a simple path of $<$ k edges from s to t }

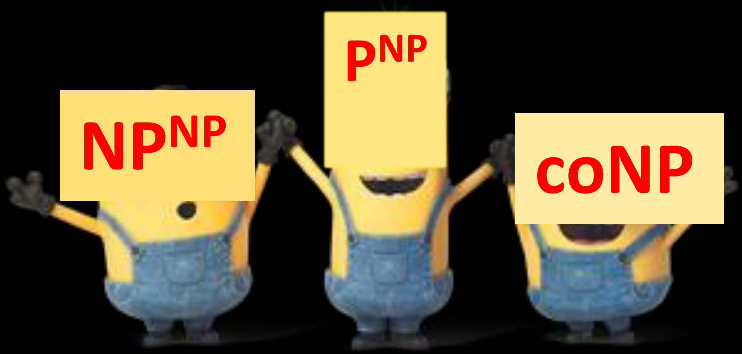
LONGEST PATH

= {(G, s, t, k) |

G has a simple path of $\geq$ k edges from s to t }

Are either of these in P? Are both of them?

coNP and Friends



(Note: any resemblance to other characters, living or animated, is purely coincidental)

NP: “Nifty Proofs”

For every L in NP,

if $x \in L$ then there is a “short proof” that $x \in L$:

$L = \{x \mid \exists y$ of poly($|x|$) length so that $V(x,y)$ accepts$\}$

But if $x \notin L$, there might not be a short proof!

There is an asymmetry between the strings in L and strings not in L.

Compare with a *recognizable* language L:

Can always verify $x \in L$ in *finite time* (a TM accepts x), but if $x \notin L$, that could be because the TM goes in an *infinite loop* on x!

Definition: coNP = { L | $\neg$L $\in$ NP }

What do coNP problems look like?

The strings NOT in L have *nifty proofs*.

Recall we can write any NP problem L in the form:

L = {x | $\exists$y of poly(|x|) length so that V(x,y) accepts}

Therefore:

$\neg$L = {x | $\neg\exists$y of poly(|x|) length so that V(x,y) accepts}

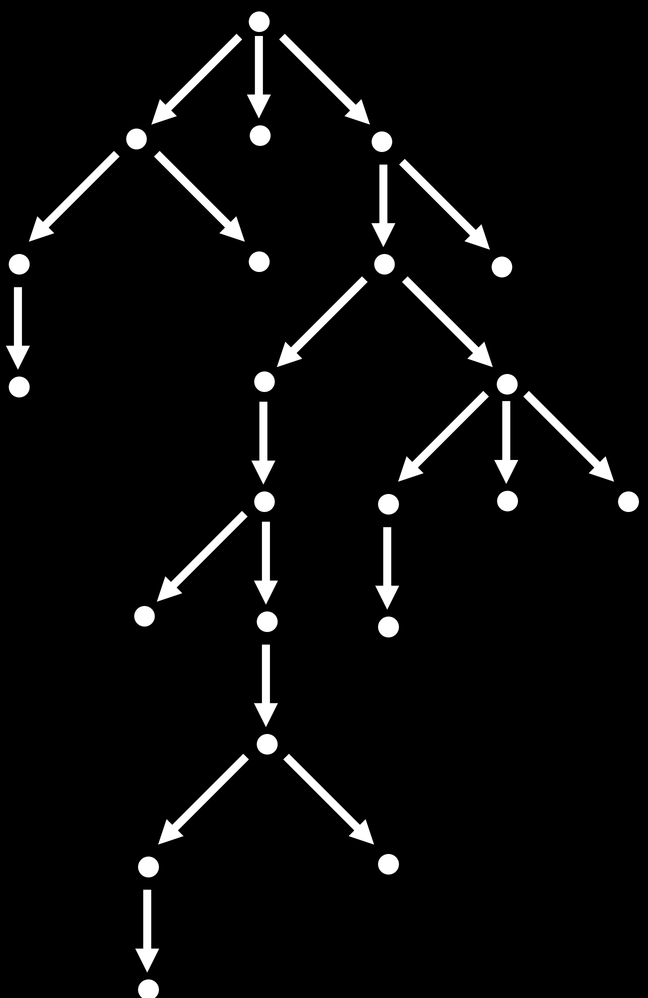
= {x | $\forall$y of poly(|x|) length, V(x,y) rejects}

Instead of using an "existentially guessing" (nondeterministic) machine,

we can define a "universally verifying" machine!

Definition: coNP = { L | ¬L ∈ NP }

What does a coNP computation look like?

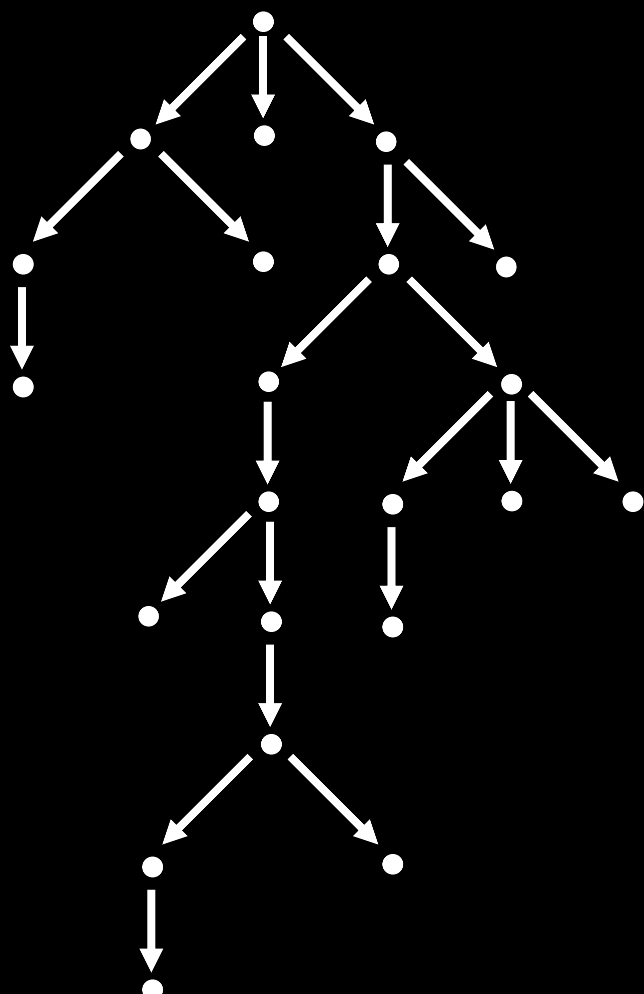


A ***co-nondeterministic*** **machine** has multiple computation paths, and has the following behavior:

- the machine **accepts**
  if ***all paths reach*** **accept state**
- the machine **rejects**
  if ***at least one path*** **reaches reject state**

Definition: coNP = { L | $\neg$L $\in$ NP }

What does a coNP computation look like?



In NP algorithms, we can use a "guess" instruction in pseudocode:

Guess string y of k|x|^k length…

and the machine accepts if some y leads to an accept state

In coNP algorithms, we can use a "try all" instruction:

Try all strings y of k|x|^k length…

and the machine accepts if every y leads to an accept state

TAUTOLOGY = { ϕ | ϕ is a Boolean formula and every variable assignment satisfies ϕ }

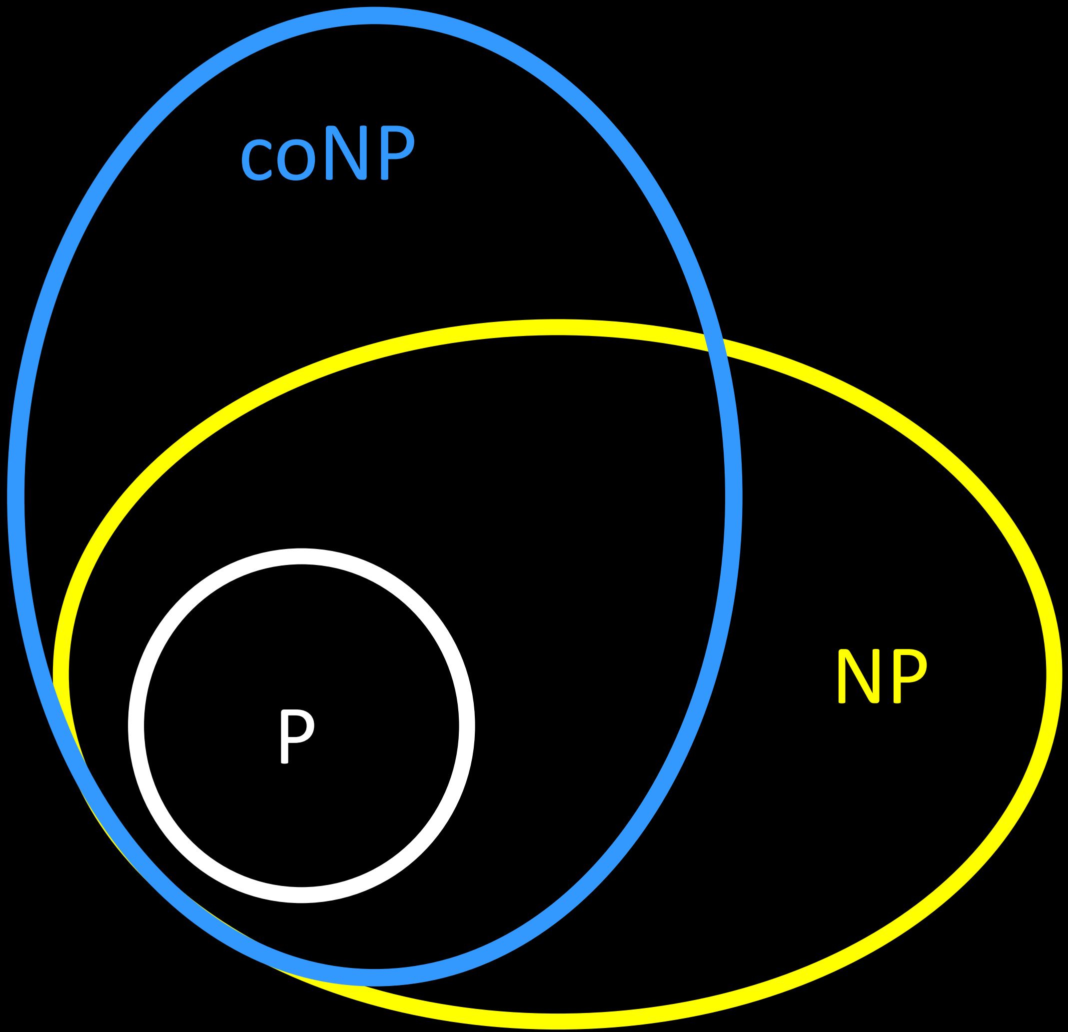
Theorem: TAUTOLOGY is in coNP

How would we write pseudocode for a coNP machine that decides TAUTOLOGY?

How would we write TAUTOLOGY as the complement of some NP language?

Is $P \subseteq coNP$?

Is NP = coNP?


coNP
NP
P

$$\text{coNP} = \{ L \mid \neg L \in \text{NP} \}$$

Definition: A language B is coNP-complete if

1. B $\in$ coNP

2. For every A in coNP, there is a polynomial-time reduction from A to B
(B is coNP-hard)

Key Trick: Can use $A \leq_P B \iff \neg A \leq_P \neg B$ to turn NP-hardness into co-NP hardness

UNSAT = { ϕ | ϕ is a Boolean formula and *no* variable assignment satisfies ϕ }

Theorem: UNSAT is coNP-complete

Proof: (1) UNSAT $\in$ coNP *(why?)*

(2) UNSAT is coNP-hard:

Let A $\in$ coNP. We show $A \leq_P$ UNSAT

Since $\neg A \in$ NP, we have $\neg A \leq_P$ 3SAT by the Cook-Levin theorem. This reduction already works!

$w \in \neg A \Rightarrow \phi_w \in$ 3SAT

$w \notin \neg A \Rightarrow \phi_w \notin$ 3SAT

$\boxed{w \notin A \Rightarrow \phi_w \notin \text{UNSAT}}$

$\boxed{w \in A \Rightarrow \phi_w \in \text{UNSAT}}$

UNSAT = { ϕ | ϕ is a Boolean formula and *no* variable assignment satisfies ϕ }

Theorem: UNSAT is coNP-complete

TAUTOLOGY = { ϕ | ϕ is a Boolean formula and *every* variable assignment satisfies ϕ }
= $\{\phi \mid \neg\phi \in$ UNSAT$\}$

Theorem: TAUTOLOGY is coNP-complete

(1) TAUTOLOGY $\in$ coNP (already shown)

(2) TAUTOLOGY is coNP-hard:

UNSAT $\leq_P$ TAUTOLOGY:
Given Boolean formula ϕ, output $\neg\phi$

$NP \cap coNP = \{ L \mid L \text{ and } \neg L \in NP \}$

L ∈ NP ∩ coNP means that
both $x \in L$ *and* $x \notin L$ have "nifty proofs"

Is P = NP ∩ coNP?

An Interesting Problem in NP $\cap$ coNP

FACTORING
= { (n, k) | n > k > 1 are integers written in binary, and there is a prime factor p of n where $k \leq p < n$ }

If FACTORING $\in$ P, we could use the algorithm to factor any integer, and break RSA!
Can binary search on k to find a prime factor of n.

Theorem: FACTORING $\in$ NP $\cap$ coNP

PRIMES = {n | n is a prime number written in binary}

Theorem (Pratt '70s): **PRIMES $\in$ NP $\cap$ coNP**

PRIMES is in P

Manindra Agrawal, Neeraj Kayal and Nitin Saxena

Ann. of Math. Volume 160, Number 2 (2004), 781-793.

Abstract

We present an unconditional deterministic polynomial-time algorithm that determines whether an input number is prime or composite.

https://en.wikipedia.org/wiki/AKS_primality_test

FACTORING

= { (n, k) | n > k > 1 are integers written in binary, there is a prime factor p of n where $k \le p < n$ }

Theorem: FACTORING ∈ NP ∩ coNP

Proof: (1) FACTORING ∈ NP

A prime factor p of n such that $p \ge k$ is a proof that (n, k) is in FACTORING

(can check primality in P, *can check p divides n in* P*)*

(2) FACTORING ∈ coNP

The prime factorization $p_1^{E1} \dots p_m^{Em}$ of n is a proof that (n, k) is not in FACTORING:

Verify each p_i is prime in P, and that $p_1^{E1} \dots p_m^{Em} = n$

Verify that for all $i=1,\dots,m$ that $p_i < k$

FACTORING

= { (n, k) | n > k > 1 are integers written in binary, there is a prime factor p of n where $k \le p < n$ }

Theorem: If FACTORING $\in$ P, then there is a polynomial-time algorithm which, given an integer n, outputs either "n is PRIME" or a prime factor of n.

Idea: Binary search for the prime factor!

Given binary integer n, initialize an interval [2,n].

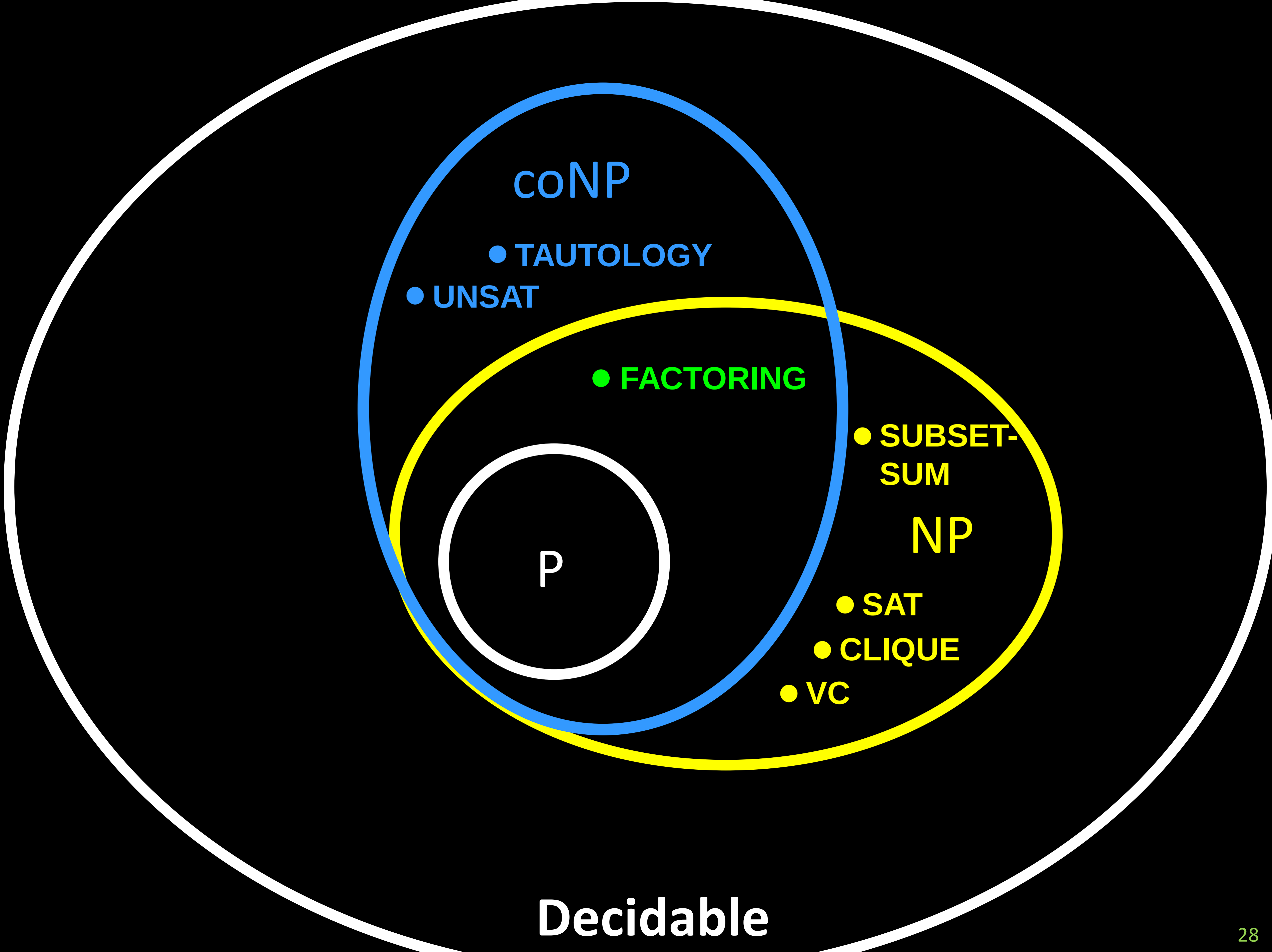
If (n, 2) is not in FACTORING then output "PRIME"

If $(n,\lceil n/2\rceil)$ is in FACTORING then

shrink interval to $[\lceil n/2\rceil,n]$ (set $k := \lceil 3n/4\rceil$)

else, shrink interval to $[2,\lceil n/2\rceil]$ (set $k := \lceil n/4\rceil$)

Keep picking k to halve the interval after each (n,k) call to FACTORING. Takes O(log n) calls to FACTORING!


coNP
TAUTOLOGY
UNSAT
FACTORING
SUBSET-SUM
NP
P
SAT
CLIQUE
VC
Decidable

NP-complete problems:

SAT, 3SAT, CLIQUE, VC, SUBSET-SUM, …

coNP-complete problems:

UNSAT, TAUTOLOGY, NOHAMPATH, …

(NP ∩ coNP)-complete problems:

Nobody knows if they exist!

P, NP, coNP can be defined in terms of specific machine models, and for every possible machine we can give a simple encoding of it.

NP ∩ coNP is *not* known to have a corresponding machine model!