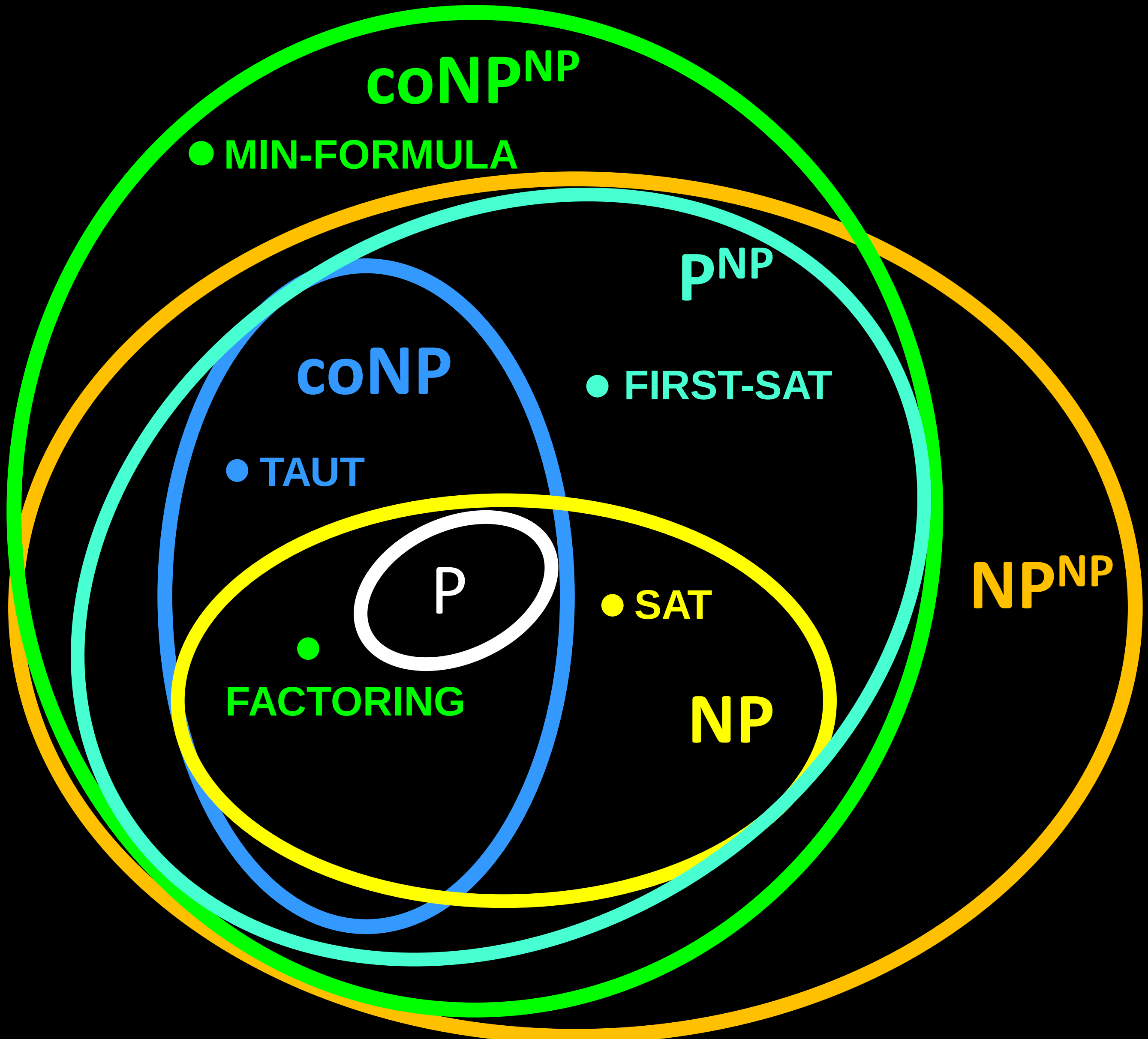
6.1400

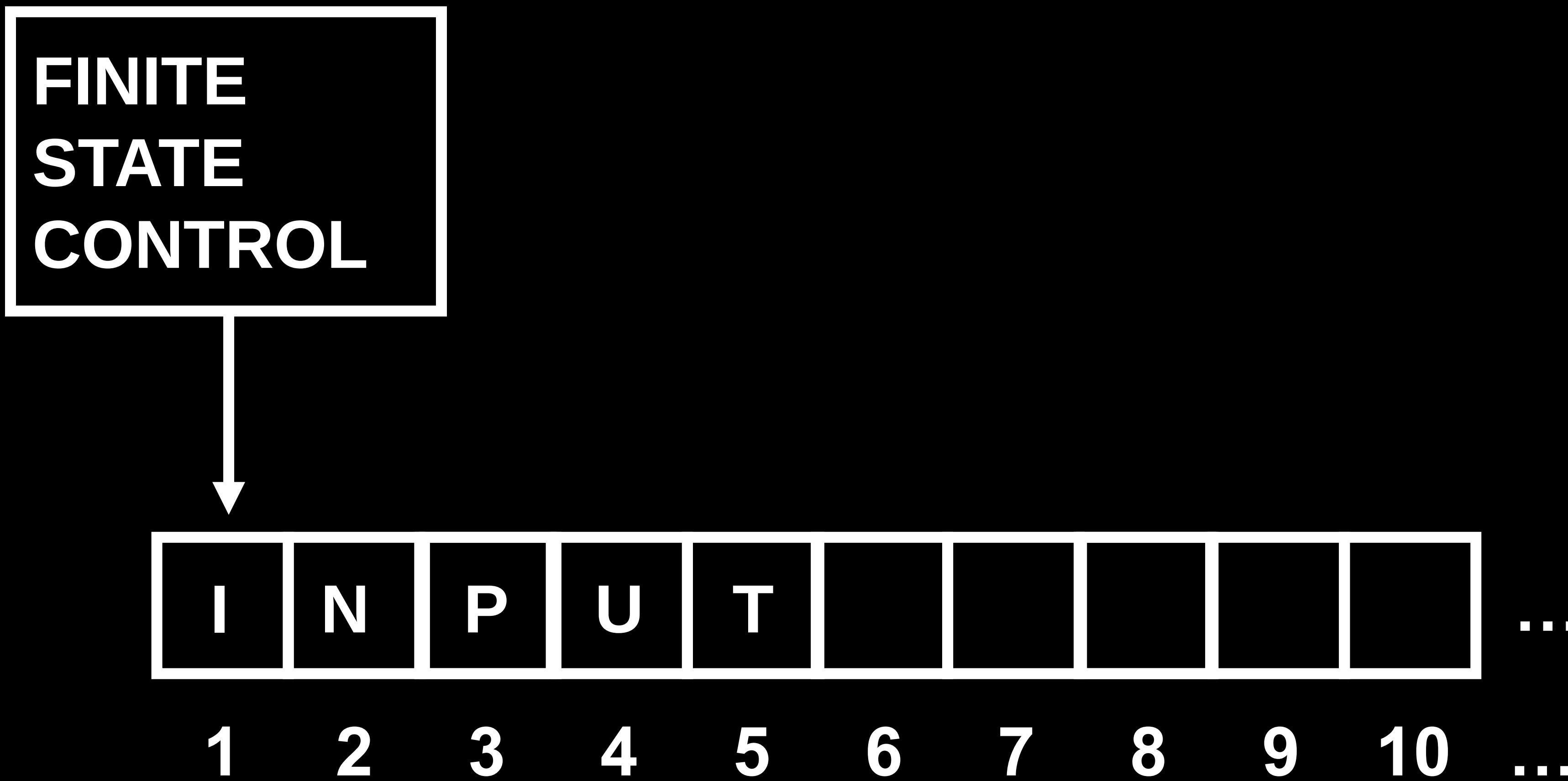
Lecture 21:
Space Complexity



$coNP^{NP}$
MIN-FORMULA
P^{NP}
coNP
FIRST-SAT
TAUT
P
SAT
NP^{NP}
FACTORING
NP

Space Problems

Measuring Space Complexity



We measure *space* complexity by finding the *largest tape index reached* during the computation

Let M be a deterministic Turing machine that only accesses a finite number of cells on each input *(not necessarily halting!)*

Definition: **The space complexity of M is the function $S : \mathbb{N} \to \mathbb{N}$, where $S(n)$ is the largest tape index reached by M on any input of length n.**

Definition: **SPACE($S(n)$) =**

{ L | L is decided by a Turing machine with O($S(n)$) space complexity}

Theorem: 3SAT ∈ SPACE(n)

Proof Idea: Given formula ϕ of length n, try all possible assignments A to the (at most n) variables. Evaluate ϕ on each A, and accept iff you find A such that ϕ(A) = 1.
All of this can be done in O(n) space.

Theorem: NTIME(t(n)) ⊆ SPACE(t(n))

Proof Idea: Try all possible computation paths of t(n) steps for an NTM on length-n input. This can be done in O(t(n)) space (store a sequence of t(n) transitions).

One Tape vs Many Tapes

Theorem: Let $s : \mathbb{N} \rightarrow \mathbb{N}$ satisfy $s(n) \geq n$, for all n.
Then every s(n) space multi-tape TM has an equivalent O(s(n)) space one-tape TM

The simulation of multitape TMs by one-tape TMs already achieves this!

Corollary: The number of tapes doesn't matter for space complexity!

One tape TMs are as good as any other model!

Space Hierarchy Theorem

Intuition: If you have more *space* to work with, then you can solve strictly more problems!

Theorem: For functions s, S : $\mathbb{N} \to \mathbb{N}$ where $s(n)/S(n) \to 0$

$$\text{SPACE}(s(n)) \subsetneq \text{SPACE}(S(n))$$

Proof Idea: Diagonalization

Make a Turing machine N that on input <M>, simulates the TM M on input <M> using up to S(|<M>|) space, *then* flips the answer.

Show L(N) is in SPACE(S(n)) but not in SPACE(s(n))

$$\mathrm{PSPACE} = \bigcup_{k \in N} \mathrm{SPACE}(n^k)$$

Since for every k, NTIME(n^k) is in SPACE(n^k), we have:

$$\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE}$$

The class PSPACE formalizes the set of problems solvable by computers with ***bounded memory***.

Fundamental (Unanswered) Question:
How does time relate to space, in computing?

SPACE(n^2) problems could potentially take much **longer** than **n^c time** to solve, for *any* c!

Intuition: You can always re-use space, but how can you re-use time?

Is P = PSPACE?

Time Complexity of SPACE[S(n)]

Let M be a halting TM with S(n) space complexity

How many time steps could M possibly take on inputs of length n? *Is there an upper bound?*

The number of time steps is at most the total number of possible *configurations*!

(If a configuration repeats, the machine is looping!)

A configuration of M specifies a head position, state, and S(n) cells of tape content. The total number of configurations is at most:

$$S(n)\ |Q|\ |\Gamma|^{S(n)} \leq 2^{O(S(n))}$$

Theorem: Let S(n) be "nice".
For every space-S(n) TM, there is a TM running in $2^{O(S(n))}$ time that decides the same language.

$$\mathrm{SPACE}(s(n)) \subseteq \bigcup_{c \in N} \mathrm{TIME}(2^{c \cdot s(n)})$$

Proof Idea: For each s(n)-space bounded TM M there is a $c > 0$ so that on all inputs x, if M runs for more than $2^{c\, s(|x|)}$ time steps on x, then *M must have repeated a configuration*, so M will never halt.

$$\text{PSPACE} = \bigcup_{k \in N} \text{SPACE}(n^k)$$

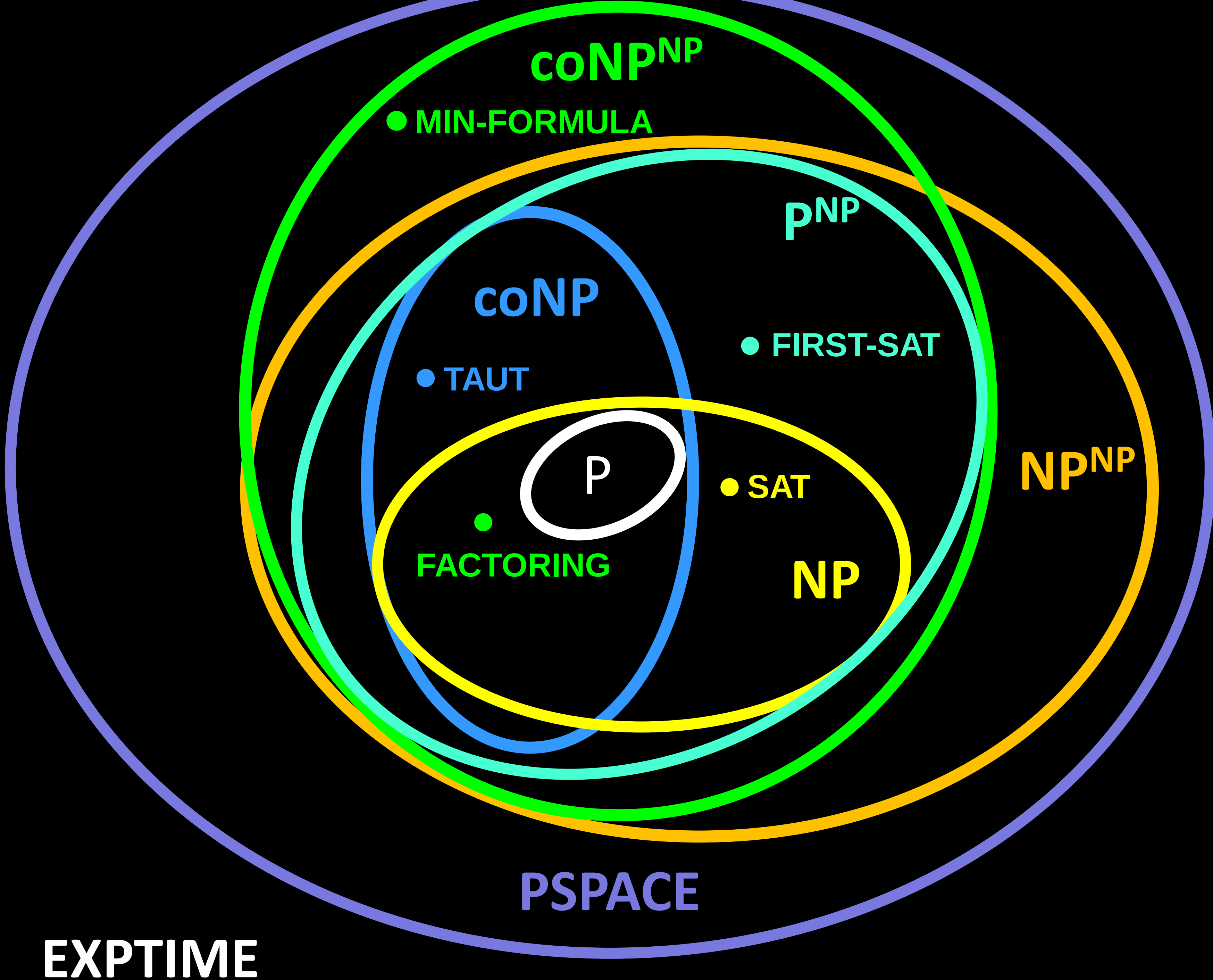
$$\text{EXPTIME} = \bigcup_{k \in N} \text{TIME}(2^{n^k})$$

$$\text{PSPACE} \subseteq \text{EXPTIME}$$

$$P \subseteq NP \subseteq PSPACE$$

Is $NP^{NP} \subseteq PSPACE$?

Is $coNP^{NP} \subseteq PSPACE$?



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$P \subseteq NP \subseteq PSPACE \subseteq EXPTIME$

Theorem: $P \neq EXPTIME$

Why? The Time Hierarchy Theorem!

$$TIME(2^n) \not\subseteq P$$

Therefore $P \neq EXPTIME$

Corollary: At least one of the following is true:
$P \neq NP$, $NP \neq PSPACE$, or $PSPACE \neq EXPTIME$

Proving any one of them would be major!

PSPACE
and Nondeterminism

Definition: SPACE(s(n)) =
{ L | L is decided by a Turing machine with
O(s(n)) space complexity}

Definition: NSPACE(s(n)) =
{ L | L is decided by a *non-deterministic*
Turing Machine with O(s(n)) space complexity}

Recall:

Space S(n) computations can be simulated in at most $2^{O(S(n))}$ time steps

$$\mathrm{SPACE}(s(n)) \subseteq \bigcup_{c \in N} \mathrm{TIME}(2^{c \cdot s(n)})$$

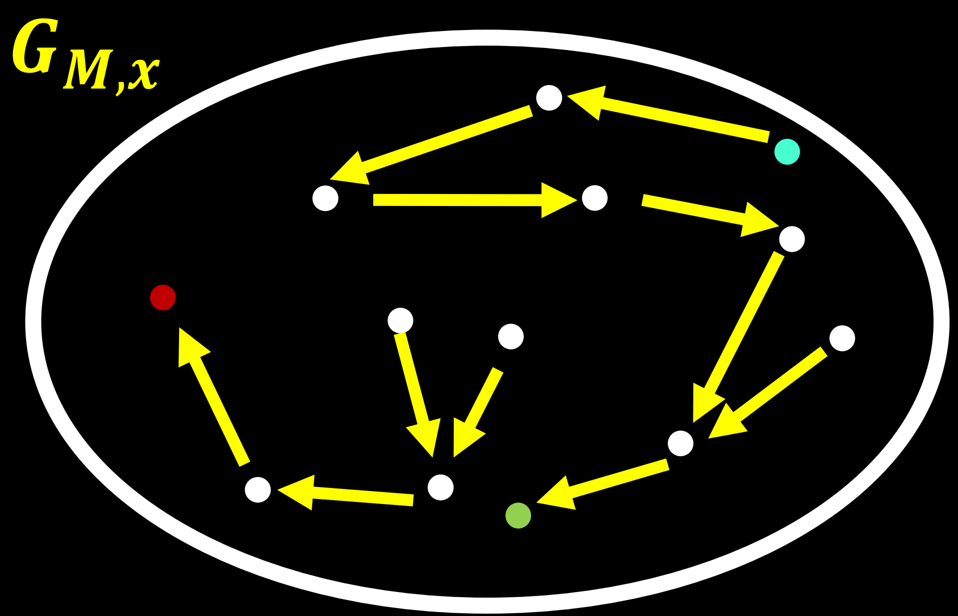
Idea: After $2^{O(s(n))}$ time steps, a s(n)-space bounded computation must have repeated a configuration, after which it will provably never halt.

Theorem:
NSPACE S(n) computations can also be simulated in at most $2^{O(S(n))}$ time steps

$$\mathrm{NSPACE}(s(n)) \subseteq \bigcup_{c \in \mathbb{N}} \mathrm{TIME}(2^{c \cdot s(n)})$$

Key Idea: Think of the problem of simulating NSPACE(s(n)) as a problem on graphs.

Def: The **configuration graph of M on x** has **nodes** C for every configuration C of M on x, and **edges** (C, C') if and only if C **yields** C'



M has space complexity $S(n)$ $\Rightarrow G_{M,x}$ has $\leq 2^{d \cdot S(|x|)}$ nodes

M is deterministic $\Rightarrow$ every node has outdegree ≤ 1

M is nondeterministic $\Rightarrow$ some nodes may have outdegree > 1

M accepts x $\Leftrightarrow$ there is a path in $G_{M,x}$ from the initial configuration node to a node in an accept state

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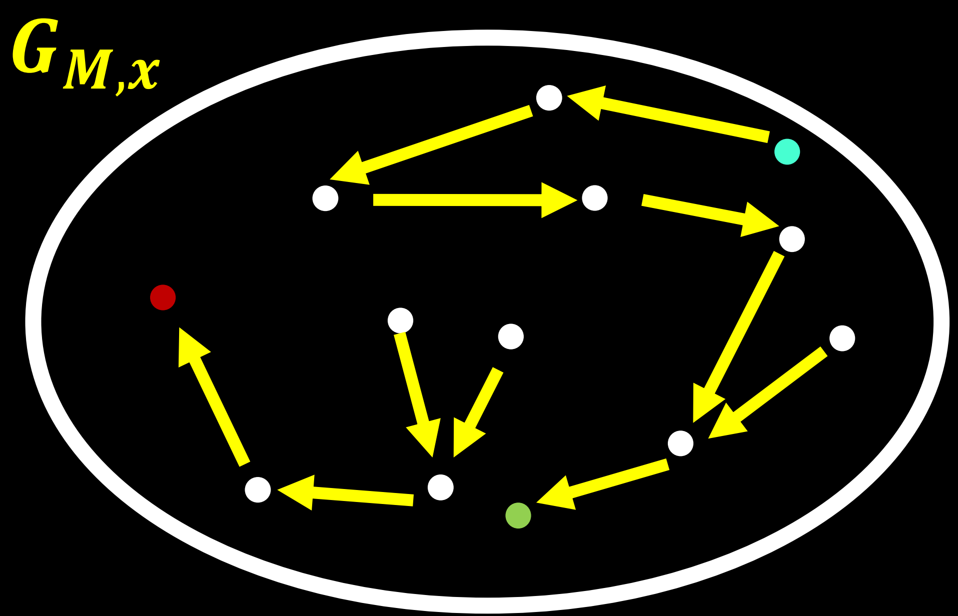

M has space complexity $S(n)$
$\Rightarrow G_{M,x}$ has $\leq 2^{d \cdot S(|x|)}$ nodes

M is deterministic
$\Rightarrow$ every node has outdegree ≤ 1

M is nondeterministic
$\Rightarrow$ some nodes may have outdegree > 1

To simulate a non-deterministic M in $2^{O(S(|x|))}$ time: do BFS in $G_{M,x}$ from the initial configuration!

$$\mathbf{PSPACE} = \bigcup_{k \in N} \mathbf{SPACE}(n^k)$$

$$\mathbf{NPSPACE} = \bigcup_{k \in N} \mathbf{NSPACE}(n^k)$$

SPACE versus NSPACE

Is $\text{NTIME}(n) \subseteq \text{TIME}(n^2)$?

Is $\text{NTIME}(n) \subseteq \text{TIME}(n^k)$ for some $k > 1$?

What about the space-bounded setting?

Is $\text{NSPACE}(s(n)) \subseteq \text{SPACE}(s(n)^k)$ for some k? Is PSPACE = NPSPACE?

Savitch's Theorem

Theorem: For functions s(n) where $s(n) \geq n$

$$\text{NSPACE}(s(n)) \subseteq \text{SPACE}(s(n)^2)$$

Proof Try:

Let N be a non-deterministic TM with space complexity s(n)

Construct a deterministic machine M that tries *every possible computation path* of N

Since each branch of N uses space at most s(n), then M uses space at most s(n)…?

Given configurations C_1 and C_2 of a $s(n)$ space machine N, and a number k (in binary), want to know if N has a computation path from C_1 to C_2 within 2^k steps

Procedure SIM(C_1, C_2, k):

If $k = 0$ then *accept* iff $C_1 = C_2$ or C_1 yields C_2 within one step.
[uses space $O(s(n))$]

If $k > 0$, then for *every* config C_m of $O(s(n))$ symbols,
if SIM(C_1,C_m,k-1) and SIM(C_m,C_2,k-1) accept
then return *accept*
return *reject* if no such C_m is found

SIM(C_1, C_2, k) has $O(k)$ levels of recursion

Each level of recursion uses $O(s(n))$ additional space.

Theorem: SIM(C_1, C_2, k) uses only $O(k \cdot s(n))$ space

Theorem: For functions s(n) where $s(n) \geq n$

$$\text{NSPACE}(s(n)) \subseteq \text{SPACE}(s(n)^2)$$

Proof:

Let N be a nondeterministic TM using s(n) space

Let $d > 0$ be such that the number of configurations of N(w) is at most $2^{d\, s(|w|)}$

Here's a deterministic $O(s(n)^2)$ space algorithm for N:

M(w): For all configurations C_a of N(w) in the accept state,
If SIM($q_0 w$, C_a, $d\, s(|w|)$) accepts, then *accept*
else *reject*

Claim: L(M) = L(N) and M uses $O(s(n)^2)$ space

Theorem: For functions s(n) where $s(n) \geq n$

$\mathrm{NSPACE}(s(n)) \subseteq \mathrm{SPACE}(s(n)^2)$

Proof:

Let N be a nondeterministic TM using s(n) space

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Why does it take only $O(s(n)^2)$ space?

$$\mathrm{PSPACE} = \bigcup_{k \in N} \mathrm{SPACE}(n^k)$$

$$\mathrm{NPSPACE} = \bigcup_{k \in N} \mathrm{NSPACE}(n^k)$$

PSPACE-complete problems

Definition: Language B is **PSPACE-complete** if:

1. B $\in$ PSPACE
2. Every A in PSPACE is **poly-time reducible** to B (i.e. B is PSPACE-hard)

Why poly-time?

Theorem: If B is PSPACE-complete and B is in P then P = PSPACE

Theorem: If B is PSPACE-complete and B is in NP then NP = PSPACE