Counting Matchings of Size k Is #W[1]-Hard

Radu Curticapean

Saarland University, Dept. of Computer Science
curticapean@cs.uni-saarland.de

Abstract. We prove #W[1]-hardness of the following parameterized counting problem: Given a simple undirected graph G and a parameter $k \in \mathbb{N}$, compute the number of matchings of size k in G.

It is known from [1] that, given an edge-weighted graph G, computing a particular weighted sum over the matchings in G is #W[1]-hard. In the present paper, we exhibit a reduction that does not require weights.

This solves an open problem from [5] and adds a natural parameterized counting problem to the scarce list of #W[1]-hard problems. Since the classical version of this problem is well-studied, we believe that our result facilitates future #W[1]-hardness proofs for other problems.

1 Introduction

Let $G = (V, E)$ be an undirected graph on n vertices. A matching M in G is a set of vertex-disjoint edges $M \subseteq E$. For $k \in \mathbb{N}$, a k-matching is a matching with $|M| = k$, and $(n/2)$-matchings are commonly referred to as perfect matchings.

Counting (Perfect) Matchings: Two natural counting problems on matchings are well-studied: The problem #PerfMatch of counting all *perfect* matchings in an input graph G, and the problem #Match of counting *all* matchings in G. The problem #PerfMatch already appeared along with the definition of the complexity class #P in [10] and was among the first problems to be proven #P-complete. In [11], the problem #Match was also proven #P-hard.

Subsequent work identified restricted graph classes on which the problems #PerfMatch and #Match are already #P-hard, as well as some tractable graph classes. For instance, #Match is already hard on planar 3-regular graphs [14], while #PerfMatch admits a polynomial-time algorithm on planar graphs [9]. This last result, and matchings in general, are also central to the new theory of holographic algorithms introduced in [12].

Parameterized Counting Complexity: In a relatively new approach to #Match, and other #P-hard problems in general, counting problems are considered as parameterized problems, see [4]. In such problems, inputs x come with an additional parameter k, and a parameterized counting problem is *fixed-parameter tractable (fpt)* in k if it can be solved in time $f(k)|x|^{O(1)}$ for a computable function f. The class #W[1] and the notion of #W[1]-hardness were both defined in [4], bridging classical counting complexity and parameterized complexity theory.

In parameterized counting problems on *graphs*, the parameter k typically either measures some notion of intricacy of the *input graph* or the intricacy of the

F.V. Fomin et al. (Eds.): ICALP 2013, Part I, LNCS 7965, pp. 352–363, 2013.

structures *to be counted*. Typical parameters associated with the input graph are, e.g., its treewidth, cliquewidth or genus. For instance, a counting analogue of Courcelle's famous theorem [2] is known [7]: Given a graph G of treewidth $\mathrm{tw}(G)$ and a formula $\phi(X)$ in monadic second-order logic over graphs with a free set variable X, counting the sets X with $G \models \phi(X)$ is fpt in $\mathrm{tw}(G)$. This implies that counting perfect (or general) matchings in G is fpt in $\mathrm{tw}(G)$.

A natural parameter associated with the structures to be counted is their *size*. This includes the results that counting k-vertex covers is fpt in k, while counting k-paths, k-cliques or k-cycles are each #W[1]-hard, all proven in [4].

Counting k-Matchings: It was conjectured in [4] that counting k-matchings on bipartite graphs is #W[1]-hard in the parameter k. The problem for general graphs is an open problem in [5]. The conjecture was later backed up by a proof [1] that counting *weighted* k-matchings is indeed #W[1]-hard: Let $G = (V, E, w)$ be an edge-weighted bipartite graph and assign to every matching $M \subseteq E$ the weight $w(M) := \prod_{e \in M} w(e)$. It was shown that, for a particular $w : E \to \mathbb{Z}$, computing the sum $\sum_M w(M)$ over matchings M in G is #W[1]-hard.

Also, the best known algorithms for counting k-matchings exhibit time bounds of the type $f(k)n^{\Theta(k)}$. Among these is [13] with a runtime of $O(2^{k+o(k)}\binom{n}{k/2})$.

Our Result: We show that counting k-matchings is #W[1]-hard on unweighted graphs without multiple edges or self-loops. It is known that weights in the sense of [1] can be simulated by parallel edges. This however creates multigraphs, and standard reductions to simple graphs fail. Our proof relies on a particular gadget construction, which is analyzed by tools from commutative algebra. This technique can probably also be applied to other counting problems.

2 Preliminaries

Parameterized Counting: Let p#Clique be the problem of counting cliques of size k in a graph G, parameterized by k. Define the class #W[1] as the set of parameterized counting problems A with $\mathsf{A} \leq^T_{fpt}$ p#Clique. Here, $\mathsf{A} \leq^T_{fpt} \mathsf{B}$ means that A admits an fpt-algorithm that solves instances (x, k) of A with oracle access to B, under the restriction that all oracle queries (y, k') feature $k' \leq g(k)$ for some computable $g : \mathbb{N} \to \mathbb{N}$. For a more formal definition, consider [4].

Polynomials: Let $\mathbf{x} = (x_1, \ldots, x_s)$ be a tuple of indeterminates and let $\mathbb{N}^{\mathbf{x}}$ be the set of monomials over $\mathbf{x}$. Given a multivariate polynomial $p \in \mathbb{Z}[\mathbf{x}]$ and $\nu \in \mathbb{N}^{\mathbf{x}}$, write $c(\nu) \in \mathbb{Z}$ for the coefficient of monomial ν in p. This gives $p = \sum_\nu c(\nu) \cdot \nu$. Note that only finitely many $c(\nu)$ are non-zero.

Let $\mathbf{x} = \mathbf{y} \dot\cup \mathbf{z}$ be a partition of the indeterminates of p. We can equivalently consider $p \in (\mathbb{Z}[\mathbf{z}])[\mathbf{y}]$. For $\nu \in \mathbb{N}^{\mathbf{y}}$, define $[\nu]p$ as the uniquely determined polynomial $H_\nu \in \mathbb{Z}[\mathbf{z}]$ in the expansion $p(\mathbf{x}) = \sum_{\theta \in \mathbb{N}^{\mathbf{y}}} H_\theta(\mathbf{z}) \cdot \theta$.

Matchings: Let $G = (V, E)$ be a graph and $k \in \mathbb{N}$. Define $\mathcal{M}_k[G]$ as the set of k-matchings of G, let $m_k := |\mathcal{M}_k[G]|$ and define $\mathcal{M}[G] := \bigcup_{k \in \mathbb{N}} \mathcal{M}_k[G]$. For a formal indeterminate X, let $M(G; X) := \sum_k m_k X^k$ be the edge-generating

matching polynomial of G. Given $u \in V$ and $M \in \mathcal{M}[G]$, we write $u \in \text{sat}(M)$ and say that u is *matched* by M if $\{u, w\} \in M$ for some $w \in V$.

2.1 Algebraic Independence

Crucial parts of our proof rely on *algebraic independence*, a notion from commutative algebra. A general introduction to this topic is given in [6].

Definition 1. *Let $P = \{p_1, \ldots, p_t\} \subseteq \mathbb{Z}[x_1, \ldots, x_s]$ be a set of polynomials and let $\mathbf{y} = (\dot{p}_1, \ldots, \dot{p}_t)$ be a tuple of indeterminates. An annihilator for P is a polynomial $A \in \mathbb{Z}[\mathbf{y}]$ which annihilates P, i.e., $A(p_1, \ldots, p_t) \equiv 0$. If the only annihilator for P is the zero polynomial, we call P algebraically independent.*

Remark 1. In the previous definition, we wrote $\mathbf{y} = (\dot{p}_1, \ldots, \dot{p}_t)$ to highlight the correspondence between indeterminates and polynomials. In this paper, expressions of the form $\dot{p}$ always denote indeterminates.

Restricting the annihilator A to linear functions without mixed-variable terms yields an alternative definition of linear independence. Algebraic independence generalizes this by allowing "polynomial" instead of only linear combinations.

We require only two ingredients from the theory of algebraic independence: The classical Jacobian criterion allows us to reduce algebraic independence to linear independence, and Lemma 1 allows us to argue about annihilators of "almost-independent" sets. A proof of Theorem 1 can be found in [3].

Theorem 1. *Let $P = \{p_1, \ldots, p_t\} \subseteq \mathbb{Z}[\mathbf{x}]$. Then P is algebraically independent iff $\text{rank}(JP) = t$, where JP denotes the Jacobian matrix $(JP)_{i,j} = \partial p_i / \partial x_j$.*

Lemma 1. *Let $\mathfrak{I} = \mathbb{Z}[\mathbf{x}]$ and let $P, Q \subseteq \mathfrak{I}$ with $P = \{p_1, \ldots, p_r\}$ and $Q = \{q_1, \ldots, q_t\}$ such that $P \cup Q$ is algebraically independent. Let $s = p_1 + \ldots + p_r$.*

Define indeterminates $\dot{s}$, $\mathbf{p} = (\dot{p}_1, \ldots, \dot{p}_r)$ and $\mathbf{q} = (\dot{q}_1, \ldots, \dot{q}_t)$, and define a ring $\mathfrak{O} := \mathbb{Z}[\dot{s}, \mathbf{p}, \mathbf{q}]$. Let $A \in \mathfrak{O}$ be an arbitrary annihilator for the set $\{s\} \cup P \cup Q$.

Let $\nu \in \mathbb{N}^{\mathbf{q}}$ be arbitrary, and consider $[\nu]A \in \mathbb{Z}[\dot{s}, \mathbf{p}]$. Applying the substitution $\dot{s} := \dot{p}_1 + \ldots + \dot{p}_r$ to $[\nu]A$ yields a polynomial $A_\nu \in \mathbb{Z}[\mathbf{p}]$ with $A_\nu \equiv 0$.

Proof. Since A annihilates $\{s\} \cup P \cup Q$, we have $A(s, p_1, \ldots, p_r, q_1, \ldots, q_t) \equiv 0$. Considering A from $(\mathbb{Z}[\dot{s}, \mathbf{p}])[\mathbf{q}]$, this equation can be rewritten as

$$\sum_{\nu \in \mathbb{N}^{\mathbf{q}}} ([\nu]A)(s, p_1, \ldots, p_r) \cdot \nu(q_1, \ldots, q_t) \equiv 0. \tag{1}$$

Note that $([\nu]A)(s, p_1, \ldots, p_r) = A_\nu(p_1, \ldots, p_r)$ since $\dot{s} := \dot{p}_1 + \ldots + \dot{p}_r$ and $s = p_1 + \ldots + p_r$. If $A_\nu \not\equiv 0$ for some ν, then (1) displays a nontrivial annihilator for $P \cup Q$ after substitution of $\dot{s}$, contradicting its independence. □

2.2 Outline of the Reduction

We prove #W[1]-hardness of counting k-matchings by a reduction from the problem p#CC of counting k-partial cycle covers, whose #W[1]-hardness was shown in [1]. Let us first define the notion of k-partial cycle covers:

Definition 2. *[1] Let $G = (V, E)$ be a directed graph and let $t \in \mathbb{N}$. A* t-partial path-cycle cover *C in G is a set $C \subseteq E$ with $|C| = t$ that consists of a vertex-disjoint union of simple paths and cycles.*

Let $\rho(C)$ be the number of paths in C. We call C a t-partial cycle *cover if $\rho(C) = 0$. The set of t-partial path-cycle covers in G with ρ paths is denoted by $\mathcal{PC}_{t,\rho}[G]$. Define $\mathcal{PC}_t[G] := \bigcup_\rho \mathcal{PC}_{t,\rho}[G]$ and extend this to $\mathcal{PC}[G] := \bigcup_t \mathcal{PC}_t[G]$. For $t, \rho \in \mathbb{N}$, let $m_{t,\rho} := |\mathcal{PC}_{t,\rho}[G]|$, if G is clear from the context.*

For compatibility with [1], define $\mathcal{C}_k[G] := \mathcal{PC}_{k,0}[G]$. In the following sections, only two parameterized counting problems will be relevant:

p#CC
Input: directed graph G, $k \in \mathbb{N}$
Parameter: k
Output: $\lvert\mathcal{C}_k[G]\rvert$

p#Match
Input: undirected graph G, $k \in \mathbb{N}$
Parameter: k
Output: $\lvert\mathcal{M}_k[G]\rvert$

It holds that p#Match $\in$ #W[1], as it is subsumed by the more general problem of counting embeddings from [4]. In the following, we sketch the reduction p#CC $\leq^T_{fpt}$ p#Match. The rest of this paper provides its details. We obtain:

Theorem 2. *The problem* p#Match *is* #W[1]*-complete.*

The reduction works as follows: To begin with, we are given a directed graph G and $k \in \mathbb{N}$ as inputs, and we wish to count the number of k-partial cycle covers in G. We are also given an oracle for p#Match that can be queried about the numbers of K-matchings in arbitrary graphs, provided that $K \leq g(k)$, where g is computable. It turns out that our queries even satisfy $K \leq 3k$, and that the reduction can be carried out in polynomial time.

The proof begins in Section 3 with the description of a particular graph transformation: First, we construct an undirected graph G' and a bijection $S : \mathcal{PC}_k[G] \to \mathcal{M}_k[G']$. Next, we apply gadgets to G' to obtain a graph $H = H(G)$ and show that, for $K = 3k$, the quantity $|\mathcal{M}_K[H]|$ can be written as a particular weighted sum over the matchings $M \in \mathcal{M}_k[G']$. The weight of M in this sum depends on the number of paths in its associated path-cycle cover $S^{-1}(M)$.

We proceed to show in Sections 3 and 4 that the weights in this sum in fact allow to distinguish matchings $M \in \mathcal{M}_k[G']$ according to the number of paths in $S^{-1}(M)$. Finally, we use this in Section 4.2 to recover the number of k-partial path-cycle covers with zero paths in G by oracle calls to p#Match.

3 The Gadget Construction

3.1 Global Construction

We want to count k-partial cycle covers in a directed graph G with an oracle for p#Match. Let $n = |V|$. First, we define a graph $S(G)$ as in [1]:

Definition 3. *[1] Given a directed graph $G = (V, E)$, replace each vertex $w \in V$ by vertices w^{in} and w^{out}, and replace each $(u, v) \in E$ by the undirected edge $\{u^{out}, v^{in}\}$. We call the resulting graph the* split graph *$S(G)$. Let $G' = S(G)$.*

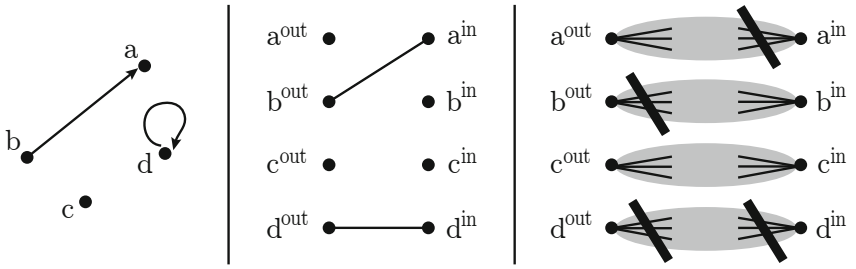

Fig. 1. (left) A partial path-cycle cover C. **(middle)** The matching $M' = S(C)$. **(right)** For $w \in V$, the gray area between $\{w^{out}, w^{in}\}$ symbolizes $\mathcal{V}_w$. Cancelled edges indicate edges in $\mathcal{V}_w$ that cannot be included into M' since $\{w^{out}, w^{in}\}$ is in-blocked, out-blocked or blocked, as seen in the first, second and fourth pair, respectively.

The graph G' is bipartite, and considering S as a function, it induces a bijection between $\mathcal{PC}_t[G]$ and $\mathcal{M}_t[G']$ for all $t \in \mathbb{N}$, as shown in [1]. Consider the left and middle part of Fig. 1 for an example. We also observe the following:

Remark 2. Let $C \in \mathcal{PC}_{t,\rho}[G]$. Since C has ρ paths, there are ρ vertices incident with only an incoming edge in C, another ρ vertices incident with only an outgoing edge, and $t - \rho$ vertices incident with both an incoming and an outgoing edge. The remaining $n - t - \rho$ vertices are not incident with any edge in C.

This translates to $M = S(C)$ as follows: Consider pairs $\{w^{out}, w^{in}\} \subseteq V(G')$, for $w \in V(G)$. There are ρ such pairs with $w^{in} \in \text{sat}(M)$ and $w^{out} \notin \text{sat}(M)$. We call such pairs *in-blocked*. There are another ρ pairs with $w^{out} \in \text{sat}(M)$ and $w^{in} \notin \text{sat}(M)$, which we call *out-blocked*. There are $t - \rho$ pairs with both $w^{out}, w^{in} \in \text{sat}(M)$, which we call *blocked*. The remaining $n - t - \rho$ pairs feature $\text{sat}(M) \cap \{w^{out}, w^{in}\} = \emptyset$. We call these pairs *free*. □

This roughly implies the following: If we can distinguish matchings $M \in \mathcal{M}_t[G']$ according to how many pairs $\{w^{out}, w^{in}\}$ occur in the above states, then we can hope to distinguish t-partial path-cycle covers $C = S^{-1}(M)$ by $\rho(C)$.

In the remaining section, we present a particular construction that achieves exactly this, as will be proven in Section 4. The construction uses a gadget, i.e., an undirected graph $\mathcal{V}$ with two special vertices u^{out} and u^{in} that can be inserted locally into G' to yield a graph $H = H(G)$.[1]

Definition 4. *Given a graph G, define a graph $H = H(G)$ as follows: First, let $G' = S(G)$. For each $w \in V(G)$, add a fresh copy $\mathcal{V}_w$ of $\mathcal{V}$ to G', identifying the vertex $w^{out} \in V(G')$ with $u^{out} \in V(\mathcal{V}_w)$, and identifying $w^{in} \in V(G')$ with $u^{in} \in V(\mathcal{V}_w)$. Note that, by construction, G' appears as a subgraph in H.*

Let $s \in V(G')$ with $s \in \{w^{out}, w^{in}\}$ for $w \in V(G)$. If $M \in \mathcal{M}[H]$ and $s \in \text{sat}(M)$, then $s \in e$ for some $e \in M$. Then either $e \in E(\mathcal{V}_w)$, in which case we call s *internally* matched, or $e \in E(G')$ and s is *externally* matched. If

[1] The actual definition of $\mathcal{V}$ is irrelevant for now and is treated in the next subsection.

s is externally matched, then all edges in M that stem from $E(\mathcal{V}_w)$ must be contained in $E(\mathcal{V}_w - s)$. Thus, when extending a matching $N \in \mathcal{M}[G']$ to some $M \in \mathcal{M}[H]$ by including edges from $\mathcal{V}_w$, we have to distinguish the state of the pair $\{w^{out}, w^{in}\}$ in N.[2] This is illustrated in the right part of Fig. 1.

We account for this by associating four restricted matching polynomials with the gadget $\mathcal{V}$. Recall that $\mathcal{V}$ features special vertices u^{out} and u^{in}. The restricted polynomials $M^A(\mathcal{V})$ are indexed by $A \subseteq \{u^{out}, u^{in}\}$ and are defined like $M(\mathcal{V})$, except that they count only matchings $M \in \mathcal{M}[\mathcal{V}]$ with $\mathrm{sat}(M) \cap \{u^{out}, u^{in}\} \subseteq A$.

Definition 5. *Let $\mathcal{V}$ be a graph with $u^{out}, u^{in} \in V(\mathcal{V})$. For $A \subseteq \{u^{out}, u^{in}\}$, let*

$$M^A(\mathcal{V}; X) := \sum_{\substack{M \in \mathcal{M}[\mathcal{V}] \\ \mathrm{sat}(M) \cap \{u^{out}, u^{in}\} \subseteq A}} X^{|M|}.$$

We consider $\mathcal{V}$ fixed. Define $B := M^{\emptyset}(\mathcal{V})$, $U := M^{\{u^{out}\}}(\mathcal{V})$, $V := M^{\{u^{in}\}}(\mathcal{V})$, and $F := M^{\{u^{out}, u^{in}\}}(\mathcal{V})$. Finally, for $t, \rho \in \mathbb{N}$ with $\rho \leq t$ and $n - t - \rho \geq 0$, define a polynomial $\mathsf{Mix}^{(t,\rho)} \in \mathbb{Z}[X]$ by $\mathsf{Mix}^{(t,\rho)} := B^{t-\rho} \cdot U^{\rho} \cdot V^{\rho} \cdot F^{n-t-\rho}$.

The polynomials $\mathsf{Mix}^{(t,\rho)}$ are crucial in Section 4 because the matching polynomial $M(H)$ can be written as a weighted sum over the path-cycle covers $C \in \mathcal{PC}[G]$ such that $C \in \mathcal{PC}_{t,\rho}[G]$ is weighted by $X^t \cdot \mathsf{Mix}^{(t,\rho)}$. This is stated in the following lemma, which can be proved using standard combinatorial arguments. Recall that $m_{t,\rho}(G) = |\mathcal{PC}_{t,\rho}[G]|$ by Definition 2.

Lemma 2. *Let G be a graph and let $H = H(G)$ as in Definition 4. Then*

$$M(H) = \sum_{0 \leq \rho \leq t \leq n} m_{t,\rho}(G) \cdot X^t \cdot \mathsf{Mix}^{(t,\rho)}.$$

We close this subsection with a remark about the coefficients of B, U, V, F:

Remark 3. Note that $[X^0]\mathsf{D} = 1$ for all $\mathsf{D} \in \{B, U, V, F\}$. Furthermore, it can be verified that $[X^1]F = [X^1](U + V - B)$ if $\{u, v\} \notin E(\mathcal{V})$. The gadget introduced in the next subsection will feature $\{u, v\} \notin E(\mathcal{V})$. □

3.2 Local Construction: The Venn Gadget

We are ready to provide an explicit construction for the gadget $\mathcal{V}$: The *Venn gadget* $\mathcal{V}(\mathbf{x})$ is an undirected graph with special vertices u^{out} and u^{in}, as shown in Fig. 2 on the next page. Its precise manifestation depends on a tuple

$$\mathbf{x} = (a_\emptyset, a_u, a_v, a_{uv},\ b_\emptyset, b_u, b_v, b_{uv},\ c_u, c_v, c_{uv}) \in \mathbb{N}^{11}. \tag{2}$$

The eleven parameters, which will be considered as indeterminates later, are named so as to reflect the particular set system represented by the gadget.

[2] This might evoke memories of *matchgates* in the readership of [12].

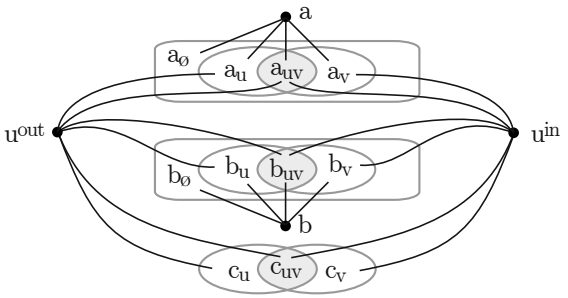

Fig. 2. The Venn gadget $\mathcal{V}$ features named vertices u^{out}, u^{in}, a and b. The other vertices are partitioned into the disjoint sets $a_\emptyset, \ldots, c_{uv}$. In this figure, an edge leading from a special vertex w into a set S symbolizes that w is adjacent to all vertices in S.

Definition 6. *Given a tuple* $\mathbf{x} \in \mathbb{N}^{11}$ *as specified in (2), the* Venn gadget $\mathcal{V}(\mathbf{x})$ *is constructed as follows from the empty graph:*

1. *Create* $(a_\emptyset + a_u + a_v + a_{uv}) + (b_\emptyset + b_u + b_v + b_{uv}) + (c_u + c_v + c_{uv})$ *fresh and unnamed vertices. Abusing notation slightly, group these vertices into sets* $a_\emptyset, \ldots, c_{uv}$ *in the obvious way.*
2. *Create a special vertex* u^{out} *adjacent to all of* $(a_u \cup a_{uv}) \cup (b_u \cup b_{uv}) \cup (c_u \cup c_{uv})$.
3. *Create a special vertex* u^{in} *adjacent to all of* $(a_v \cup a_{uv}) \cup (b_v \cup b_{uv}) \cup (c_v \cup c_{uv})$.
4. *Create a vertex* a *adjacent to all of* $a_\emptyset \cup a_u \cup a_v \cup a_{uv}$.
5. *Create a vertex* b *adjacent to all of* $b_\emptyset \cup b_u \cup b_v \cup b_{uv}$.

Remark 4. Note that constructing $\mathcal{V}(\mathbf{x})$ for different $\mathbf{x} \in \mathbb{N}^{11}$ yields different graphs. Thus, using the gadget $\mathcal{V}(\mathbf{x})$ to construct the graph H in Definition 4 in fact yields a graph $H = H(\mathbf{x})$ that also depends on $\mathbf{x}$.

Furthermore, when considering $\mathbf{x}$ as a tuple of indeterminates, the matching polynomials B, U, V, F associated with $\mathcal{V}$, introduced in Definition 5, are easily seen to be elements in $\mathbb{Z}[X, \mathbf{x}]$. Equivalently, we can define $\mathfrak{I} := \mathbb{Z}[\mathbf{x}]$ and say that $B, U, V, F \in \mathfrak{I}[X]$, where X denotes a formal generating variable. □

We now consider the coefficients of the polynomials $B, U, V, F \in \mathfrak{I}[X]$ from Remark 4. Note that these coefficients are elements of $\mathfrak{I} = \mathbb{Z}[\mathbf{x}]$, and thus in turn polynomials. We show that the set of coefficients is "almost" algebraically independent, in the sense that it allows Lemma 1 to be invoked.

First observe that $\deg(B) = 2$, $\deg(U) = \deg(V) = 3$ and $\deg(F) = 4$, as these are the maximum cardinalities of matchings counted by B, U, V, F, respectively. These four polynomials therefore feature at most 16 non-zero coefficients in total. For $\mathsf{D} \in \{B, U, V, F\}$, abbreviate $\mathsf{D}_i := [X^i]\mathsf{D} \in \mathfrak{I}$.

Furthermore, note that $B_0 = V_0 = U_0 = F_0 = 1$ by Remark 3. We will ignore these four coefficients from now on, for reasons that will become clear in Section 4. Let $\mathcal{Y}$ be the set of all *other* coefficients of B, U, V, F. For convenience:

$$\mathcal{Y} := \{B_1, B_2,\ U_1, U_2, U_3,\ V_1, V_2, V_3,\ F_1, F_2, F_3, F_4\}.$$

Additionally, let $\mathcal{B} := \{B_1, U_1, V_1, F_1\}$. Note that $F_1 = U_1 + V_1 - B_1$ by Remark 3. The set $\mathcal{B}$ is thus algebraically (and even linearly) dependent. We now consider the set $\mathcal{Y}' := \mathcal{Y} \setminus \{F_1\}$. After computing the elements in $\mathcal{Y}'$ explicitly and verifying that $\det(J\mathcal{Y}') \not\equiv 0$, we obtain the following lemma as a corollary from Theorem 1:

Lemma 3. *The set $\mathcal{Y}' \subseteq \mathfrak{I}$ is algebraically independent.*

We can now apply Lemma 1 verbatim to obtain the following corollary. It states a restriction on annihilators for $\mathcal{Y}$ which will be used in Section 4.1.

Corollary 1 (of Lemma 1). *Let $P := \mathcal{B} \setminus \{F_1\}$ and $Q := \mathcal{Y} \setminus \mathcal{B}$. By Lemma 3, the set $P \cup Q = \mathcal{Y}'$ is algebraically independent. Recall that $F_1 = U_1 + V_1 - B_1$.*

Define indeterminates $\dot{F}_1$, $\mathbf{p} = (\dot{B}_1, \dot{U}_1, \dot{V}_1)$ and $\mathbf{q}$, where $\mathbf{q}$ represents Q, and let $\mathbf{y} = (\dot{F}_1, \mathbf{p}, \mathbf{q})$. Let $\mathfrak{O} = \mathbb{Z}[\mathbf{y}]$ and let $A \in \mathfrak{O}$ annihilate $\mathcal{Y} = \{F_1\} \cup P \cup Q$.

Let $\theta^* = \dot{B}_2^b$ with $b > 0$ and consider $[\theta^*]A \in \mathbb{Z}[\dot{F}_1, \mathbf{p}]$. Applying the substitution $\dot{F}_1 := \dot{U}_1 + \dot{V}_1 - \dot{B}_1$ to $[\theta^*]A$ yields a polynomial $A_{\theta^*} \in \mathbb{Z}[\mathbf{p}]$ with $A_{\theta^*} \equiv 0$.

4 Analysis of the Graph Construction

Recall that we wish to determine $m_{k,0}$, where $m_{t,\rho}$ denotes the number of t-partial path-cycle covers with ρ paths in G. We fix k and $K := 3k$. We also fix $\mathbf{y}$ and $\mathfrak{O} = \mathbb{Z}[\mathbf{y}]$ as in Corollary 1, as well as $\mathbf{x}$ and $\mathfrak{I} = \mathbb{Z}[\mathbf{x}]$ as in Remark 4.

The indeterminates in $\mathbf{y}$ correspond to $\mathcal{Y}$ from Section 3.2. We extend this view by considering the polynomials B, U, V, F and $\mathsf{Mix}^{(t,\rho)} \in \mathbb{Z}[X]$ from Definition 5 formally as elements from $\mathfrak{O}[X]$, writing $\mathsf{Mix}_{\mathfrak{O}}^{(t,\rho)}$ to make this explicit:

Definition 7. *For $\mathsf{D} \in \{B, U, V, F\}$, let $\mathsf{D}_{\mathfrak{O}} = \sum_{i=1}^{\deg(\mathsf{D})} \dot{\mathsf{D}}_i X^i \in \mathfrak{O}[X]$. Define $\mathsf{Mix}_{\mathfrak{O}}^{(t,\rho)} \in \mathfrak{O}[X]$ exactly as $\mathsf{Mix}^{(t,\rho)}$ in Definition 5, but replace any D by $\mathsf{D}_{\mathfrak{O}}$.*

Let $\mathsf{Mix}_{\mathfrak{O}} \in \mathfrak{O}^{(K+1)\times(K+1)}$ be the matrix whose entry at (t, ρ) is $[X^{K-t}]\mathsf{Mix}_{\mathfrak{O}}^{(t,\rho)}$ for $0 \leq \rho \leq t \leq K$, and 0 else. Also write $\mathsf{Mix}_{\mathfrak{O}}$ for the set of its entries.

We similarly define $M_{\mathfrak{O}}(H) \in \mathfrak{O}[X]$ by formally replacing coefficients of Venn gadgets with indeterminates from $\mathbf{y}$. Extending Lemma 2, we obtain:

Lemma 4. *Let $H = H(G)$ according to Definition 4. For matrices A, B of the same dimensions, let $A \odot B := \sum_{ij} A_{ij} B_{ij}$. Then*

$$[X^K]M_{\mathfrak{O}}(H) = \underbrace{\begin{pmatrix} [X^K]\mathsf{Mix}_{\mathfrak{O}}^{(0,0)} & \dots & [X^0]\mathsf{Mix}_{\mathfrak{O}}^{(K,0)} \\ & \ddots & \vdots \\ & & [X^0]\mathsf{Mix}_{\mathfrak{O}}^{(K,K)} \end{pmatrix}}_{=\mathsf{Mix}_{\mathfrak{O}}} \odot \begin{pmatrix} m_{0,0} & \dots & m_{K,0} \\ & \ddots & \vdots \\ & & m_{K,K} \end{pmatrix}.$$

This yields a formal "linear combination" of the quantities $m_{t,\rho}$ with coefficients from $\mathfrak{O}$. For $t = k$ and $0 \leq \rho \leq k$, the interesting quantities $m_{k,\rho}$ appear in it as

$$[X^K]M_{\mathfrak{O}}(H) = \ldots + m_{k,0}[X^{2k}]\mathsf{Mix}_{\mathfrak{O}}^{(k,0)} + \ldots + m_{k,k}[X^{2k}]\mathsf{Mix}_{\mathfrak{O}}^{(k,k)} + \ldots \quad (3)$$

In Section 4.1, we substitute the polynomials $\mathcal{Y} \subseteq \mathfrak{I}$ from Section 3.2 into the indeterminates $\mathbf{y}$, yielding a matrix $\mathsf{Mix}_{\mathfrak{I}} \in \mathfrak{I}^{(K+1)\times(K+1)}$. We show that, after this substitution, the polynomial $p^* := [X^{2k}]\mathsf{Mix}_{\mathfrak{I}}^{(k,0)}$ associated with $m_{k,0}$ in (3) is *special*, in the sense that it cannot be written as a linear combination (with rational coefficients) of the other polynomials in $\mathsf{Mix}_{\mathfrak{I}}$.

In Section 4.2, we show that a linear system of equations in the unknowns $m_{t,\rho}$ can be set up from (3) by evaluating[3] the entries of $\mathsf{Mix}_{\mathfrak{I}}$ on $\xi \in \mathbb{N}^{11}$ and using oracle calls on graphs derived by the gadget construction from Section 3. This system will feature $O(k^{11})$ linear equations, whose integer coefficients can be computed in time $n^{O(1)}$. Furthermore, the specialness of p^* will imply that the system can be solved unambiguously for $m_{k,0}$. This proves Theorem 2.

4.1 The Polynomial p^* Is Special

We consider expansions of the polynomials $p \in \mathsf{Mix}_{\mathfrak{O}}$ into monomials over $\mathbf{y}$. This is used to show that, after substitution of $\mathcal{Y}$ from Section 3.2 into $\mathbf{y}$, the polynomial $p^* = [X^{2k}]\mathsf{Mix}_{\mathfrak{I}}^{(k,0)}$ associated with $m_{k,0}$ satisfies the following:

Theorem 3. *Substitute $\mathcal{Y}$ into $\mathbf{y}$ in all of $\mathsf{Mix}_{\mathfrak{O}}$ to obtain the matrix $\mathsf{Mix}_{\mathfrak{I}}$. Then $p^* = [X^{2k}]\mathsf{Mix}_{\mathfrak{I}}^{(k,0)}$ is not in the span of the other entries in $\mathsf{Mix}_{\mathfrak{I}}$. Formally, if*

$$\sum_{0\leq\rho\leq t\leq K} \alpha_{t,\rho} \cdot [X^{K-t}]\mathsf{Mix}_{\mathfrak{I}}^{(t,\rho)} \equiv 0, \tag{4}$$

with $\alpha_{t,\rho} \in \mathbb{Q}$ for all $0 \leq \rho \leq t \leq K$, then $\alpha_{k,0} = 0$.

This theorem will be proven at the end of this subsection. We first consider polynomials $p \in \mathsf{Mix}_{\mathfrak{O}}$ and require some notation for the set of monomials *appearing* in p. Recall that $\mathfrak{O} = \mathbb{Z}[\mathbf{y}]$, and note that $[\theta]p \in \mathbb{Z}$ if $p \in \mathfrak{O}$ and $\theta \in \mathbb{N}^{\mathbf{y}}$.

Definition 8. *For $p \in \mathfrak{O}$, let $\mathfrak{M}p = \{\theta \in \mathbb{N}^{\mathbf{y}} \mid [\theta]p \neq 0\}$. For $P \subseteq \mathfrak{O}$, define $\mathfrak{M}P = \bigcup_{p\in P} \mathfrak{M}p$. If $\theta \in \mathbb{N}^{\mathbf{y}}$ and $\theta \in \mathfrak{M}P$, we say that θ* appears *in P.*

Our proof of Theorem 3 proceeds as follows: We first identify a special monomial $\theta^* \in \mathbb{N}^{\mathbf{y}}$ and show that, among all $p \in \mathsf{Mix}_{\mathfrak{O}}$, the monomial θ^* appears only in $p = p^*$. Using this, we show that containment of p^* in the span of the other polynomials yields an annihilator for $\mathcal{Y}$ that contradicts Corollary 1.

To begin with, we define several quantities associated with monomials in $\mathbb{N}^{\mathbf{y}}$:

Definition 9. *Let $\theta \in \mathbb{N}^{\mathbf{y}}$ and observe that θ is of the form*

$$\theta = (\dot{B}_1^{b_1}\dot{B}_2^{b_2})(\dot{U}_1^{u_1}\dot{U}_2^{u_2}\dot{U}_3^{u_3})(\dot{V}_1^{v_1}\dot{V}_2^{v_2}\dot{V}_3^{v_3})(\dot{F}_1^{f_1}\dot{F}_2^{f_2}\dot{F}_3^{f_3}\dot{F}_4^{f_4}).$$

Define $\mathrm{td}(\theta) := \sum_{i=1}^{4} i(b_i + u_i + v_i + f_i)$. *Let $\Theta := \mathfrak{M}\mathsf{Mix}_{\mathfrak{O}}$. For $\ell \in \mathbb{N}$, let $\Theta_\ell := \Theta \cap \{\theta \mid \mathrm{td}(\theta) = \ell\}$. Let* $\mathrm{occ}(\theta) := (\sum_i b_i, \sum_i u_i, \sum_i v_i, \sum_i f_i)$ *and write* $\mathrm{occ}_B(\theta)$ *for the first entry of* $\mathrm{occ}(\theta)$.

[3] Recall that $\mathfrak{I} = \mathbb{Z}[\mathbf{x}]$, where $\mathbf{x}$ is the tuple of 11 indeterminates from (2) in Section 3.2. Thus, evaluating $\mathsf{Mix}_{\mathfrak{I}}^{(t,\rho)}(\xi)$ at $\xi \in \mathbb{N}^{11}$ yields an integer value.

Example 1. Let $\theta = \dot{B}_1^1 \dot{B}_2^2 \dot{U}_2^4 \dot{V}_2^5 \dot{F}_1^6$. Then $\mathrm{td}(\theta) = 1 \cdot (1+6) + 2 \cdot (2+4+5) = 29$ and $\mathrm{occ}(\theta) = (3,4,5,6)$. Furthermore, we have $\mathrm{occ}_B(\theta) = 3$.

This notation is used for the statement of the following lemma, which follows from a relatively straightforward application of the multinomial theorem.

Lemma 5. *Let $0 \leq t \leq K$. For $a, b_1, \ldots, b_\ell \in \mathbb{N}$ with $s := \sum_i b_i \leq a$, let $\binom{a}{b_1,\ldots,b_\ell} = \frac{a!}{b_1!\ldots b_\ell!(a-s)!}$. With $\theta \in \mathbb{N}^{\mathcal{Y}}$ written as in Definition 9, we have*

$$[X^{K-t}]\mathsf{Mix}_{\mathfrak{D}}^{(t,\rho)} = \sum_{\theta \in \Theta_{K-t}} \underbrace{\binom{t-\rho}{b_1,b_2}\binom{\rho}{u_1,u_2,u_3}\binom{\rho}{v_1,v_2,v_3}\binom{n-t-\rho}{f_1,f_2,f_3,f_4}}_{=:\lambda_{t,\rho}(\theta)} \theta.$$

Corollary 2. A monomial $\theta \in \Theta$ appears in $[X^{K-t}]\mathsf{Mix}_{\mathfrak{D}}^{(t,\rho)}$ iff $\mathrm{td}(\theta) = K - t$ and $\lambda_{t,\rho}(\theta) \neq 0$. The second condition is true iff $\mathrm{occ}(\theta) \leq (t-\rho, \rho, \rho, n-t-\rho)$, where $\leq$ is considered component-wise.

We now define the *special monomial* $\theta^* := \dot{B}_2^k$ and show that it appears only in the previously defined special polynomial $p^* = [X^{2k}]\mathsf{Mix}_{\mathfrak{D}}^{(k,0)}$.

Lemma 6. *If $\theta \in \Theta$ contains $\theta^* = \dot{B}_2^k$ as a factor, then $\theta = \theta^*$. Furthermore, if θ^* appears in $p \in \mathsf{Mix}_{\mathfrak{D}}$, then $p = p^*$. In fact, we have $[\theta^*]p^* = 1$.*

Proof. If $\theta \in \Theta$ contains $\dot{B}_2^k$, then $\mathrm{td}(\theta) \geq 2k$. Since $\theta \in \Theta$, it must appear in $[X^{K-t}]\mathsf{Mix}_{\mathfrak{D}}^{(t,\rho)}$ for some $0 \leq \rho \leq t \leq K$. Then $K - t \geq \mathrm{td}(\theta)$ by Corollary 2. Recall that $K = 3k$, implying $t \leq k$. Since θ contains $\dot{B}_2^k$, we have $\mathrm{occ}_B(\theta) \geq k$. But by Corollary 2, we also have $\mathrm{occ}_B(\theta) \leq t - \rho$.

The last two inequalities and $t \leq k$ imply $\rho = 0$ and $\mathrm{occ}_B(\theta) = k$. Thus θ appears only in p^*. But then $\mathrm{td}(\theta) = 2k$, and thus $\theta = \theta^*$. Finally, $[\theta^*]p^* = \lambda_{k,0}(\theta^*) = 1$ follows independently from Lemma 5. □

This allows us to finish the subsection with the promised proof of Theorem 3.

Proof (of Theorem 3). Assume there were coefficients $\alpha_{t,\rho}$ satisfying (4) with $\alpha_{k,0} \neq 0$. With $\lambda_{t,\rho}(\theta)$ from Lemma 5, write $[X^{K-t}]\mathsf{Mix}_{\mathfrak{D}}^{(t,\rho)} = \sum_{\theta \in \Theta} \lambda_{t,\rho}(\theta) \cdot \theta$ and rearrange (4) to obtain

$$A := \left(\alpha_{k,0} \cdot \sum_{\theta \in \Theta} \lambda_{k,0}(\theta) \cdot \theta \right) + \sum_{\theta \in \Theta} \theta \sum_{\substack{0 \leq \rho \leq t \leq K \\ (t,\rho) \neq (k,0)}} \alpha_{t,\rho} \cdot \lambda_{t,\rho}(\theta) \equiv 0. \tag{5}$$

By Lemma 6, the monomial $\theta^* = \dot{B}_2^k$ appears only in the parentheses and has $\lambda_{k,0}(\theta^*) = 1$. Regrouping (5) yields $A = \alpha_{k,0} \cdot \theta^* + \sum_{\theta \neq \theta^*} \mu(\theta) \cdot \theta$, for new coefficients μ, with the property that $A(B_1, \ldots, F_4) \equiv 0$.

Also by Lemma 6, the only monomial in A that contains θ^* is θ^* itself. Therefore, A is a nontrivial annihilator for the set $\mathcal{Y}$ from Section 3.2, with the property that $[\theta^*]A = \alpha_{k,0} \in \mathbb{Q}$ is non-zero. Corollary 1 then leads to a contradiction: Since $[\theta^*]A \neq 0$ is constant, it is unaffected by the substitution $\dot{F}_1 := \dot{U}_1 + \dot{V}_1 - \dot{B}_1$, thus contradicting $A_{\theta^*} \equiv 0$ from Corollary 1. □

4.2 Deriving Linear Equations from Mix

In this subsection, we complete the proof of Theorem 2. For this, we substitute the elements in $\mathcal{Y}$ from Section 3.2 into $\mathsf{Mix}_{\mathfrak{O}}$. Using the gadget $\mathcal{V}(\mathbf{x})$, we can evaluate the resulting polynomials $\mathsf{Mix}_{\mathfrak{I}}^{(t,\rho)}$ to yield integer values.

Definition 10. *For $\xi \in \mathbb{N}^{11}$, let $\mathsf{Mix}(\xi) \in \mathbb{Z}^{(K+1)\times(K+1)}$ be the matrix obtained from $\mathsf{Mix}_{\mathfrak{I}}$ by evaluating each of its entries $p \in \mathfrak{I}$ at ξ.*

For $\Xi = (\xi_1, \ldots, \xi_D)$ with $\xi_i \in \mathbb{N}^{11}$, let $\mathsf{Mix}(\Xi) \in \mathbb{Z}^{D\times(K+1)^2}$ be such that the i-th row of $\mathsf{Mix}(\Xi)$ contains the entries of $\mathsf{Mix}(\xi_i)$ as a row vector. Consider columns of $\mathsf{Mix}(\Xi)$ to be indexed by pairs (t,ρ) and write $A^{(t,\rho)}$ for column (t,ρ).

Remark 5. If $|\Xi| \leq n^{O(1)}$ and all entries of Ξ have bit-length $n^{O(1)}$, then $\mathsf{Mix}(\Xi)$ can be computed in time $n^{O(1)}$: It holds by Definition 5 that $\mathsf{Mix}_{\mathfrak{O}}^{(t,\rho)} \in \mathfrak{O}[X]$ is the product of n polynomials, each of degree ≤ 4. Any $[X^\ell]\mathsf{Mix}_{\mathfrak{O}}^{(t,\rho)} \in \mathfrak{O}$ can therefore be computed, the elements in $\mathcal{Y}$ can be substituted into it, and the resulting $p \in \mathfrak{I}$ can be evaluated at any ξ_i, all in time $n^{O(1)}$.

In the following, we fix $\Xi = (\xi_1, \ldots, \xi_D)$, with $D = (K+1)^{11}$, to some enumeration of the grid $\{0, \ldots, K\}^{11}$. Furthermore, if $B \in \mathbb{Z}^{\ell\times b^2}$ is a matrix whose columns are indexed by pairs (i,j), and $C \in \mathbb{Z}^{b\times b}$, let $B \odot C \in \mathbb{Z}^\ell$ be defined by $(B \odot C)_t = \sum_{ij} B_{t,(i,j)} C_{ij}$. It can be checked that Lemma 4 implies

$$\mathsf{Mix}(\Xi) \odot \begin{pmatrix} m_{0,0} & \ldots & m_{K,0} \\ & \ddots & \vdots \\ & & m_{K,K} \end{pmatrix} = \begin{pmatrix} [X^K]M(H(\xi_1)) \\ \vdots \\ [X^K]M(H(\xi_D)) \end{pmatrix}, \tag{6}$$

with $H(\xi)$ for $\xi \in \mathbb{N}^{11}$ as in Remark 4. Recall that $[X^K]M(H(\xi)) \in \mathbb{N}$ counts the K-matchings in $H(\xi)$. Since $K = 3k$, we can thus evaluate the right-hand side of (6) by D oracle queries of the form $(H(\xi), K)$ to $\mathsf{p\#Match}$.

We consider (6) as a linear system of equations in the unknowns $m_{t,\rho}$. By Remark 5, $\mathsf{Mix}(\Xi)$ can be evaluated in time $n^{O(1)}$. This implies that a solution to (6) can also be found in time $n^{O(1)}$. The final and crucial step towards the proof of Theorem 2 now consists of showing that all solutions to (6) agree on their values for $m_{k,0}$. For this, we build upon Theorem 3 to show that column $(k,0)$ of $\mathsf{Mix}(\Xi)$ is not contained in the linear span of its other columns.

First, we require a generalization of the fact that every univariate polynomial $p \in \mathbb{Z}[x]$ of degree d that vanishes at $d+1$ points has $p \equiv 0$. This is stated in the following lemma, a corollary of the classical Schwartz-Zippel lemma [8,15].

Lemma 7. *Let $p \in \mathbb{Z}[x_1, \ldots, x_s]$ be a polynomial with $\deg(p) \leq d$. If $p(\xi) = 0$ holds for all $\xi \in \{0, \ldots, d\}^s$, then $p \equiv 0$.* □

From this, we obtain the last missing step for the proof of Theorem 2.

Lemma 8. *If $\sum_{t,\rho} \alpha_{t,\rho} \cdot A^{(t,\rho)} = 0$ for coefficients $\alpha_{t,\rho} \in \mathbb{Q}$, then $\alpha_{k,0} = 0$.*

Proof. Observe that $\deg(p) \leq K$ for $p \in \mathsf{Mix}_{\mathfrak{Y}}$: All monomials θ appearing in $p \in \mathsf{Mix}_{\mathfrak{O}}$ have $\mathrm{td}(\theta) \leq K$, and it can be verified that substituting $\mathcal{Y}$ into $\mathbf{y}$ then yields polynomials of degree $\leq K$. Also recall that $\mathfrak{Y} = \mathbb{Z}[\mathbf{x}]$ with $|\mathbf{x}| = 11$.

Assume there were coefficients $\alpha_{t,\rho}$ with $\alpha_{k,0} \neq 0$ and $\sum_{t,\rho} \alpha_{t,\rho} \cdot A^{(t,\rho)} = 0$. Then $q = \sum_{t,\rho} \alpha_{t,\rho} \cdot [X^{K-t}]\mathsf{Mix}_{\mathfrak{Y}}^{(t,\rho)}$ vanishes on $\{0, \ldots, K\}^{11}$. Thus $q \equiv 0$ by Lemma 7, contradicting Theorem 3 because $\alpha_{k,0} \neq 0$. □

Acknowledgements. The author wishes to thank Mingji Xia for sharing ideas that were fundamental for the Venn gadget from Section 3.2, Markus Bläser for mentioning algebraic independence in the right moment, and an anonymous reviewer, whose comments helped improving the presentation of the paper.

References

1. Bläser, M., Curticapean, R.: Weighted counting of k-matchings is #W[1]-hard. In: Thilikos, D.M., Woeginger, G.J. (eds.) IPEC 2012. LNCS, vol. 7535, pp. 171–181. Springer, Heidelberg (2012)
2. Courcelle, B.: The monadic second-order logic of graphs. I. Recognizable sets of finite graphs. Information and Computation 85(1), 12–75 (1990)
3. Ehrenborg, R., Rota, G.-C.: Apolarity and canonical forms for homogeneous polynomials. European Journal of Combinatorics 14(3), 157–181 (1993)
4. Flum, J., Grohe, M.: The parameterized complexity of counting problems. SIAM Journal on Computing, 538–547 (2002)
5. Flum, J., Grohe, M.: Parameterized Complexity Theory. Springer-Verlag New York, Inc., Secaucus (2006)
6. Hartshorne, R.: Algebraic geometry. Springer (1977)
7. Makowsky, J.: Algorithmic uses of the Feferman-Vaught theorem. Annals of Pure and Applied Logic 126, 159–213 (2004)
8. Schwartz, J.: Fast probabilistic algorithms for verification of polynomial identities. J. ACM 27(4), 701–717 (1980)
9. Temperley, H., Fisher, M.: Dimer problem in statistical mechanics - an exact result. Philosophical Magazine 6(68) (1961) 1478–6435
10. Valiant, L.: The complexity of computing the permanent. Theoretical Computer Science 8(2), 189–201 (1979)
11. Valiant, L.: The complexity of enumeration and reliability problems. SIAM Journal on Computing 8(3), 410–421 (1979)
12. Valiant, L.: Holographic algorithms. In: Proceedings of the 45th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2004, pp. 306–315 (2004)
13. Vassilevska, V., Williams, R.: Finding, minimizing, and counting weighted subgraphs. In: Proceedings of the 41st Annual ACM Symposium on Theory of Computing, STOC 2009, pp. 455–464. ACM, New York (2009)
14. Xia, M., Zhang, P., Zhao, W.: Computational complexity of counting problems on 3-regular planar graphs. Theor. Comp. Sc. 384(1), 111–125 (2007)
15. Zippel, R.: Probabilistic algorithms for sparse polynomials. In: Ng, K.W. (ed.) EUROSAM 1979 and ISSAC 1979. LNCS, vol. 72, pp. 216–226. Springer, Heidelberg (1979)