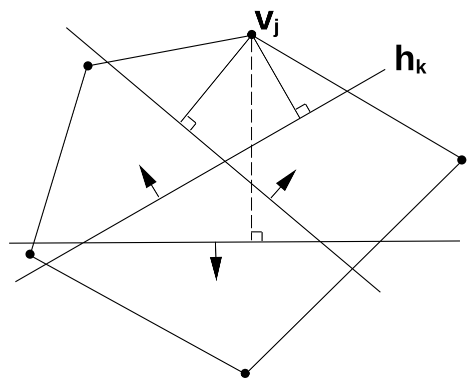
A Conservative $O(mnd)$-Time Infeasibility Test for Linear Programs Posed Within a Convex Region

SETH TELLER, MIT

The following is a simple observation leading to a "fast infeasibility test" for linear programs posed inside convex regions for which a vertex description is known. For a set of m constraints and n vertices in d dimensions, the test requires $O(m \cdot n \cdot d)$ time.

The test is conservative; if it succeeds, the posed LP is known to be infeasible within the specified region. If the test fails, the LP must be solved using other means.

Theorem. Given n points $\mathbf{v}_j,\ 1 \le j \le n$, and m hyperplanes $\mathbf{h}_k,\ 1 \le k \le m$, such that $\forall\, j,\ \sum_{k=1}^{m}(\mathbf{h}_k \cdot \mathbf{v}_j) < 0$, there exists no point $\mathbf{p}$ in *conv*$(\mathbf{v}_j)$ such that $\forall\, k,\ \mathbf{h}_k \cdot \mathbf{p} \ge 0$. (Here, the inner product represents the signed distance between a point and plane.)



Proof (by contradiction). Suppose there exists a point $\mathbf{p}$ in *conv*$(\mathbf{v}_j)$ such that

$$\forall\, k, \mathbf{h}_k \cdot \mathbf{p} \ge 0.$$

By convexity,

$$\mathbf{p} = \sum_{j=1}^{n} c_j \mathbf{v}_j;\ \ \forall\, j, c_j \ge 0;\ \text{and}\ \sum_{j=1}^{n} c_j = 1.$$

Substituting and summing over k,

$$\sum_{k=1}^{m} \mathbf{h}_k \cdot \sum_{j=1}^{n} c_j \mathbf{v}_j \ge 0.$$

Moving the inner product inside the j summation,

$$\sum_{k=1}^{m} \sum_{j=1}^{n} c_j (\mathbf{h}_k \cdot \mathbf{v}_j) \ge 0,$$

and exchanging summation order yields

$$\sum_{j=1}^{n} c_j \left(\sum_{k=1}^{m} (\mathbf{h}_k \cdot \mathbf{v}_j) \right) \ge 0,$$

a contradiction, since the c_j are nonnegative and not all zero. □

Seth Teller, Princeton CS Dept., Spring 1994