“Cookbook bounds” for 1 Master recurrences

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Abstract

In analysis of algorithms, recurrences with more than one recursive term can arise since a problem can be divided into subproblems of different size. The usual Master Theorem provides a tight bound of a given recurrence when only one recursive term is given. We analyze the general case providing a bound for this kind of recurrences.

1 Introduction

The Master Theorem provides a “cookbook” method for solving the following “Master recurrence”

$$T(n) = aT(\frac{n}{b}) + F(n) \tag{1}$$

where $a \geq 1$ and $b > 1$ are constants and $F(n)$ is an asymptotically non decreasing positive function. In all our recurrences we assume without losing generality, that the initial condition has the form $T(n) = C$ for all $n > n_0 > 0$. In addition, every time we refer to $F(n)$ the above restriction must be hold.
The recurrence (1) describes the running time of an algorithm that divides a problem of size n into a subproblems, each of size $\frac{n}{b}$ solved recursively in time $T(\frac{n}{b})$, and where the function $F(n)$ represents the cost of dividing the problem into subproblems and combining them.

As it is known, the Master Theorem implies the comparison of $F(n)$ with the watershed function $n^{\log_b a}$ representing the homogeneous solution of (1) which can be also interpreted as the cost of doing approximately $a^{\log_b n}$ subproblems of size $O(1)$.
Intuitively, the asymptotic solution of the recurrence is determined by the larger of the two functions: the cost of the leaves or the cost of all the internal nodes of the recursion tree.
We provide some results useful to deal with 1 recurrences and an upper bound, in some case tight, for the following "1 Master recurrence":

$$T(n) = a_1 T(\frac{n}{b_1}) + a_2 T(\frac{n}{b_2}) + \cdots + a_m T(\frac{n}{b_m}) + F(n) \qquad (2)$$

with $a_i \geq 1$ and $b_i > 1$ constants, and $b_1 < b_i$ for $1 < i \leq m$.
In order to proceed, we look for a bound, as tight as possible, of the homogeneous solution of the (2), that is its new watershed function.
By standard techniques we can transform (2) into the following one

$$t(k) = a_1 t(k - \epsilon_1) + a_2 t(k - \epsilon_2) + \ldots + a_m t(k - \epsilon_m) + f(k) \qquad (3)$$

with $\epsilon_1 = 1$ and $\epsilon_1 < \epsilon_i$ for $1 < i \leq m$.
It is difficult to provide a tight bound for the homogeneous solution for the (3) because of the complex relationship among a_i and b_i with $i = 1 \ldots m$. Therefore, before eliminating one or more of its recursive terms, we have to analyze these relationships. Sometimes we can 0 (3) into a recurrence of the form (1) with $a = G(a_1, \ldots, a_m, b_1, \ldots, b_m)$. But the analysis of this, carries a variety of cases and computations that takes us too far from our desire to provide a "cookbook" bound. On the other hand, it is obvious that the more accuracy we require the more computations we need.

2 Dealing with real recurrences

Lemma 1. *Let $t(k)$ be defined by the equation (3), with ϵ_i positive real constant for $1 \leq i \leq m$ and $f(k) = O(\sum_{i=1}^{k-1} f(i))$.*
If $\lfloor \epsilon_i \rfloor \geq 1$ and $s(k)$ is defined by the following equation

$$s(k) = a_1 s(k - \lfloor \epsilon_1 \rfloor) + a_2 s(k - \lfloor \epsilon_2 \rfloor) + \ldots + a_m s(k - \lfloor \epsilon_m \rfloor) + f(k)$$

then $t(k) = \Theta(s(k))$.
Similarly, if $r(k)$ *is defined by the following equation*

$$r(k) = a_1 r(k - \lceil\epsilon_1\rceil) + a_2 r(k - \lceil\epsilon_2\rceil) + \ldots + a_m r(k - \lceil\epsilon_m\rceil) + f(k)$$

then, $t(k) = \Theta(r(k))$.
Therefore, the order of the (3) is not affected by rounding the argument of its recursive terms "by ceilings" or "by floors", provided the "recursive arguments" less than k. lemma

Proof

Since $t(k)$ is a non decreasing function, we have $t(k) \leq s(k)$.
Claim: $s(k) = \Omega(t(k))$ that is: $\exists C > 0$ such that $Cs(k) \leq t(k)$.
By a property of the floors and since $f(k)$ is non decreasing, $s(k-1) \leq t(k)$.
Then, we have to show that $\exists D_1 > 0$ such that $s(k) \leq D_1 s(k-1)$.
Since $\lfloor\epsilon_i\rfloor \geq 1$ for $i = 1, \ldots, m$ we can infer that

$$\frac{s(k)}{s(k-1)} = a_1 \frac{s(k - \lfloor\epsilon_1\rfloor)}{s(k-1)} + \ldots + a_m \frac{s(k - \lfloor\epsilon_m\rfloor)}{s(k-1)} + \frac{f(k)}{s(k-1)}$$

$$\leq a_1 + \ldots + a_m + \frac{f(k)}{\sum_{i=1}^{k-1} f(i)}$$

By the hypotheses $\frac{f(k)}{\sum_{i=1}^{k-1} f(i)} = O(1)$, then $t(k) = \Theta(s(k))$.

The proof of the second part of the Lemma is very similar to the previous one. Since $f(k)$ is not decreasing and by a property of ceilings we have $t(k-1) \leq r(k) \leq t(k)$.
Claim: $t(k) = \Omega(r(k))$.
Then we have to show that $\exists D_2 > 0$ such that $D_2 t(k) \leq t(k-1)$.
If $\epsilon_i \geq 1$ for $1/leqi/leqm$, we proceed as in the first part, else provided $\epsilon = \min(\epsilon_1, \ldots, \epsilon_m)$, we have

$$\frac{t(k)}{t(k-1)} \leq (a_1 + \cdots + a_m)\frac{t(k-\epsilon)}{t(k-1)} + \frac{f(k)}{t(k-1)}$$

$$< (a_1 + \cdots + a_m)^2 (1 + \frac{\int_\epsilon^1 f(k-x)dx}{t(k-1)}) + \frac{f(k)}{t(k-1)}$$

Therefore $t(k) = \Theta(r(k))$.□

Remark

If $f(k)$ is at least doubly exponential in the (3), then the hypothesis of the

lemma does not hold.
Claim: Under this hypothesis $t(k) = \Theta(f(k))$.
Proof: let ϵ be $\min(\epsilon_1, \ldots, \epsilon_m)$, then

$$t(k) \le t_1(k) = (a_1 + \ldots + a_m)t_1(k - \epsilon) + f(k)$$

If $\epsilon \ge 1$ we have:

$$t(k) \le t_2(k) = (a_1 + \ldots + a_m)t_2(k - 1) + f(k)$$

Then $t(k) = O(\sum_{i=0}^{k}(a_1 + \ldots + a_m)^i f(k - i))$ and since $f(k)$ is doubly exponential, $t(k) = \Theta(f(k)$.

Lemma1 allows us to compute the solution of any 1 recurrence as we discuss later, by transforming it into another recurrence whose arguments are integer, and then by computing the roots of its characteristic polynomial. This transformation allows us to refer to the characteristic polynomial since otherwise we should deal with polynomials with real exponents.

Example:
Let $T(n)$ be defined by the following equation

$$T(n) = 2T(\frac{n}{2}) + 3T(\frac{n}{5}) + 4T(\frac{n}{20}) + 5T(\frac{n}{25}) + n$$

By the transformation $k := \log_2 n$ and by Lemma1 we get the following recurrence:

$$r(k) = 2r(k - 1) + 3r(k - 2) + 9T(k - 4) + 2^k$$

The order of $T(n)$ is obtained applying the known techniques to $r(k)$ that imply the computation of the roots of its characteristic polynomial

$$x^4 - 2x^3 - 3x^2 - 9 = 0$$

2.1 Translations

Some consequences helpful to deal with 1 recurrences can be drawn from Lemma1.

Lemma 2. *Let $t(k)$ be defined by (3) and $s(k)$ by the following equation*

$$s(k) = a_1 s(k - \epsilon_1 + c) + \ldots + a_m s(k - \epsilon_m + c) + f(k)$$

If ϵ_i and c are real positive numbers with $0 < c < \min(\epsilon_1, \ldots, \epsilon_m)$ and $f(k) = O(\sum_{i=1}^{k-1} f(i))$, then $t(k) = \Theta(s(k))$.
lemma

Proof
The proof is similar to the Lemma1.
In this case $s(k - c) \leq t(k) \leq s(k)$.
Claim: $s(k) = \Omega(t(k))$.
We have to show that $\exists D > 0$ such that $Ds(k) \leq s(k - c)$.
From the hypothesis we can infer:

$$\frac{s(k)}{s(k-c)} \leq a_1 \frac{s(k - \epsilon_1 + c)}{s(k-c)} + \cdots + a_m \frac{s(k - \epsilon_m + c)}{s(k-c)} + \frac{f(k)}{s(k-c)}$$

Let $\epsilon = \min(\epsilon_1, \ldots, \epsilon_m)$

$$\frac{s(k)}{s(k-c)} \leq (a_1 + \cdots + a_m) \frac{s(k - \epsilon + c)}{s(k-c)} + \frac{f(k)}{s(k-c)}$$

If $c \leq \frac{\epsilon}{2}$ then

$$\frac{s(k)}{s(k-c)} \leq (a_1 + \ldots + a_m) + \frac{f(k)}{s(k-c)}$$

and we proceed in the same way as done for Lemma 1, else we have

$$\frac{s(k)}{s(k-c)} < (a_1 + \cdots + a_m)^{2c+1} (1 + \frac{\int_{\epsilon-c}^{c} f(k-x)dx}{s(k-c)} + \frac{f(k)}{s(k-c)}$$

Since $\frac{\int_{\epsilon-c}^{c} f(k-i)}{s(k-c)} = \Theta(1)$, then $t(k) = \Theta(s(k))$.$\Box$

Example
We can apply Lemma1 and Lemma2 to compute the order of recurrences

defined by the equation (3) with $|\epsilon_i - \epsilon_j| < 1$ for $i \neq j$.
Let $T(n)$ be the following recurrence

$$T(n) = 2T(\sqrt[4]{n} - 1) + 4T(\sqrt[4]{n} - 2) + 10T(\sqrt[4]{n} - 5) + 16T(\sqrt[4]{n} - 8) + \log n$$

By the transformation $n := 2^{2^k}$ and by applying Lemma2 with $c = 1$ and Lemma1, for all $n > 256$ we have $T(n) = \Theta(r(k))$ with $r(k)$ defined by the following equation

$$r(k) = 32r(k-1) + 2^k = \Theta(32^k)$$

Then $T(n) = \Theta(\log n^5)$.

2.2 A more general case

Applying Lemma1 we can handle the following recurrence

$$T(n) = a_1 T(n - c_1) + \cdots + a_m T(n - c_m) + d_1 T(\frac{n}{b_1}) + \cdots + d_s T(\frac{n}{b_s}) + F(n)$$

provided $F(n) = O(\sum_{i=0}^{n-1} F(i))$.
Let $b_1 = \min(b_1, \ldots, b_s)$ and $n_0 = \max(\frac{c_1 b_1}{b_1 - 1}, \ldots, \frac{c_m b_1}{b_1 - 1})$.
By the 0 $s(k) := t(b_1^k)$ and since $k - 1 < \log_{b_1}(n - c) < k$ for $n > n_0$, we have $s(k-1) \leq t(n - c_i) \leq s(k)$.
Applying Lemma1 we have $T(n) = \Theta(r(k))$ with $r(k)$ be defined by the following equation

$$r(k) = (a_1 + \cdots + a_m + d_1) r(k-1) + d_2 t(k - \lceil \alpha_2 \rceil) + \cdots + d_s t(k - \lceil \alpha_s \rceil) + f(k)$$

Example:
Let $T(n)$ be defined by the following equation:

$$T(n) = T(n-2) + T(n-4) + T(n-6) + 2T(\frac{n}{5}) + 3T(\frac{n}{10}) + 3T(\frac{n}{15}) + n$$

By the transformation $k := \log_5 n$ and by Lemma1 $T(n)$ has the same order of the following recurrence $r(k)$ for all $n > \frac{15}{2}$

$$r(k) = 5r(k-1) + 6r(k-2) + 5^k = \Theta(6^k)$$

Then $T(n) = \Theta(n^{\log_5 6})$.

3 Recurrences with two recursive terms

Let us analyze now recurrences of the form (2) but involving only two recursive terms.

3.1 Case $b_1 \leq b_2 < b_1^2$

Under this hypothesis we provide a tight upper bound.

Lemma 3. *Let $a_1, a_2 \leq 1$ and $b_1, b_2 > 1$ be constants and $b_1 \leq b_2 \leq b_1^2$. If $F(n) = O(\sum_{i=0}^{n-1} F(i))$ and $T(n)$ is defined by the following recurrence*

$$T(n) = a_1 T(\frac{n}{b_1}) + a_2 T(\frac{n}{b_2}) + F(n) \qquad (4)$$

then $T(n)$ can be bounded 0 as follows:
1. *If $F(n) = O(n^{\log_{b_1}(a_1+a_2)-\varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_{b_1}(a_1+a_2)})$.*
2. *If $F(n) = \Theta(n^{\log_{b_1}(a_1+a_2)})$, then $T(n) = \Theta(n^{\log_{b_1}(a_1+a_2)} \log_{b_1} n)$.*
3. *If $F(n) = \Omega(n^{\log_{b_1}(a_1+a_2)+\varepsilon})$ for some constant $\varepsilon > 0$ and if $(a_1+a_2)F(\frac{n}{b_1}) \leq cF(n)$ for $c < 1$, then $T(n) = \Theta(F(n))$. lemma*

Proof

Let $t(n)$ be defined by the following recurrence

$$t(n) = (a_1 + a_2)t(\frac{n}{b_1}) + F(n)$$

Claim: $T(n) = \Theta(t(n))$
By the transformation $k = \log_{b_1} n$ we get the following recurrence

$$\tau(k) := a_1\tau(k-1) + a_2\tau(k-\alpha) + f(k)$$

with $\alpha = log_{b_1} b_2$.
Since $\lfloor \alpha \rfloor = 1$ and by Lemma1 we get the claim.
Then applying the Master Theorem to $t(n)$ we prove Lemma3. $\square$

3.2 Case $b_2 \geq b_1^2$

In this case we look for an upper bound as tight as possible for the homogeneous solution of (4).

By the transformation $k = \log_{b_1} n$ and applying Lemma1, the (4) is transformed into the following recursion

$$\tau(k) := a_1\tau(k-1) + a_2\tau(k-\alpha) + f(k) \tag{5}$$

with $\alpha = \lfloor \log_{b_1} b_2 \rfloor$.
We can determine the order of the (4) applying the following known technique, that is by computing the characteristic polynomial of the (5).

$$x^i - a_1 x^{i-1} - a_2 = 0$$

Fact 1. *Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_kx^{n-k}$ be the characteristic equation of the recurrence*

$$a_0T_n + a_1T_{n-1} + \ldots + a_kT_{n-k} = 0 \tag{6}$$

*with $n \geq k$ and a_i constants. Let the roots of $f(x)$, over the complex numbers, be r_i and their respective multiplicity be q_i with $i = 1, \ldots, m$.
Then, any solution to (6) is of the form $\sum_{i=1}^{m}(r_i^n \sum_{j=0}^{q_i-1} c_{ij}n^j)$ where the c_{ij} are constants.*

Since to provide a solution of (5) would imply the computation of the roots of the characteristic polynomial and we are looking for "cookbook bounds", we want to improve the trivial bound bound for the homogeneous solution of (5) $n^{\log_{b_1}(a_1+a_2)}$.
Many bounds on the roots of a given polynomial are known. The following seems most appropriate for us, even if there are cases in which it is less tight than the others.

Fact 2. *(Cauchy) Let α be any root 19 the polynomial*

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0$$

then $|\alpha| < 1 + \frac{\max(|a_0|, \ldots, |a_{n-1}|)}{|a_n|}$

From Fact2 we can draw the following:

Lemma 4. *Let* γ *be* $\max(a_1, a_2)$.
Then an upper bound of (4) is given by $O(n^{\log_{b_1}(\gamma+1)})$ *lemma*

Remark
This bound can be considered a good approximation if $a_1 \geq a_2$. In this case $T(n) = O(n^{\log_{b_1}(a_1+1)})$ and $T(n) = \Omega(n^{log_{b_1} a_1})$.
The bigger a_i is with respect to a_1 and $log_{b_1} a_i$ with respect to $log_{b_1} a_1$ the less tight this bound is.
We could apply other polynomial root bounds, such as the following Cauchy bound:

$$|\alpha| \leq \max\left(\frac{n|a_{n-1}|}{|a_n|}, \sqrt{\frac{n|a_{n-2}|}{|a_n|}}, \ldots, \sqrt[n]{\frac{n|a_0|}{a_n}}\right)$$

This bound, easy to apply to recursions with only two terms involves too many computations in the general case since we should compute the characteristic polynomial and reorder its terms.

4 General Case

We can easily apply what discussed before, to recurrences of the form (2) by computing $\lfloor \log_{b_1} b_i \rfloor$ for $1/leqi/leqm$.

Lemma 5. *Let* $T(n)$ *be defined by the (2).*
Provided the hypotheses of lemma3 and $\lfloor \log_{b_1} b_i \rfloor = 1$ *for* $1 < i \leq m$, $T(n)$ *can be bounded 0 as follows:*
1. *If* $F(n) = O(n^{log_{b_1}(a_1+\ldots+a_m)-\varepsilon})$ *for some constant* $\varepsilon > 0$, *then* $T(n) = \Theta(n^{log_{b_1}(a_1+\ldots+a_2)})$.
2. *If* $F(n) = \Theta(n^{log_{b_1}(a_1+\ldots+a_m)})$, *then* $T(n) = \Theta(n^{log_{b_1}(a_1+\ldots+a_m)} log_{b_1} n)$.
3. *If* $F(n) = \Omega(n^{log_{b_1}(a_1+\ldots+a_m)+\varepsilon})$ *for some constant* $\varepsilon > 0$ *and if* $(a_1 + \ldots + a_m)F(\frac{n}{b_1}) \leq cF(n)$ *for* $c < 1$, *then* $T(n) = \Theta(F(n))$. *lemma*

If the above hypothesis $\lfloor \log_{b_1} b_i \rfloor = 1$, is not satisfied, we can group together all the terms with the same value $\lfloor \log_{b_1} b_i \rfloor$ obtaining a new recurrence. In order to compute an upper bound, we can apply the Cauchy bound to the characteristic polynomial relative to the latter recurrence.

Lemma 6. *Let* γ *be the maximum coefficient of the characteristic polynomial of the new recursion. An upper bound of (2) is given by* $O(n^{\log_{b_1}(\gamma+1)})$ *lemma*

5 Conclusion

We have seen so far a way to compute the order of 1 recurrence of this form

$$T(n) = a_1T(n - c_1) + \cdots + a_mT(n - c_m) + d_1T(\frac{n}{b_1}) + \cdots + d_sT(\frac{n}{b_s}) + F(n)$$

Let $b_1 < b_i$ for $1 < i \leq s$, if $\lfloor \log_{b_1} b_i \rfloor = 1$ then $T(n) = \Theta(r(k))$ with $k := \log_{b_1} n$ and $r(k)$ be defined by the following equation

$$r(k) = (a_1 + \cdots + a_m + d_1 + \cdots + d_s)r(k - 1) + f(k)$$

Otherwise we can apply the known techniques to the following recurrence

$$s(k) = (a_1 + \cdots + a_m + d_1)s(k - 1) + d_2s(k - k_2) + \cdots + d_ss(k - k_s) + f(k)$$

with $k_i = \lceil \log_{b_1} b_i \rceil$.
We can extend what have done, to recurrences whose coefficients are not constants provided them to be functions bounded by a constant.

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